Journal for Geometry and Graphics
Volume 6 (2002), No. 1, 17–26.

Isometric Invariants of Conics in the Isotropic Plane — Classification of Conics

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Abstract. A real affine plane $\mathbb{A}_2$ is called an isotropic plane $\mathbb{I}_2$, if in $\mathbb{A}_2$ a metric is induced by an absolute figure (f, F), consisting of the line f at infinity of $\mathbb{A}_2$ and a point $F \in f$.
This paper gives a complete classification of the second order curves in the isotropic plane $\mathbb{I}_2$. Although conics in $\mathbb{I}_2$ have been investigated earlier, e.g. in the standard text-book of H. Sachs [8] or in the paper of Makarowa [7], this paper offers a new method based on Linear Algebra. The definition of invariants of a conic with respect to the group of motions in $\mathbb{I}_2$ makes it possible to determine the type of a conic without reducing its equation to canonical form. The obtained results are summarized in an overview table.
Such an approach can also be understood as an example of classifying quadratic forms in n-dimensional spaces with non-regular metric, e.g. quadrics or pencils of quadrics in the double isotropic space $\mathbb{I}_3^{(2)}$ [2].

Key Words: conics, plane isotropic geometry

MSC 2000: 51N25

1. Isotropic plane

The isotropic plane $\mathbb{I}_2(\mathbb{R})$ is a real affine plane $\mathbb{A}_2$ where the metric is introduced by a real line element (f, F) [8, 9]. The ordered pair (f, F) with $F \in f$ is called *absolute figure* of the isotropic plane $\mathbb{I}_2(\mathbb{R})$. In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0, \tag{i}$$

the absolute figure is determined by $f \equiv x_0 = 0$, and $F = (0 : 0 : 1)$. Any line through F is called *isotropic*.

All projective transformations that keep the absolute figure fixed form a 5-parametric group

$$\mathbf{G}_5 \begin{cases} \bar{x} = c_1 + c_4 x \\ \bar{y} = c_2 + c_3 x + c_5 y \end{cases}, \qquad c_1, \ldots, c_5 \in \mathbb{R}, \ c_4, c_5 \neq 0. \tag{ii}$$

ISSN 1433-8157/\$ 2.50

We call it the *group of similarities* of the isotropic plane $\mathbb{I}_2$.

After defining in $\mathbb{I}_2$ the usual metric quantities such as the distance between two points or the angle between two lines, we look for the subgroup of those transformations in $\mathbf{G}_5$ which preserve these quantities. In such a way one obtains mappings of the form

$$\mathbf{G}_3 \begin{cases} \bar{x} = c_1 + x \\ \bar{y} = c_2 + c_3 x + y. \end{cases} \tag{iii}$$

Their group is called the *motion group* of the isotropic plane $\mathbb{I}_2$. Hence, the group of isotropic motions consists of *translations* and *rotations*, that is

$$\mathbf{T} \begin{cases} \bar{x} = c_1 + x \\ \bar{y} = c_2 + y \end{cases} \qquad \text{and} \qquad \mathbf{R} \begin{cases} \bar{x} = x \\ \bar{y} = c_3 x + y. \end{cases} \tag{iv}$$

In the affine model, any rotation is understood as stretching along the y-axis.

2. Conic equation

The general second-degree equation in two variables can be written in the form

$$F(x,y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0 \tag{1}$$

where $a_{11}, \ldots, a_{00} \in \mathbb{R}$ and at least one of the coefficients $a_{11}, a_{12}, a_{22} \neq 0$ [3]. The solutions of the equation (1) represent the locus of points in a plane which is called a *conic section* or simply a *conic*.

Using the matrix notation, we have

$$\begin{aligned} F(x,y) &\equiv \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \\ &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2 \begin{bmatrix} a_{01} & a_{02} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + a_{00} = 0 \end{aligned} \tag{2}$$

where

$$A := \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \qquad \text{and} \qquad \sigma := \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \tag{3}$$

are real, symmetric matrices. In the sequel we will use

$$\mathbf{A} := |A| = \det A, \quad \boldsymbol{\sigma} := |\sigma| = \det \sigma \ \text{ and } \ \mathbf{A}_{ij}(i,j = 0,1,2), \tag{4}$$

where $\mathbf{A}_{ij}$ are the minors of the matrix A.

The goal is to determine the invariants of conics with respect to the group $\mathbf{G}_3$ of motions in $\mathbb{I}_2$. For that purpose let's apply on the conic equation (1) the mapping from $\mathbf{G}_3$ given by

$$\begin{cases} x = \bar{x} \\ y = \alpha\bar{x} + \bar{y} \end{cases}, \qquad \alpha \in \mathbb{R}. \tag{5}$$

One obtains

$$F(\bar{x},\bar{y}) \equiv \begin{bmatrix} 1 & \bar{x} & \bar{y} \end{bmatrix} \begin{bmatrix} \bar{a}_{00} & \bar{a}_{01} & \bar{a}_{02} \\ \bar{a}_{01} & \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{02} & \bar{a}_{12} & \bar{a}_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix} = 0, \tag{6}$$

where

$$\begin{array}{lll} \bar{a}_{00} = a_{00}, & \bar{a}_{01} = a_{01} + a_{02}\alpha, & \bar{a}_{02} = a_{02}, \\ \bar{a}_{11} = a_{11} + 2a_{12}\alpha + a_{22}\alpha^2, & \bar{a}_{12} = a_{12} + a_{22}\alpha, & \bar{a}_{22} = a_{22}. \end{array} \tag{7}$$

This yields

$$\overline{\mathbf{A}} = \mathbf{A},\ \overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma},\ \overline{\mathbf{A}}_{11} = \mathbf{A}_{11},$$

and directly from (7)

$$\bar{a}_{22} = a_{22}, \quad \bar{a}_{02} = a_{02}.$$

For example,

$$\overline{\boldsymbol{\sigma}} = \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{12} & \bar{a}_{22} \end{vmatrix} = \begin{vmatrix} a_{11} + 2a_{12}\alpha + a_{22}\alpha^2 & a_{12} + a_{22}\alpha \\ a_{12} + a_{22}\alpha & a_{22} \end{vmatrix} = \ldots = a_{11}a_{22} - a_{12}^2 = \boldsymbol{\sigma}.$$

3. Diagonalization of the quadratic form

The quadratic form included in eq. (1) is the homogenous polynomial of the second degree

$$K(x, y) := a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{8}$$

The question is whether and when it is possible to obtain $\bar{a}_{12} = 0$ using transformations of the group $\mathbf{G}_3$. It can be seen from (7) that

$$\bar{a}_{12} = a_{12} + a_{22}\alpha \quad \Longrightarrow \quad \alpha = -\frac{a_{12}}{a_{22}}, \quad a_{22} \neq 0. \tag{9}$$

One obtains

$$K(\bar{x}, \bar{y}) = \begin{bmatrix} \bar{x} & \bar{y} \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & 0 \\ 0 & \bar{a}_{22} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \tag{10}$$

where

$$\bar{a}_{11} = \frac{a_{11}a_{22} - a_{12}^2}{a_{22}} = \frac{\boldsymbol{\sigma}}{a_{22}}, \qquad \bar{a}_{22} = a_{22}. \tag{11}$$

4. Isotropic values of the matrix

Definition 1 *Let* $\sigma := \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ *be any real symmetric matrix. Then for* $\boldsymbol{\sigma} := \det \sigma$ *the values*

$$\mathbf{I}_1 := \frac{\boldsymbol{\sigma}}{\alpha_{22}}, \qquad \mathbf{I}_2 := a_{22}, \quad (a_{22} \neq 0), \tag{12}$$

are called isotropic values *of the matrix* σ.

Definition 2 *We say that the real symmetric* 2×2*-matrix* σ *allows the isotropic diagonalization if there is a matrix* $G = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$ *such that* $G^T \sigma G$ *is a diagonal matrix, i.e.,*

$$G^T \sigma G = \begin{bmatrix} \mathbf{I}_1 & 0 \\ 0 & \mathbf{I}_2 \end{bmatrix}, \tag{13}$$

where $\mathbf{I}_1, \mathbf{I}_2$ *are the isotropic values of the matrix* σ. *We say that the matrix* G *isotropically diagonalizes* σ.

It can be seen from what has been shown earlier that the following propositions are valid:

Proposition 1 *The isotropic values* $\mathbf{I}_1, \mathbf{I}_2$ *as well as their product* $\mathbf{I}_1\mathbf{I}_2 = \boldsymbol{\sigma}$ *are invariants with respect to the group* $\mathbf{G}_3$ *of motions in the isotropic plane* $\mathbb{I}_2$.

Proposition 2 *Let* σ *be a matrix from Definition 1. Then there is a matrix*

$$G = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \quad \textit{with} \quad \alpha = -\frac{a_{12}}{a_{22}}$$

which under the condition $a_{22} \neq 0$ *isotropically diagonalizes* σ.

Proposition 3 *It is always possible to reduce the quadratic form (8) by an isotropic motion to the canonical form (10) except for*

$$\begin{array}{ll} \text{a)} & a_{22} = 0, \quad \boldsymbol{\sigma} \neq 0; \ \textit{and} \\ \text{b)} & a_{22} = 0, \quad \boldsymbol{\sigma} = 0. \end{array} \tag{14}$$

The meaning of the main invariants is as follows:

$$\begin{array}{rcl} a_{22} \neq 0 & \Longleftrightarrow & \text{conic section without isotropic direction,} \\ \boldsymbol{\sigma} \neq 0 & \Longleftrightarrow & \text{conic section with center,} \\ \mathbf{A} = 0 & \Longleftrightarrow & \text{degenerate conic section.} \end{array}$$

5. Isotropic classification of conics

1st family of conics

Let's assume that it is possible to reduce the quadratic form in the conic equation (1) to the canonical form (10). This implies according to Propositions 1 and 2 that either $a_{22} \neq 0$, $\boldsymbol{\sigma} \neq 0$, or $a_{22} \neq 0$, $\boldsymbol{\sigma} = 0$.

I. Let's consider $a_{22} \neq 0$, $\boldsymbol{\sigma} \neq 0$: The isotropic values are

$$\mathbf{I}_1 = \frac{\boldsymbol{\sigma}}{\alpha_{22}} \neq 0, \quad \mathbf{I}_2 = a_{22} \neq 0,$$

wherefrom it is possible to write down the conic equation (1) in the form

$$F(\bar{x}, \bar{y}) \equiv \mathbf{I}_1 \bar{x}^2 + \mathbf{I}_2 \bar{y}^2 + 2\bar{a}_{01}\bar{x} + 2\bar{a}_{02}\bar{y} + \bar{a}_{00} = 0. \tag{15}$$

After a translation of the coordinate system in $\bar{x}$- and $\bar{y}$-direction we have

$$F(\bar{\bar{x}}, \bar{\bar{y}}) \equiv \mathbf{I}_1 \bar{\bar{x}}^2 + \mathbf{I}_2 \bar{\bar{y}}^2 + t = 0. \tag{16}$$

One computes

$$t = \frac{\bar{a}_{00}\mathbf{I}_1\mathbf{I}_2 - \bar{a}_{01}^2\mathbf{I}_2 - \bar{a}_{02}^2\mathbf{I}_1}{\mathbf{I}_1\mathbf{I}_2} = \frac{\mathbf{A}}{\boldsymbol{\sigma}}. \tag{17}$$

Hence t is invariant. Let's introduce:

$$a := \sqrt{\left|\frac{t}{\mathbf{I}_1}\right|}, \qquad b := \sqrt{\left|\frac{t}{\mathbf{I}_2}\right|}. \tag{18}$$

The values a and b shall be called *isotropic semiaxes*.

The possibilities for the conic sections with equation (16) are:

(i) $\operatorname{sgn}\mathbf{I}_1 = \operatorname{sgn}\mathbf{I}_2 = \operatorname{sgn} t$ and $t \neq 0$:
The conic equation (16) under conditions (i) can be written in the form

$$\frac{\bar{\bar{x}}^2}{a^2} + \frac{\bar{\bar{y}}^2}{b^2} = -1. \tag{19}$$

The conic is called *imaginary ellipse* meeting the conditions $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} > 0$, $a_{22} \neq 0$, $a_{22}\mathbf{A} > 0$.

(ii) $\operatorname{sgn}\mathbf{I}_1 = \operatorname{sgn}\mathbf{I}_2 \neq \operatorname{sgn} t$ and $t \neq 0$:

$$\frac{\bar{\bar{x}}^2}{a^2} + \frac{\bar{\bar{y}}^2}{b^2} = 1. \tag{20}$$

This is a *real ellipse* meeting the conditions $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} > 0$, $a_{22} \neq 0$, and $a_{22}\mathbf{A} < 0$.

(iii) $\operatorname{sgn}\mathbf{I}_1 = \operatorname{sgn}\mathbf{I}_2$ and $t = 0$:

$$\mathbf{I}_1\bar{\bar{x}}^2 + \mathbf{I}_2\bar{\bar{y}}^2 = 0. \tag{21}$$

This is the equation of an *imaginary pair of straight lines.* The corresponding conditions are $\mathbf{A} = 0$, $\boldsymbol{\sigma} > 0$, $a_{22} \neq 0$.

(iv) $\operatorname{sgn}\mathbf{I}_1 \neq \operatorname{sgn}\mathbf{I}_2$ and $t\mathbf{I}_2 > 0$:

$$\frac{\bar{\bar{x}}^2}{a^2} - \frac{\bar{\bar{y}}^2}{b^2} = 1, \tag{22}$$

We call this conic a *first type hyperbola* obeying $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} < 0$, $a_{22} \neq 0$, $a_{22}\mathbf{A} < 0$.

(v) $\operatorname{sgn}\mathbf{I}_1 \neq \operatorname{sgn}\mathbf{I}_2$ and $t\mathbf{I}_2 < 0$:
(16) turns into the form

$$-\frac{\bar{\bar{x}}^2}{a^2} + \frac{\bar{\bar{y}}^2}{b^2} = 1, \tag{23}$$

that we call the equation of a *second type hyperbola* meeting $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} < 0$, $a_{22} \neq 0$, $a_{22}\mathbf{A} > 0$.

(vi) $\operatorname{sgn}\mathbf{I}_1 \neq \operatorname{sgn}\mathbf{I}_2$ and $t = 0$:

$$|\mathbf{I}_1|\,\bar{\bar{x}}^2 - |\mathbf{I}_2|\,\bar{\bar{y}}^2 = 0. \tag{24}$$

This represents a pair of *intersecting straight lines* and we have $\mathbf{A} = 0$, $\boldsymbol{\sigma} < 0$, $a_{22} \neq 0$.

In order to figure out the *geometrical meaning of the isotropic semiaxes* (18) we note:
Let's consider a conic section equation $F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + t = 0$, supposing that the conic's center coincides with the origin of the coordinate system. From $x = 0$ and $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + t = 0$ we obtain

$$y = \pm\sqrt{-\frac{t}{\mathbf{I}_2}}.$$

Hence $2b$ is the length of the cross section of the conic along the y-axis and invariant with respect to isotropic motions.

The condition
$$\frac{\partial F}{\partial y} \equiv 2a_{12}x + 2a_{22}y = 0 \quad \text{implies} \quad y = -\frac{a_{12}}{a_{22}}x.$$

This reveals that the slope of the obtained straight line equals the parameter α of the isotropic rotation used to obtain a canonical form of the conic.

The conditions

$$y = -\frac{a_{12}}{a_{22}} \quad \text{and} \quad a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + t = 0 \quad \text{give} \quad x = \pm\sqrt{-\frac{t}{\mathbf{I}_1}}\,.$$

So $2a$ is the length of the cross section with the straight line $y = \alpha x$.

II. Let's now consider $a_{22} \neq 0$, $\boldsymbol{\sigma} = 0$:
The isotropic values of the matrix σ are

$$\mathbf{I}_1 = \frac{\boldsymbol{\sigma}}{\alpha_{22}} = 0, \qquad \mathbf{I}_2 = a_{22} \neq 0$$

which implies that one linear term can be eliminated by a translation. , i.e.,

$$F(\bar{\bar{x}}, \bar{\bar{y}}) \equiv \mathbf{I}_2\bar{\bar{y}}^2 + 2\bar{a}_{01}\bar{\bar{x}} + t = 0. \tag{25}$$

One computes

$$\mathbf{A} = -\bar{a}_{01}^2\mathbf{I}_2, \quad \mathbf{A}_{11} = t\mathbf{I}_2. \tag{26}$$

We distinguish

(vii) If $\bar{a}_{01} \neq 0$, the equation (25) represents a *parabola* meeting $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} = 0$, $a_{22} \neq 0$.

(viii) $\operatorname{sgn}\mathbf{I}_2 \neq \operatorname{sgn} t$ and $\bar{a}_{01} = 0$:

$$|\mathbf{I}_2|\,\bar{\bar{y}}^2 - |t| = 0\,. \tag{27}$$

This is a pair of *parallel lines* obeying $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} \neq 0$, $\mathbf{A}_{11} < 0$.

(ix) $\operatorname{sgn}\mathbf{I}_2 = \operatorname{sgn} t$ and $\bar{a}_{01} = 0$:

$$\mathbf{I}_2\bar{\bar{y}}^2 + t = 0. \tag{28}$$

This is the equation of an *imaginary pair of parallel lines* with $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} \neq 0$, $\mathbf{A}_{11} > 0$.

(x) Under $t = 0$ and $\bar{a}_{01} = 0$ the equation (25) can be written in the form

$$\bar{\bar{y}}^2 + t = 0 \tag{29}$$

and represents two *coinciding parallel lines* obeying $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} \neq 0$, $\mathbf{A}_{11} = 0$.

We are going to demonstrate at one example that the conditions on the invariants characterize the types uniquely: Let us presume $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} < 0$, $a_{22} \neq 0$ and $a_{22}\mathbf{A} < 0$: We conclude from $\boldsymbol{\sigma} < 0$ either $\mathbf{I}_1 > 0$ and $\mathbf{I}_2 < 0$ or $\mathbf{I}_1 < 0$ and $\mathbf{I}_2 > 0$. This results in

$$\mathbf{I}_1 > 0 \;\text{ and }\; \mathbf{I}_2 < 0 \;\Longrightarrow\; \left\{\begin{array}{l} \mathbf{I}_2 < 0 \\ \mathbf{I}_2 A = a_{22}A < 0 \\ A > 0 \end{array}\right\} \;\Longrightarrow\; t < 0$$

$$\mathbf{I}_1 < 0 \;\text{ and }\; \mathbf{I}_2 > 0 \;\Longrightarrow\; \left\{\begin{array}{l} \mathbf{I}_2 > 0 \\ \mathbf{I}_2 A = a_{22}A < 0 \\ A < 0 \end{array}\right\} \;\Longrightarrow\; t > 0$$

Thus we have got

$$\operatorname{sgn}\mathbf{I}_1 \neq \operatorname{sgn}\mathbf{I}_2 \;\text{ and }\; t\mathbf{I}_2 > 0,$$

and this is the condition (iv). All the other cases can be checked in a similar way.

Let's assume furtheron that it is not possible to diagonalize the quadratic form in the conic equation (1). Then according to Proposition 2 we have to distinguish a) $a_{22} = 0$, $\sigma \neq 0$ or b) $a_{22} = 0$, $\sigma = 0$.

2nd family of conics

The conditions a) $a_{22} = 0$, $\boldsymbol{\sigma} \neq 0$ imply $\boldsymbol{\sigma} = -a_{12}^2 < 0$. The conic equation (1) is of the form

$$\begin{aligned} F(x,y) &\equiv a_{11}x^2 + 2a_{12}xy + 2a_{01}x + 2a_{02}y + a_{00} = \\ &= \left[\begin{array}{cc} x & y \end{array}\right] \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & 0 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] + 2\left[\begin{array}{cc} a_{01} & a_{02} \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] + a_{00} = 0 \end{aligned} \tag{30}$$

After the isotropic rotation with $\alpha = -\dfrac{a_{11}}{2a_{12}}$ and a translation in direction of the $\bar{x}$ and $\bar{y}$ axes we obtain

$$F(\bar{\bar{x}}, \bar{\bar{y}}) \equiv 2a_{12}\bar{\bar{x}}\bar{\bar{y}} + t = 0 \tag{31}$$

with the remaining coefficients

$$a_{12} = \sqrt{-\boldsymbol{\sigma}}, \quad t = \bar{a}_{00} - \frac{2\bar{a}_{01}\bar{a}_{02}}{a_{12}} = \frac{\mathbf{A}}{\boldsymbol{\sigma}}\,. \tag{32}$$

Corollary 1 *For any conic obeying the conditions a) of Proposition 2 there is an isotropic motion to reduce its equation to the form (31).*

The possible cases to distinguish are

(xi) Under $t \neq 0$ equation (31) represents a *special hyperbola* with $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} < 0$, $a_{22} = 0$.

(xii) Under $t = 0$ we get from (31)

$$\bar{\bar{x}}\bar{\bar{y}} = 0. \tag{33}$$

This is a pair of *intersecting straight lines* from which one is isotropic. The invariants are $\mathbf{A} = 0$, $\boldsymbol{\sigma} < 0$, $a_{22} \neq 0$.

3rd family of conics

Let's assume that the conditions b) of Proposition 2 are fulfilled, that is $a_{22} = 0$, $\boldsymbol{\sigma} = 0$, and let's assume $a_{11} \neq 0$ which implies $a_{12} = 0$. The conic section equation (1) has now the form

$$\begin{aligned} F(x,y) &\equiv a_{11}x^2 + 2a_{01}x + 2a_{02}y + a_{00} = \\ &= \left[\begin{array}{cc} x & y \end{array}\right] \left[\begin{array}{cc} a_{11} & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] + 2\left[\begin{array}{cc} a_{01} & a_{02} \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] + a_{00} = 0. \end{aligned} \tag{34}$$

We can always transform it into

$$F(\bar{x}, \bar{y}) \equiv a_{11}\bar{x}^2 + 2\bar{a}_{02}\bar{y} + t = 0, \tag{35}$$

where

$$\mathbf{A} = -a_{02}^2\, a_{11}, \quad \mathbf{A}_{11} = -a_{12}^2, \quad \mathbf{A}_{22} = t\, a_{11} \tag{36}$$

are invariants.

Corollary 2 *For any conic section obeying the conditions b) of Proposition 2 and with* $a_{11} \neq 0$ *there is an isotropic motion to reduce its equation to the form (35).*

We distinguish

(xiii) Under $a_{02} \neq 0$ we get with (35) the equation of a *parabolic circle* with $\mathbf{A} \neq 0$, $\boldsymbol{\sigma} = 0$, $a_{22} = 0$.

(xiv) Under $\operatorname{sgn} a_{11} \neq \operatorname{sgn} t$ and $a_{02} = 0$ the equation (35) turns into

$$|a_{11}|\,\bar{x}^2 - |\,t| = 0 \tag{37}$$

which represents a pair of *isotropic lines* obeying $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} = 0$, $\mathbf{A}_{22} < 0$.

(xv) $\operatorname{sgn} a_{11} = \operatorname{sgn} t$ and $a_{02} = 0$:

$$|a_{11}|\,\bar{x}^2 + |\,t| = 0 \tag{38}$$

gives an *imaginary pair of isotropic lines* with $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} = 0$, $\mathbf{A}_{22} > 0$.

(xvi) Under $a_{11} = 0$ and $t = 0$ eq. (35) can be written in the form

$$\bar{x}^2 = 0, \tag{39}$$

which represents two *coinciding isotropic lines* and $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} = 0$, $\mathbf{A}_{22} = 0$, $a_{11} \neq 0$.

4th family of conics

Suppose the conditions b) of the Proposition 2 are fulfilled, that is $a_{22} = 0$ and $\boldsymbol{\sigma} = 0$, and let's assume $a_{11} = 0$. The conic section equation (1) has now the form

$$F(x,y) \equiv \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & 0 & 0 \\ a_{02} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = 0, \tag{40}$$

that is in homogenous coordinates $(x_0 : x_1 : x_2)$

$$\begin{aligned} F(x_0, x_1, x_2) &\equiv [x_0\ x_1\ x_2] \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & 0 & 0 \\ a_{02} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \\ &= a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 = x_0\,(a_{00}x_0 + 2a_{01}x_1 + 2a_{02}x_2) = 0, \end{aligned} \tag{41}$$

where all the remaining coefficients are invariants.

Corollary 3 *For any conic section obeying the conditions b) of Proposition 2 and with $a_{11} = 0$ there is always an isotropic motion to transform its equation into the form (41). This reveals that the conic consists of two straight lines including the absolute line f.*

The possibilities for the other line are the following:

(xvii) Under $a_{02} \neq 0$ the second line of (41) is non-isotropic. The invariants are $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} = 0$, $\mathbf{A}_{11} \neq 0$.

(xviii) Under $a_{02} = 0$ and $a_{01} \neq 0$ we get from (41)

$$x_0x_1 = 0. \tag{42}$$

Now the second line is isotropic. We note $\mathbf{A} = 0$, $\boldsymbol{\sigma} = 0$, $a_{22} = 0$, $\mathbf{A}_{11} = 0$, $a_{11} \neq 0$, $\mathbf{A}_{22} = 0$.

(xix) If $a_{02} = 0$, $a_{01} = 0$ and $a_{00} \neq 0$, eq. (41) takes the form

$$x_0^2 = 0, \tag{43}$$

this is f as a *double line* with $\mathbf{A} = \boldsymbol{\sigma} = a_{22} = \mathbf{A}_{11} = a_{11} = \mathbf{A}_{22} = 0$, $a_{00} \neq 0$.

<table>
<tr><td rowspan="10">$a_{22} \neq 0$</td><td rowspan="3">$\boldsymbol{\sigma} > 0$</td><td rowspan="2">$\mathbf{A} \neq 0$</td><td>$a_{22}\mathbf{A} > 0$</td><td colspan="4">imaginary ellipse</td></tr>
<tr><td>$a_{22}\mathbf{A} < 0$</td><td colspan="4">real ellipse</td></tr>
<tr><td>$\mathbf{A} = 0$</td><td colspan="5">imaginary pair of straight lines</td></tr>
<tr><td rowspan="3">$\boldsymbol{\sigma} < 0$</td><td rowspan="2">$\mathbf{A} \neq 0$</td><td>$a_{22}\mathbf{A} > 0$</td><td colspan="4">2nd type hyperbola</td></tr>
<tr><td>$a_{22}\mathbf{A} < 0$</td><td colspan="4">1st type hyperbola</td></tr>
<tr><td>$\mathbf{A} = 0$</td><td colspan="5">pair of intersecting straight lines</td></tr>
<tr><td rowspan="4">$\boldsymbol{\sigma} = 0$</td><td>$\mathbf{A} \neq 0$</td><td colspan="5">parabola</td></tr>
<tr><td rowspan="3">$\mathbf{A} = 0$</td><td>$\mathbf{A}_{11} < 0$</td><td colspan="4">pair of parallel lines</td></tr>
<tr><td>$\mathbf{A}_{11} > 0$</td><td colspan="4">imaginary pair of parallel lines</td></tr>
<tr><td>$\mathbf{A}_{11} = 0$</td><td colspan="4">two coinciding parallel lines</td></tr>
<tr><td rowspan="10">$a_{22} = 0$</td><td rowspan="2">$\boldsymbol{\sigma} < 0$</td><td>$\mathbf{A} \neq 0$</td><td colspan="5">special hyperbola</td></tr>
<tr><td>$\mathbf{A} = 0$</td><td colspan="5">pair of lines, one is an isotropic line</td></tr>
<tr><td rowspan="8">$\boldsymbol{\sigma} = 0$</td><td>$\mathbf{A} \neq 0$</td><td colspan="5">parabolic circle</td></tr>
<tr><td rowspan="7">$\mathbf{A} = 0$</td><td>$\mathbf{A}_{22} < 0$</td><td colspan="4">pair of istropic lines</td></tr>
<tr><td>$\mathbf{A}_{22} > 0$</td><td colspan="4">imaginary pair of istropic lines</td></tr>
<tr><td>$\mathbf{A}_{22} = 0$</td><td>$a_{11} \neq 0$</td><td colspan="3">two coinciding istropic lines</td></tr>
<tr><td>$\mathbf{A}_{11} \neq 0$</td><td colspan="4">straight line $+ f$</td></tr>
<tr><td rowspan="3">$\mathbf{A}_{11} = 0$</td><td>$\mathbf{A}_{22} \neq 0$</td><td rowspan="3">$a_{11} = 0$</td><td colspan="2">isotropic line $+ f$</td></tr>
<tr><td rowspan="2">$\mathbf{A}_{22} = 0$</td><td>$a_{00} \neq 0$</td><td>double f</td></tr>
<tr><td>$a_{00} = 0$</td><td>all points $\in \mathbb{I}_2$</td></tr>
</table>

Table 1: Classification of second order curves in $\mathbb{I}_2$

(xx) Under $a_{02} = a_{01} = a_{00} = 0$ we get in (41) the zero polynomial. The conditions are $\mathbf{A} = \boldsymbol{\sigma} = a_{22} = \mathbf{A}_{11} = a_{11} = \mathbf{A}_{22} = a_{00} = 0$.

Thus, we have partitioned the set of conic sections k of the isotropic plane $\mathbb{I}_2$ into four families according to their relation to the absolute figure (f, F):

1st family: The conic section k doesn't contain any isotropic direction, i.e., $F \notin k$;

2nd family: The conic section k has one isotropic direction, i.e., $F \in k$;

3rd family: The conic section k has a double isotropic direction;

4th family: The conic section k contains an absolute line f.

We conclude with

Proposition 4 *In the isotropic plane $\mathbb{I}_2$ there are 20 different types of conic sections to distinguish with respect to the group $\mathbf{G}_3$ of motions (see Table 1).*

Acknowledgments

The author is grateful to the referees for their valuable suggestions.

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Received September 12, 2001; final form June 9, 2002