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Algorithm for D-V cells and fundamental domains, $\mathbb{E}^4$ space groups with broken translations in the icosahedral family

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Abstract. As a continuation of our earlier work [2] we extend our algorithm for $\mathbb{E}^4$-space groups in the icosahedral family, where non-lattice translations (broken translations) occur as well. So we obtain new 4-polytopes as fundamental domains from D-V cells of crystallographic orbits. We illustrate our situations by some characteristic examples. A computer program will produce further results.

Key Words: crystallographic group, four-dimensional space group, icosahedral family, fundamental domain, Dirichlet-Voronoi cell

MSC 2000: 20H15, 51F15

1. Introduction

As an example, we start with the crystallographic group $G = \mathbf{P4_1}$ that acts discontinuously and freely on the Euclidean 3-space $\mathbb{E}^3$. Therefore, the orbit space $\mathbb{E}^3/\mathbf{P4_1}$ is a (compact) Euclidean space form.

We choose an origin O and a base $\overrightarrow{OE_1} = \mathbf{e_1}$, $\overrightarrow{OE_2} = \mathbf{e_2}$, $\overrightarrow{OE_3} = \mathbf{e_3}$ for the lattice $\mathbf{L_G}$ with Gramian

$$(\langle \mathbf{e}_i; \mathbf{e}_j \rangle) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{1}$$

where $c > 0$ is an arbitrary constant; as usual $\langle \, ; \, \rangle$ denotes the Euclidean scalar product in $\mathbb{E}^3$.

A generating screw motion $s(\mathbf{s}, \mathbf{S})$ for $G = \mathbf{P4_1}$ will be given by the linear transformation $\mathbf{S} : \mathbb{E}^3 \to \mathbb{E}^3$

$$\mathbf{S}(\mathbf{e_1}\,\mathbf{e_2}\,\mathbf{e_3}) := (\mathbf{Se_1}\,\mathbf{Se_2}\,\mathbf{Se_3}) = (\mathbf{e_1}\,\mathbf{e_2}\,\mathbf{e_3}) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and by the *"broken translation" vector* $\mathbf{s} := \overrightarrow{OO^s}$

$$\mathbf{s} := (\mathbf{e_1}\,\mathbf{e_2}\,\mathbf{e_3}) \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \end{pmatrix}, \tag{3}$$

that does not belong to the lattice $\mathbf{L_G}$. Hence the s-image $Y := X^s$ of point X will be expressed by the position vectors $\overrightarrow{OX} = \mathbf{x} = \mathbf{e}_i x^i$, $\mathbf{e}_j y^j = \mathbf{y} = \overrightarrow{OY} = \overrightarrow{OX^s} = \mathbf{x^s} = \mathbf{Sx} + \mathbf{s} = \mathbf{e}_j \mathbf{s}_i^j x^i + \mathbf{e}_j \mathbf{s}^j$. In homogeneous coordinates:

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} \tag{4}$$

We use the row-column multiplication convention. We can see that the generating translations of $\mathbf{L_G}$ are

$$p_1(\mathbf{e_1}, \mathbf{1}), \quad p_2(\mathbf{e_1}, \mathbf{1}) = s\,p_1\,s^{-1}, \quad p_3(\mathbf{e_3}, \mathbf{1}) = s^4 \tag{5}$$

where $\mathbf{1}$ denotes the identity matrix. For illustration we construct the D-V (Dirchlet-Voronoi) polyhedron $\mathcal{D}_O$ belonging to the origin O. From this one can imagine the whole fundamental tiling by D-V polyhedra according to the G-orbit of O:

$$O^G := \{O^G \in \mathbb{E}^3 : g \in G\}. \tag{6}$$

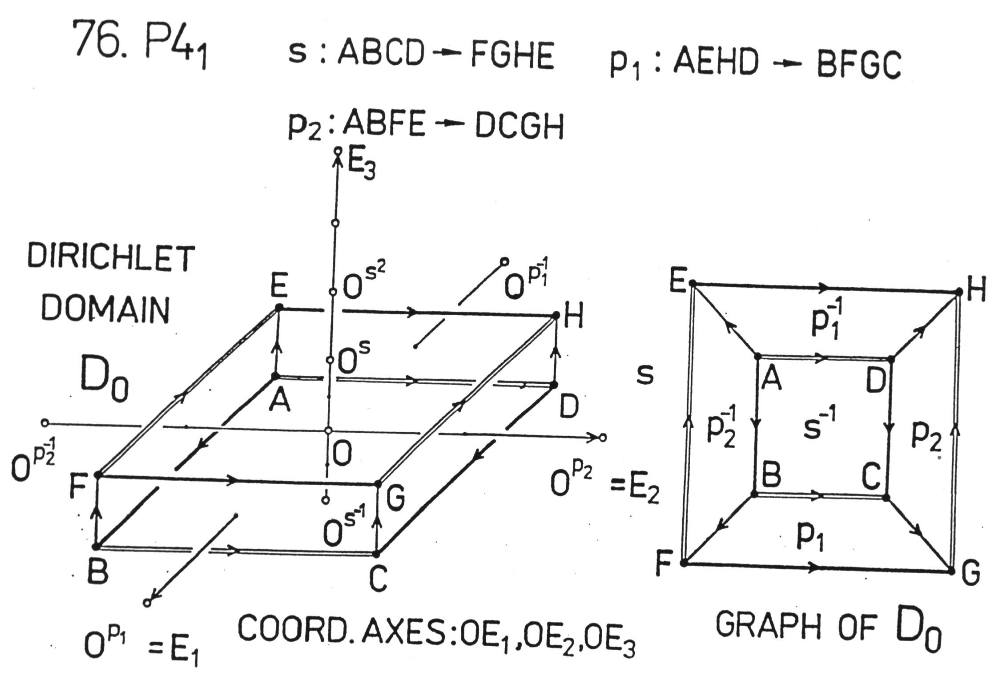


Figure 1: Dirichlet-Voronoi polyhedron of O with respect to $\mathbf{P4_1}$

In Fig. 1 the Dirichlet domain

$$\mathcal{D}_O := \{X \in \mathbb{E}^3 : \rho(O,X) \leq \rho(O^g, X) \text{ for every } g \in G\} \tag{7}$$

has been described. Here $\rho(,)$ denotes the distance function. $\mathcal{D}_O$ is a quadratic column whose opposite side faces are paired by the $\mathbf{4_1}$ screw motion $s(\mathbf{s}, \mathbf{S})$ and by the translations p_1 and p_2, respectively. For example, the face $f_{s^{-1}} := ABCD$, lying on the symmetry plane O and $O^{s^{-1}}$, is mapped by s onto the face $f_s := FGHE$, lying on the symmetry plane O^s and O; moreover, the domain $\mathcal{D}_O$ is mapped by s onto the domain $\mathcal{D}_{O^s}$ joining $\mathcal{D}_O$ along f_s. Of course, the inverse screw motion s^{-1} maps the face $f_s := FGHE$ onto the face $f_{s^{-1}} := ABCD$ and the domain $\mathcal{D}_O$ is mapped by s^{-1} onto the domain $\mathcal{D}_{O^{s^{-1}}}$ joining $\mathcal{D}_O$ along $f_{s^{-1}}$. In the SCHLEGEL diagram of $\mathcal{D}_O$ we omit f's from the face symbols. This diagram will be referred to in the following as the *graph* of the domain [12].

The pairing isometries, or shortly *identifications*, generate the group $G = \mathbf{P4_1}$. This identifications induce a partition of the edges into classes of oriented segments such that a segment does not contain two G-equivalent points in its interior. Consider, e.g. the segment equivalence class

$$\Longrightarrow\!= \{FE; AD; BC; GH\}. \tag{8}$$

This involves the relation

$$p_2^{-1} s p_1 s^{-1} = \mathbf{1} \tag{9}$$

by going around any edge of the class in the tiling [12], in the sense of POINCARÉ. Similarly we get the other relations

$$\longrightarrow\!- \quad p_2^{-1} p_1^{-1} p_2 p_1 = \mathbf{1}, \quad \blacktriangleright\!\!- \quad p_1 s p_2 s^{-1} = \mathbf{1} \tag{10}$$

and a presentation of $G = \mathbf{P4_1}$.

In this paper we shall discuss analogous crystallographic space groups of the Euclidean space $\mathbb{E}^4$, belonging to the icosahedral family [3].

The icosahedral crystal family has two BRAVAIS lattices with Gramians $g_{ij} = g_{ji} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle$

$$\frac{1}{4}\begin{pmatrix} 4a & -a & -a & -a \\ -a & 4a & -a & -a \\ -a & -a & 4a & -a \\ -a & -a & -a & 4a \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 2a & a & a & a \\ a & 2a & a & a \\ a & a & 2a & a \\ a & a & a & 2a \end{pmatrix}, \tag{11}$$

inverse to each other up to a positive constant factor a, for the primitive and its SN centred (seitenflächen-nebendiagonal-zentriert) lattices, respectively [3]. Each vector basis $\mathbf{e}_i$ $(i = 1, \dots, 4)$ spans the corresponding integer vector 4-lattice

$$\mathbf{L} = \{\mathbf{l} = \mathbf{e}_i l^i \in \mathbb{E}^4 : (l^i) \in \mathbb{Z}^4\} \text{ (EINSTEIN's summing convention).} \tag{12}$$

The translational parts $\mathbf{a} + \mathbf{L}$ to an integer linear part $\mathbf{A}$ are determined up to lattice translations, and $\mathbf{a}$ depends on choosing the origin O of the coordinate system $(O; \mathbf{e}_i)$ of the space $\mathbb{E}^4$.

Our purpose in this paper is to discuss space groups $\Gamma = \{\alpha(\mathbf{a}, \mathbf{A})\}$ where the vector system

$$\mathbf{A} \mapsto \mathbf{a} + \mathbf{L} \quad \text{to} \quad \alpha : \mathbf{x} \mapsto \mathbf{a} + \mathbf{A}\mathbf{x} \tag{13}$$

contains *broken* (non integer) translations $\mathbf{a}$ (then $\mathbf{a}$ may have rational $(\mathbb{Q})$ coordinates with small denominators).

Our illustrating two space groups will show the main steps of our algorithm.

2. Fundamental domain for space group XXII.31/04/02

XXII.31/04/02 $= \Gamma_4$ is a space group for the SN centred lattice in (1) [3]. The point group Γ_{04} of space group Γ_4 has 120 linear transforms, and can be generated by

$$\boldsymbol{\gamma}_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \boldsymbol{\gamma}_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \boldsymbol{\gamma}_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

of order 4, 5, 6, respectively. First, the vector system here is trivial $\boldsymbol{\gamma}_j \mapsto \mathbf{o} + \mathbf{L}$ for $\boldsymbol{\gamma}_4$, $\boldsymbol{\gamma}_5$, $\boldsymbol{\gamma}_6$ and so for the point group Γ_{04}.

For any point $P \in \mathbb{E}^4$, we take its orbit under Γ

$$\Gamma P := \{\boldsymbol{\gamma} P \in \mathbb{E}^4 \ : \ \boldsymbol{\gamma} \in \Gamma\} \tag{14}$$

and the D-V cell of the kernel point P to its Γ-orbit

$$\mathcal{D}_\Gamma(P) = \{X \in \mathbb{E}^d \ : \ \rho(P, X) \le \rho(\boldsymbol{\gamma} P, X) \ \text{ for each } \ \boldsymbol{\gamma} \in \Gamma\}. \tag{15}$$

Here $\rho(,)$ — the distance function — can be expressed by the scalar product $\langle , \rangle$, i.e., by the Gramian. Only finitely many $\boldsymbol{\gamma}$'s occur in forming $\mathcal{D}_\Gamma(P)$.

If $Stab_\Gamma(P) = \mathbf{1}$ is trivial, then $\mathcal{D}_\Gamma(P) = \mathcal{F}_\Gamma(P)$ is a fundamental domain for Γ. If not, then

$$\mathcal{F}_\Gamma := \mathcal{D}_\Gamma(P) \cap \mathcal{D}_{Stab(P)}(Q) \tag{16}$$

where $Q \in \mathbb{E}^4$ ist not fixed at $Stab(P)$. Then Q is called a *second kernel.*

2.1. Algorithm for the fundamental domain $\mathcal{F}_4$ of space group Γ_4

First: We determine the equations of *bisector 3-planes of a kernel, say the origin O* in the role of P, *and its Γ_4-images, now the lattice points with* $0, 1, -1$ *coordinates are sufficient for us.* Then we obtain the 3-faces (facets) of $\mathcal{D}(O)$.

Second: We choose a further kernel point $P_1(3; 2; 1; 0)$ to obtain the 3-plane bisectors for these transforms (the kernel always lies in negative halfspaces of the former *3-planes*). Thus we get a pyramidal domain $\mathcal{F}_{04}$ (with apex O) that will contain $\mathcal{F}_4$.

Third: We start e.g. with five suitable equations of 3-planes and determine the *5 vertices* of a starting simplex (*5-cell*). We take a new 3-plane, substitute into its equation the coordinates of all vertices of the convex *4-dimensional polyhedron* (polytopes), being in process. If at least one vertex exists in the positive halfspace of the 3-plane, then we cut the polyhedron with the 3-plane, otherwise drop it, and take a new 3-plane listed by increasing distance from the kernel. So we obtain the equations of 3-planes and the vertices of the fundamental domain.

Finally, the fundamental domain $\mathcal{F}_4$ will be a famous COXETER *simplex* for $\Gamma_4 = 31/04/02$. Among the **5** bisector 3-faces, one bisector is to the origin and the lattice point $(1; 0; 0; 0)$, further **4** 3-faces appertain to $\mathcal{F}_{04}$. We have chosen this example for simplicity. We shall see in Sect. 3 that a fundamental domain can have a cumbersome structure to be described by computer, in general.

Table 1: Pairing on fundamental domain $\mathcal{F}_4$

Source 3-plane and its points	*Pairing transform by 5×5 homogenous matrices*	*Image 3-plane and its points*
$2x+y+z+w-1=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(3;3;-2;-2)$ $\frac{1}{5}(4;-1;-1;-1)$	μ_1 $\begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+y+z+w-1=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(3;3;-2;-2)$ $\frac{1}{5}(4;-1;-1;-1)$
$y-x=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(3;3;-2;-2)$ $(0;0;0;0)$	μ_2 $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$y-x=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(3;3;-2;-2)$ $(0;0;0;0)$
$z-y=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(4;-1;-1;-1)$ $(0;0;0;0)$	μ_3 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$z-y=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(4;-1;-1;-1)$ $(0;0;0;0)$
$w-z=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(3;3;-2;-2)$ $\frac{1}{5}(4;-1;-1;-1)$ $(0;0;0;0)$	μ_4 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$w-z=0$ $\frac{1}{5}(1;1;1;1)$ $\frac{1}{5}(3;3;-2;-2)$ $\frac{1}{5}(4;-1;-1;-1)$ $(0;0;0;0)$
$-x-y-z-2w=0$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(3;3;-2;-2)$ $\frac{1}{5}(4;-1;-1;-1)$ $(0;0;0;0)$	μ_5 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-x-y-z-2w=0$ $\frac{1}{5}(2;2;2;-3)$ $\frac{1}{5}(3;3;-2;-2)$ $\frac{1}{5}(4;-1;-1;-1)$ $(0;0;0;0)$

2.2. Pairings on fundamental domain $\mathcal{F}_4$

For pairing the bisector 3-planes of $\mathcal{F}_4$, and in general, we consider those images of the kernel which are under transforms of inverse types $(\mathbf{a}, \mathbf{A})$, $(-\mathbf{A}^{-1}\mathbf{a},\ \mathbf{A}^{-1})$ in the space group scheme (3). This will be applied first for $\mathcal{F}_{Stab(P)}$, then the further bisectors of $\mathcal{F}_4$.

Remark 1: In case of our fundamental domain $\mathcal{F}_4$ each bisector 3-planes is paired with itself by 3-plane reflection.

Remark 2: In our case we used two kernel points, but just any P (near O not lying in the reflection planes) with non-integer coordinates would be an appropriate kernel for $\mathcal{F}_4$.

A *flag* of a convex polyhedron in $\mathbb{E}^d$ consists of a $(d-1)$-dimensional facet f_1^{d-1}; then an

Table 2: Flag structure of fundamental domain $\mathcal{F}_4$

3-faces	(1) $2x+y+z+w-1=0$ $[1,2,3,4]$ $\begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(2) $y-x=0$ $[1,2,3,5]$ $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(3) $z-y=0$ $[1,2,4,5]$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(4) $w-z=0$ $[1,3,4,5]$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(5) $-x-y-z-2w=0$ $[2,3,4,5]$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

2-faces	$(1,2)$ $[1,2,3]$	$(1,3)$ $[1,2,4]$	$(1,4)$ $[1,3,4]$	$(1,5)$ $[2,3,4]$	$(2,3)$ $[1,2,5]$	$(2,4)$ $[1,3,5]$	$(2,5)$ $[2,3,5]$	$(3,4)$ $[1,4,5]$	$(3,5)$ $[2,4,5]$	$(4,5)$ $[3,4,5]$
Edges	$(1,2,3)$ $[1,2]$	$(1,2,4)$ $[1,3]$	$(1,2,5)$ $[2,3]$	$(1,3,4)$ $[1,4]$	$(1,3,5)$ $[2,4]$	$(1,4,5)$ $[3,4]$	$(2,3,4)$ $[1,5]$	$(2,3,5)$ $[2,5]$	$(2,4,5)$ $[3,5]$	$(3,4,5)$ $[4,5]$

Points	$(1,2,3,4)$ $[1]$ $\frac{1}{5}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}$	$(1,2,3,5)$ $[2]$ $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 2 \\ -3 \\ 5 \end{bmatrix}$	$(1,2,4,5)$ $[3]$ $\frac{1}{5}\begin{bmatrix} 3 \\ 3 \\ -2 \\ -2 \\ 5 \end{bmatrix}$	$(1,3,4,5)$ $[4]$ $\frac{1}{5}\begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \\ 5 \end{bmatrix}$	$(2,3,4,5)$ $[5]$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

incident $(d-2)$-face f_1^{d-2}, as intersection of two $(d-1)$-facets

$$f_1^{d-2} = f_1^{d-1} \cap f_2^{d-1}, \dots, \tag{17}$$

then an incident $(d-k)$-face as intersection

$$f_1^{d-k} = f_1^{d-k+1} \cap f_k^{d-1}, \quad 1 \le k \le d. \tag{18}$$

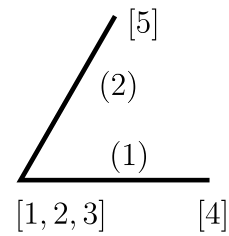
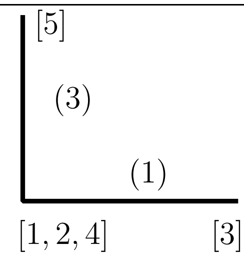
Thus the flag

$$F_1 := (f_1^0, f_1^1, \dots, f_1^{(d-2)}, f_1^{(d-1)}) \tag{19}$$

is a d-tuple of consecutively incident faces.

Any relation of Γ_4, in general, can be described by walking from $\mathcal{F}_4$ through its image domains by crossing side facets and returning to $\mathcal{F}_4$.

Table 3: Relations to 2-face classes in fundamental domain $\mathcal{F}_4$ (two examples)

2-face class	*Relator transform, exponent*	*Diagram of 2-domain*
$(1,2)$ $[1,2,3]$	$\mu_1 \cdot \mu_2 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{12} = 3$	
$(1,3)$ $[1,2,4]$	$\mu_1 \cdot \mu_3 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{13} = 2$	

3. Fundamental domain for space group XXII.31/05/02/002

XXII.31/05/02/002 $= \Gamma_{5t}$ is the richest space group for the SN centred lattice with broken translations [3]. The 120 transform of its point group Γ_{05t} and all transforms of Γ_{5t} itself can be generated by homogeneous 5×5 matrices, as usual

$$\boldsymbol{\gamma}_{4t} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2/3 \\ 0 & -1 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & -1 & 2/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\gamma}_{5t} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 2/3 \\ 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{\gamma}_{6t} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1/3 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Table 4: Pairing on pyramidal domain $\mathcal{F}_{Stab(O)} := \mathcal{F}$

Source 3-plane and its points	*Pairing transform*	*Image 3-plane and its points*
(1) $-x-y-z=0$ $\frac{1}{39}(-5;2;3;16)$, $\frac{1}{27}(-8;5;3;8)$ $\frac{1}{27}(-3;-5;8;8)$, $\frac{1}{27}(-3;1;2;11)$ $\frac{1}{27}(-3;-1;4;10)$, $\frac{1}{15}(2;-4;2;2)$ $\frac{1}{15}(-2;4;-2;2)$, $\frac{1}{15}(1;-3;2;3)$ $\frac{1}{15}(-3;1;2;6)$, $\frac{1}{12}(-1;3;-2;2)$ $\frac{1}{12}(-3;2;1;4)$, $\frac{1}{6}(1;-1;0;1)$ $\frac{1}{6}(1;0;-1;1)$, $\frac{1}{6}(0;1;-1;1)$ $\frac{1}{6}(-1;1;0;2)$, $\frac{1}{3}(-1;0;1;1)$ $\frac{1}{3}(0;0;0;1)$, $(0;0;0;0)$	$\mathbf{z_{12}}$ $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ indirect	(2) $-x-y-w=0$ $\frac{1}{39}(-3;-2;16;5)$, $\frac{1}{27}(-3;-5;8;8)$ $\frac{1}{27}(-8;5;8;3)$, $\frac{1}{27}(-2;-1;11;3)$ $\frac{1}{27}(-4;1;10;3)$, $\frac{1}{15}(-2;4;2;-2)$ $\frac{1}{15}(2;-4;2;2)$, $\frac{1}{15}(-2;3;3;-1)$ $\frac{1}{15}(-2;-1;6;3)$, $\frac{1}{12}(2;-3;2;1)$ $\frac{1}{12}(-1;-2;4;3)$, $\frac{1}{6}(0;1;1;-1)$ $\frac{1}{6}(1;0;1;-1)$, $\frac{1}{6}(1;-1;1;0)$ $\frac{1}{6}(0;-1;2;1)$, $\frac{1}{3}(-1;0;1;1)$ $\frac{1}{3}(0;0;1;0)$, $(0;0;0;0)$
(3) $-y-z-w=0$ $\frac{1}{15}(2;-4;2;2)$, $\frac{1}{15}(2;4;-2;-2)$ $\frac{1}{12}(2;3;-2;-1)$, $\frac{1}{12}(2;-3;2;1)$ $\frac{1}{6}(1;-1;0;1)$, $\frac{1}{6}(1;1;0;-1)$ $\frac{1}{6}(1;0;-1;1)$, $\frac{1}{6}(1;0;1;-1)$ $\frac{1}{6}(1;1;-1;0)$, $\frac{1}{6}(1;-1;1;0)$ $(0;0;0;0)$	$\mathbf{z_{33}}$ $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ involutive indirect	(3) $-y-z-w=0$ $\frac{1}{15}(2;4;-2;-2)$, $\frac{1}{15}(2;-4;2;2)$ $\frac{1}{12}(2;-3;2;1)$, $\frac{1}{12}(2;3;-2;-1)$ $\frac{1}{6}(1;1;0;-1)$, $\frac{1}{6}(1;-1;0;1)$ $\frac{1}{6}(1;0;1;-1)$, $\frac{1}{6}(1;0;-1;1)$ $\frac{1}{6}(1;-1;1;0)$, $\frac{1}{6}(1;1;-1;0)$ $(0;0;0;0)$
(4) $-2x-y-z-w=0$ $\frac{1}{45}(-14;11;11;6)$, $\frac{1}{45}(-14;11;6;11)$ $\frac{1}{27}(-8;5;3;8)$, $\frac{1}{27}(-8;5;8;3)$ $\frac{1}{15}(-2;8;-2;-2)$, $\frac{1}{15}(-2;4;-2;2)$ $\frac{1}{15}(-2;4;2;-2)$, $\frac{1}{15}(-4;6;1;1)$ $\frac{1}{9}(-3;2;2;2)$, $\frac{1}{3}(-1;0;1;1)$ $(0;0;0;0)$	$\mathbf{r_{44}}$ $\begin{pmatrix} -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ involutive direct	(4) $-2x-y-z-w=0$ $\frac{1}{45}(-14;11;6;11)$, $\frac{1}{45}(-14;11;11;6)$ $\frac{1}{27}(-8;5;8;3)$, $\frac{1}{27}(-8;5;3;8)$ $\frac{1}{15}(-2;8;-2;-2)$, $\frac{1}{15}(-2;4;2;-2)$ $\frac{1}{15}(-2;4;-2;2)$, $\frac{1}{15}(-4;6;1;1)$ $\frac{1}{9}(-3;2;2;2)$, $\frac{1}{3}(-1;0;1;1)$ $(0;0;0;0)$
(5) $-x-y-2z-w=0$ $\frac{1}{15}(-2;8;-2;-2)$, $\frac{1}{15}(2;4;-2;-2)$ $\frac{1}{15}(-2;4;-2;2)$, $\frac{1}{15}(1;6;-4;1)$ $\frac{1}{15}(2;2;-3;2)$, $\frac{1}{12}(-1;3;-2;2)$ $\frac{1}{12}(2;3;-2;-1)$, $\frac{1}{6}(1;0;-1;1)$ $\frac{1}{6}(1;1;-1;0)$, $\frac{1}{6}(0;1;-1;1)$ $(0;0;0;0)$	$\mathbf{r_{55}}$ $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ involutive direct	(5) $-x-y-2z-w=0$ $\frac{1}{15}(-2;8;-2;-2)$, $\frac{1}{15}(-2;4;-2;2)$ $\frac{1}{15}(2;4;-2;-2)$, $\frac{1}{15}(1;6;-4;1)$ $\frac{1}{15}(2;2;-3;2)$, $\frac{1}{12}(2;3;-2;-1)$ $\frac{1}{12}(-1;3;-2;2)$, $\frac{1}{6}(1;0;-1;1)$ $\frac{1}{6}(0;1;-1;1)$, $\frac{1}{6}(1;1;-1;0)$ $(0;0;0;0)$
(6) $-x-y-z-2w=0$ $\frac{1}{15}(-2;8;-2;-2)$, $\frac{1}{15}(2;4;-2;-2)$ $\frac{1}{15}(-2;4;2;-2)$, $\frac{1}{15}(2;2;2;-3)$ $\frac{1}{6}(1;1;0;-1)$, $\frac{1}{6}(1;0;1;-1)$ $\frac{1}{6}(0;1;1;-1)$, $(0;0;0;0)$	$\mathbf{r_{66}}$ $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ involutive direct	(6) $-x-y-z-2w=0$ $\frac{1}{15}(-2;8;-2;-2)$, $\frac{1}{15}(-2;4;2;-2)$ $\frac{1}{15}(2;4;-2;-2)$, $\frac{1}{15}(2;2;2;-3)$ $\frac{1}{6}(0;1;1;-1)$, $\frac{1}{6}(1;0;1;-1)$ $\frac{1}{6}(1;1;0;-1)$, $(0;0;0;0)$

3.1. Algorithm for the fundamental domain $\mathcal{F}_{5t}$ of space group Γ_{5t}

First: We determine all broken translations of Γ_{5t} in increasing distances, say, to the origin O, at the same time we obtain the transforms without translations $\gamma_{05} \in \Gamma_{Stab(O)} < \Gamma_{05t}$.

Second: We determine the equations of *bisector 3-planes of the origin O and its broken translates.*

Third: We choose the second kernel point $P_1\,(1;1;1;1)$, and determine the pyramidal domain $\mathcal{F}_{Stab(O)}$ with γ_{05}'s fixing O.

Fourth: We apply the algorithm in the third point of the previous chapter, and we obtain the equations of 3-planes and the vertices of the fundamental domain.

Our fundamental domain $\mathcal{F}_{5t}$ will have **20** *bisector 3-faces and* **45** *proper vertices, but there will be geometric faces with different "algebraic" parts and with additional vertices, should be listed in the following pairings (but no more detailed here).*

3.2. Pairings on fundamental domain $\mathcal{F}_{5t}$

– Pairing of 3-faces of $\mathcal{F}_{5t}$: The method is described at the algorithm for pairing the 3-faces of $\mathcal{F}_4$.

– The algorithm for pairing 3-faces of the pyramidal domain $\mathcal{F}_{Stab(O)} := \mathcal{F}$ is analogous, too. Now we shall have much more complicated face structure from which we tabulate the most important facts (see Table 4).

3.3. Geometric description of relations for $\mathcal{F}_{Stab(O)} := \mathcal{F}$

The **12** *2-faces* of $\mathcal{F}$ are in four equivalence classes:

1. (2,1), (2,4), (4,1) with relation $\mathbf{z_{12}} \cdot \mathbf{z_{12}} \cdot \mathbf{r_{44}} = \mathbf{1}$;

2. (1,3), (3,6), (6,2) with relation $\mathbf{z_{12}} \cdot \mathbf{z_{33}} \cdot \mathbf{r_{66}} = \mathbf{1}$;

3. (1,5), (5,3), (3,2) with relation $\mathbf{z_{12}} \cdot \mathbf{r_{55}} \cdot \mathbf{z_{33}} = \mathbf{1}$;

4. (4,5), (5,6), (6,4) with relation $\mathbf{r_{44}} \cdot \mathbf{r_{66}} \cdot \mathbf{r_{55}} = \mathbf{1}$.

We only indicate the so-called Poincaré algorithm from [12] which describes how to get a defining relation to a $(d-2)$-face class by walking around a representant $(d-2)$-face in a d-dimensional fundamental tiling ($d = 4$ now). Table 5 shows this, e.g. to the first equivalence class of 2-face (1,2), containing the common vertices [1], [2], [14], [19] of 3-faces (1) and (2) in Table 4.

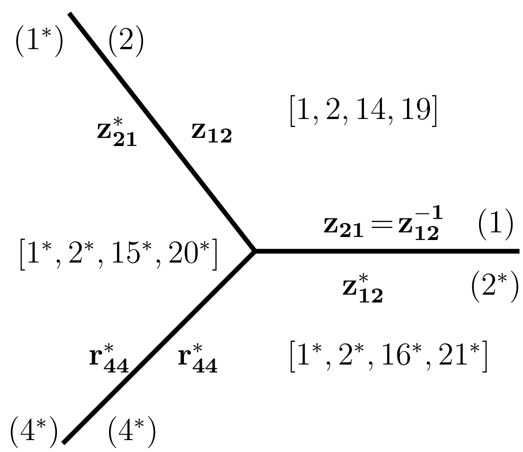
This table says that the generating transform $\mathbf{z_{12}}$ maps the 3-face (1) of (1,2) into (2) and the above vertices onto [1], [2], [16], [21] of (2), in common with the 3-face (4). Vice-versa this 2-face (2,4) of $\mathcal{F}_{Stab(O)}$ and its domain are mapped under $\mathbf{z_{21}} = \mathbf{z_{12}^{-1}}$ onto $[1] = [1^*]$, $[2] = [2^*]$, $[14] = [16^*]$, $[19] = [21^*]$ and onto the domain of $(2^*, 4^*)$, $*$ means $\mathbf{z_{21}}$-image, (2^*) joins (1).

Furthermore, the 3-face (4) with vertices [1], [2], [16], [21] of 2-face (2,4) will be mapped onto itself by the involutive direct transform $\mathbf{r_{44}}$, but $\mathbf{r_{44}} : [1] \mapsto [1], [2] \mapsto [2], [16] \mapsto [15], [21] \mapsto [20]$. The images are common vertices with 3-face (1). Now $[1^*] = [1]$, $[2^*] = [2]$, $[15^*] = [14]$, $[20^*] = [19]$ also hold where $*$ means now the mapping $\mathbf{z_{21}} \cdot \mathbf{r_{44}}$, the inverse of $\mathbf{r_{44}} \cdot \mathbf{z_{12}}$. The image $(4^*, 1^*)$ of the 2-domain (4,1) joins the former domain $(2^*, 4^*)$ at (4^*).

Finally, this last 3-face (1) with [1], [2], [15], [20] of 2-face (4,1) will be mapped onto the 3-face (2) again by $\mathbf{z_{12}}$, and $\mathbf{z_{12}} : [1] \mapsto [1], [2] \mapsto [2], [15] \mapsto [14], [20] \mapsto [19]$. The images are common with 3-face (1), and we have turned back to the starting 2-face (1,2) by the transform $\mathbf{z_{12}} \cdot \mathbf{r_{44}} \cdot \mathbf{z_{12}}$ and by its inverse $\mathbf{z_{21}} \cdot \mathbf{r_{44}} \cdot \mathbf{z_{21}}$ as well.

Table 5: Relations to 2-face classes in fundamental domain $\mathcal{F}$
Equivalence class **1**

<table>
<tr><th>2-face class</th><th>Relator transform, exponent
Diagram of 2-domain</th></tr>
<tr><td>(2) $-x-y-w=0$
(1) $-x-y-z=0$
[1] [2] [14] [19]
$\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \frac{1}{3}\begin{bmatrix}-1\\0\\1\\1\\3\end{bmatrix} \frac{1}{15}\begin{bmatrix}2\\-4\\2\\2\\15\end{bmatrix} \frac{1}{27}\begin{bmatrix}-3\\-5\\8\\8\\27\end{bmatrix}$
$\downarrow \mathbf{z_{12}}$</td><td>$(\mathbf{z_{21}} \cdot \mathbf{r_{44}} \cdot \mathbf{z_{21}})^{\nu_1} = \mathbf{1}$

$\nu_1 = 1$</td></tr>
<tr><td>(2) $-x-y-w=0$
(4) $-2x-y-z-w=0$
[1] [2] [16] [21]
$\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \frac{1}{3}\begin{bmatrix}-1\\0\\1\\1\\3\end{bmatrix} \frac{1}{15}\begin{bmatrix}-2\\4\\2\\-2\\15\end{bmatrix} \frac{1}{27}\begin{bmatrix}-8\\5\\8\\3\\27\end{bmatrix}$
$\downarrow \mathbf{r_{44}}$
<hr>(4) $-2x-y-z-w=0$
(1) $-x-y-z=0$
[1] [2] [15] [20]
$\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \frac{1}{3}\begin{bmatrix}-1\\0\\1\\1\\3\end{bmatrix} \frac{1}{15}\begin{bmatrix}-2\\4\\-2\\2\\15\end{bmatrix} \frac{1}{27}\begin{bmatrix}-8\\5\\1\\8\\27\end{bmatrix}$
$\downarrow \mathbf{z_{12}}$</td><td>
</td></tr>
</table>

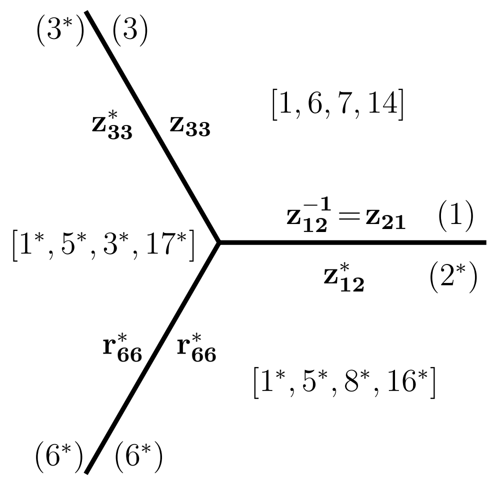
The 2-face domains by angles, and the matrix product as well, show the equivalent cycle relations

$$\begin{array}{lllllll}\mathbf{z_{12}} \cdot \mathbf{r_{44}} \cdot \mathbf{z_{12}} = \mathbf{1} & \Longleftrightarrow & \mathbf{z_{12}} \cdot \mathbf{z_{12}} \cdot \mathbf{r_{44}} = \mathbf{1} & \Longleftrightarrow & \mathbf{r_{44}} \cdot \mathbf{z_{12}} \cdot \mathbf{z_{12}} = \mathbf{1} & \Longleftrightarrow \\ \mathbf{z_{21}} \cdot \mathbf{r_{44}} \cdot \mathbf{z_{21}} = \mathbf{1} & \Longleftrightarrow & \mathbf{r_{44}} \cdot \mathbf{z_{21}} \cdot \mathbf{z_{21}} = \mathbf{1} & \Longleftrightarrow & \mathbf{z_{21}} \cdot \mathbf{z_{21}} \cdot \mathbf{r_{44}} = \mathbf{1}, & \end{array}$$

depending on the starting 2-face in the class and on the walking direction. Now $\nu_1 = 1$ is the exponent of the cycle transform. This also means geometrically that the angle sum of the 2-face class equals 2π.

Lateron, we shall get also a rotation, e.g. a halfturn, as cycle transform with its order, e.g. 2, respectively, in the exponent of the cycle transform (see [12] for a complete discussion).

Table 5: (cont. 1) Equivalence class **2**

2-face class	*Relator transform, exponent* *Diagram of 2-domain*
(1) $-x-y-z=0$ (3) $-y-z-w=0$ [1] $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ [6] $\frac{1}{6}\begin{bmatrix}1\\0\\-1\\1\\6\end{bmatrix}$ [7] $\frac{1}{6}\begin{bmatrix}1\\-1\\0\\1\\6\end{bmatrix}$ [14] $\frac{1}{15}\begin{bmatrix}2\\-4\\2\\2\\15\end{bmatrix}$ $\downarrow \mathbf{z_{33}}$	$\mathbf{z_{33}\cdot r_{66}\cdot z_{12}=1}$ $\nu_2=1$
(3) $-y-z-w=0$ (6) $-x-y-z-2w=0$ [1] $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ [5] $\frac{1}{6}\begin{bmatrix}1\\0\\1\\-1\\6\end{bmatrix}$ [3] $\frac{1}{6}\begin{bmatrix}1\\1\\0\\-1\\6\end{bmatrix}$ [17] $\frac{1}{15}\begin{bmatrix}2\\4\\-2\\-2\\15\end{bmatrix}$ $\downarrow \mathbf{r_{66}}$ (6) $-x-y-z-2w=0$ (2) $-x-y-w=0$ [1] $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ [5] $\frac{1}{6}\begin{bmatrix}1\\0\\1\\-1\\6\end{bmatrix}$ [8] $\frac{1}{6}\begin{bmatrix}0\\1\\1\\-1\\6\end{bmatrix}$ [16] $\frac{1}{15}\begin{bmatrix}-2\\4\\2\\-2\\15\end{bmatrix}$ $\downarrow \mathbf{z_{21}}$	

3.3.1. Presentation of Γ_{05t}

$$\Gamma_{05t} = (\mathbf{z_{21}},\ \mathbf{z_{33}},\ \mathbf{r_{44}},\ \mathbf{r_{55}},\ \mathbf{r_{66}-z^2_{33}},\ \mathbf{r^2_{44}},\ \mathbf{r^2_{55}},\ \mathbf{r^2_{66}},\ \mathbf{z_{12}\cdot z_{12}\cdot r_{44}},\ \mathbf{z_{12}\cdot z_{33}\cdot r_{66}},\ \mathbf{z_{12}\cdot r_{55}\cdot z_{33}},\ \mathbf{r_{44}\cdot r_{66}\cdot r_{55}}).$$

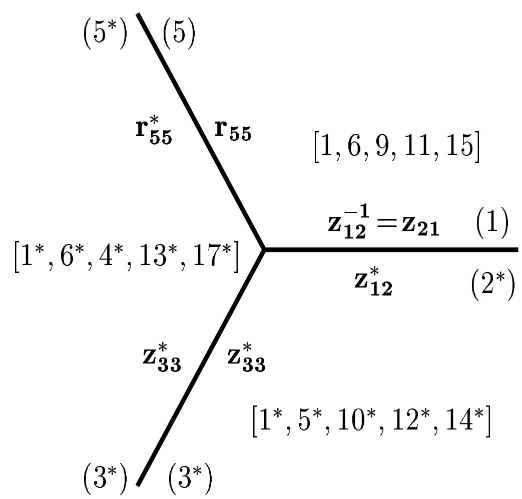
After simplification:

$$\Gamma_{05t} = \big(\mathbf{z_{12}},\ \mathbf{z_{33}-z^4_{12}},\ \mathbf{z^2_{33}},\ (\mathbf{z_{12}\cdot z_{33}})^{\mathbf{2}},\ (\mathbf{z_{33}\cdot z_{12}})^{\mathbf{2}}\big).$$

Analogously to the pairing tables 4 and 5, one can read all the 2-face classes and all defining relations for Γ_{5t} by the Poincaré algorithm.

So we get also a presentation for the space group Γ_{5t}. Instead of this we illustrate our $\mathcal{F}_{5t}$ on the computer screen in some positions. See the Dirichlet-Voronoi domain of space group **31/05/02/002** in the starting projection where two axes and some 3-faces lie

Table 5: (cont. 2) Equivalence class **3**

2-face class	*Relator transform, exponent* *Diagram of 2-domain*
(1) $-x-y-z=0$ (5) $-x-y-2z-w=0$ $[1]\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ $[6]\ \frac{1}{6}\begin{bmatrix}1\\0\\-1\\1\\6\end{bmatrix}$ $[9]\ \frac{1}{6}\begin{bmatrix}0\\1\\-1\\1\\6\end{bmatrix}$ $[11]\ \frac{1}{12}\begin{bmatrix}-1\\3\\-2\\2\\12\end{bmatrix}$ $[15]\ \frac{1}{15}\begin{bmatrix}-2\\4\\-2\\2\\15\end{bmatrix}$ $\downarrow \mathbf{r_{55}}$	$\mathbf{r_{55} \cdot z_{33} \cdot z_{12} = 1}$ $\nu_3 = 1$
(5) $-x-y-2z-w=0$ (3) $-y-z-w=0$ $[1]\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ $[6]\ \frac{1}{6}\begin{bmatrix}1\\0\\-1\\1\\6\end{bmatrix}$ $[4]\ \frac{1}{6}\begin{bmatrix}1\\1\\-1\\0\\6\end{bmatrix}$ $[13]\ \frac{1}{12}\begin{bmatrix}2\\3\\-2\\-1\\12\end{bmatrix}$ $[17]\ \frac{1}{15}\begin{bmatrix}2\\4\\-2\\-2\\15\end{bmatrix}$ $\downarrow \mathbf{z_{33}}$	
(3) $-y-z-w=0$ (2) $-x-y-w=0$ $[1]\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ $[5]\ \frac{1}{6}\begin{bmatrix}1\\0\\1\\-1\\6\end{bmatrix}$ $[10]\ \frac{1}{6}\begin{bmatrix}1\\-1\\1\\0\\6\end{bmatrix}$ $[12]\ \frac{1}{12}\begin{bmatrix}2\\-3\\2\\1\\12\end{bmatrix}$ $[14]\ \frac{1}{15}\begin{bmatrix}2\\-4\\2\\2\\15\end{bmatrix}$ $\downarrow \mathbf{z_{21}}$	

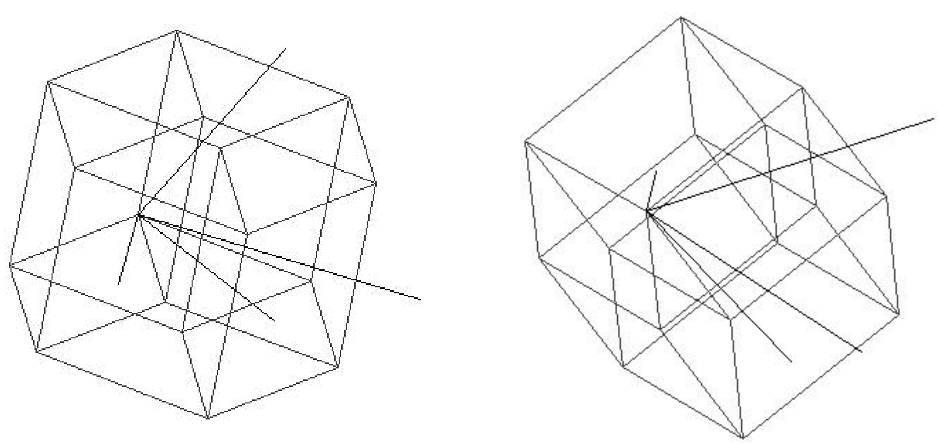
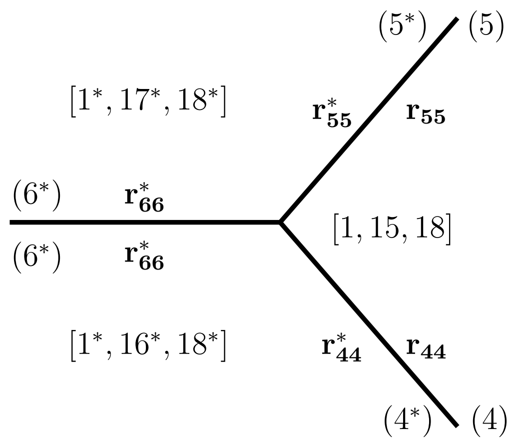


Figure 2: The JGG logo as a fundamental domain of **27/01/01**

Table 5: (cont. 3) Equivalence class **4**

2-face class	*Relator transform, exponent* *Diagram of 2-domain*
(4) $-2x-y-z-w=0$ (5) $-x-y-2z-w=0$ [1] [15] [18] $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ $\frac{1}{15}\begin{bmatrix}-2\\4\\-2\\2\\15\end{bmatrix}$ $\frac{1}{15}\begin{bmatrix}-2\\8\\-2\\-2\\15\end{bmatrix}$ $\downarrow \mathbf{r_{55}}$	$\mathbf{r_{55}} \cdot \mathbf{r_{66}} \cdot \mathbf{r_{44}} = \mathbf{1}$ $\nu_4 = 1$
(5) $-x-y-2z-w=0$ (6) $-x-y-z-2w=0$ [1] [17] [18] $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ $\frac{1}{15}\begin{bmatrix}2\\4\\-2\\-2\\15\end{bmatrix}$ $\frac{1}{15}\begin{bmatrix}-2\\8\\-2\\-2\\15\end{bmatrix}$ $\downarrow \mathbf{r_{66}}$ (6) $-x-y-z-2w=0$ (4) $-2x-y-z-w=0$ [1] [16] [18] $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ $\frac{1}{15}\begin{bmatrix}-2\\4\\2\\-2\\15\end{bmatrix}$ $\frac{1}{15}\begin{bmatrix}-2\\8\\-2\\-2\\15\end{bmatrix}$ $\downarrow \mathbf{r_{44}}$	

in projection positions (see Fig. 3). The other figures are constructed by rotations from the starting domain.

In Fig. 2 we present the JGG logo, the 4-cube, as a fundamental domain of **27/01/01** whose lattice is specialized by the free parameters of its Gramian.

Acknowledgments

We thank István Prok for his computer implementations of figures.

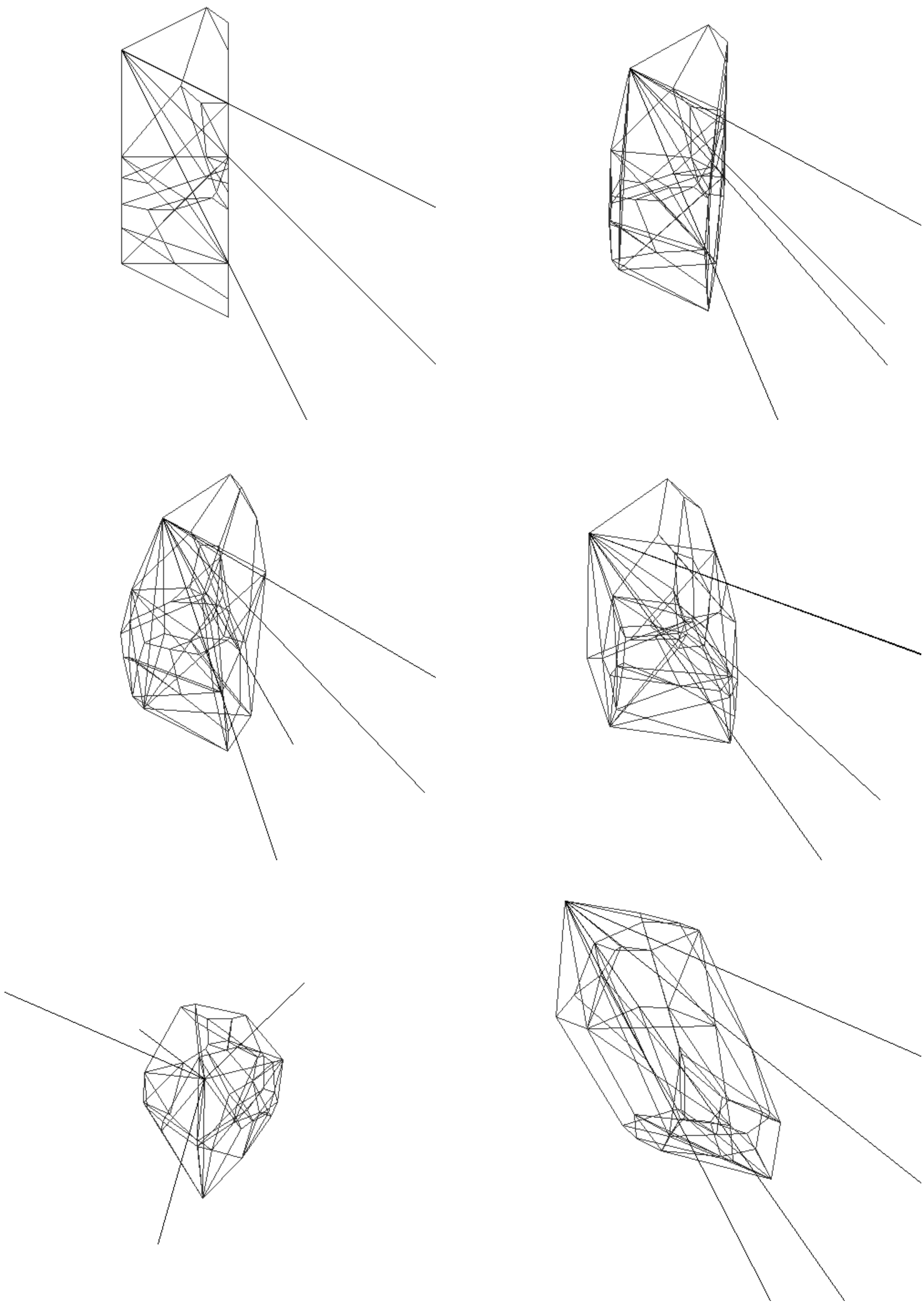


Figure 3: Dirichlet-Voronoi domain of the space group Γ_{5t} in different positions

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