Journal for Geometry and Graphics
Volume 6 (2002), No. 1, 27–35.

Two-parametric Motions in the Lobatchevski Plane

Marta Hlavová

Department of Technical Mathematics, Faculty of Mechanical Engineering CTU
Karlovo nám. 13, 120 00 Praha, Czech Republic
email: hlavova@marian.fsik.cvut.cz

Abstract. In this paper we classify two-parametric motions in the Lobatchevski plane L_2. These motions are surfaces on the Lie group $SO(2,1)$. In the first part the basic properties of motions in L_2 are derived and it turns out that the kinematical space belonging to these motions is locally the space $SO(2,2)/SO(2,1)$ realized as the unit quadric with signature $(2,2)$ in the vector space $\mathbb{R}_4$. The remaining part contains explicit expressions and graphic representations of surfaces induced by motions with constant invariants. We also present some special cases — developable surfaces.

Key Words: two-parameter motion, hyperbolic geometry, Lobatchevski plane

MSC 2000: 53A17, 51N25

1. Introduction

Let us consider vector space $V = \mathbb{R}_3$ with scalar product $(x,y) = x_1y_1 + x_2y_2 - x_3y_3$ and vector product $x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2)$ for any $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ from V. We can choose an orthonormal base $\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3\}$ in V ($\mathrm{e}_1^2 = \mathrm{e}_2^2 = 1$, $\mathrm{e}_3^2 = -1$, $(\mathrm{e}_i,\mathrm{e}_j) = 0$ for $i \neq j$) for which $\mathrm{e}_1 \times \mathrm{e}_2 = -\mathrm{e}_3, \mathrm{e}_2 \times \mathrm{e}_3 = \mathrm{e}_1, \mathrm{e}_3 \times \mathrm{e}_1 = \mathrm{e}_2$. The *Lobatchevski (hyperbolic) plane* L_2 is the projective plane P_2, realized as one-dimensional subspaces of V.

Planar hyperbolic geometry is the geometry of L_2 with respect to the group of projective transformations preserving the so-called absolute conic $x_1^2 + x_2^2 - x_3^2 = 0$ (see [4]). This group is isomorphic to the Lie group $G = SO(2,1)$ realized as a group of 3×3 matrices acting on L_2 by matrix multiplication. Each element g of $SO(2,1)$ satisfies the condition $J = g^T J g$, where J is the matrix representation of the quadratic form $x_1^2 + x_2^2 - x_3^2$,

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

ISSN 1433-8157/\$ 2.50

The 3-dimensional Lie algebra of the group $SO(2,1)$, denoted by $\mathfrak{so}(2,1)$, is interpreted as the tangent space in the unit element of the group G and every element X from $\mathfrak{so}(2,1)$ is of the form

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$$

or $X^v = (x_1, x_2, x_3)$ in vector notation. The operation of Lie bracket in $\mathfrak{so}(2,1)$ is the same as the vector product introduced above, $[X,Y] = X^v \times Y^v$. For the invariant Killing form we obtain $\mathrm{K}(X,Y) = 2(X^v, Y^v)$, where (X^v, Y^v) means the scalar product introduced above. From this we see that the Lie algebra $\mathfrak{so}(2,1)$ can be identified with the vector space V with the vector product introduced above. We also notice that introduced operations are similar to the usual operations of vector algebra, which correspond to spherical kinematics (see [1]). This analogy will be also used in what follows.

2. Motion in a homogeneous space

By a *two-parametric motion* in L_2 we mean a 2-dimensional motion in the group $G = SO(2,1)$, which is an immersion g of a 2-dimensional manifold X into G. Let us choose two copies of L_2 and in each of them a fixed orthonormal frame: $\bar{\mathcal{R}}_0$ in moving plane $\bar{L}_2$ and $\mathcal{R}_0$ in fixed plane L_2. The group $G = SO(2,1)$ acts as a group of linear maps from $\bar{L}_2$ into L_2 by the rule $g(\bar{\mathcal{R}}_0) = \mathcal{R}_0 g$. Then for fixed elements g_1, g_2 in G we consider motions $g_1 g g_2^{-1}$ and g equivalent.

This shows that we have to consider the homogeneous space $G_0 = G \times G/\mathrm{Diag}(G \times G)$, where $\mathrm{Diag}(G \times G)$ is the group of all elements (g,g) for g from G. We consider the action of $G \times G$ on G given by the rule $(g_1, g_2)g = g_1 g g_2^{-1}$ and this shows that G can be identified with G_0 because the isotropy group of the unit element e of the G is the group $\mathrm{Diag}(G \times G)$ (see [3]). This homogeneous space G_0 is called the *kinematical space* of the group G. By a 2-dimensional motion in G we now understand an immersion of a 2-dimensional manifold X into G_0 and the properties of the motion in G are properties of an immersion of X into G_0.

The classical example is the geometry of the group $SO(3)$ realized as the group of motions of the unit sphere S_2 in $\mathbb{E}_3$, the homogeneous space G_0 of which is locally the space $SO(4)/SO(3)$ and it is the 3-dimensional elliptic space (see [1, 2]).

In our case, $SO(2,1) \times SO(2,1)$ is locally isomorphic with $SO(2,2)$, which is the group of projective transformations in P_3 preserving the quadric with signature $(2,2)$. This follows that the homogeneous space G_0 is locally the space $SO(2,2)/SO(2,1)$ of dimension 3. We can realize this space as the unit quadric with the signature $(2,2)$ in the projective space P_3 realized as one-dimensional subspaces of vector space $\mathbb{R}_4$. Because the space $SO(2,2)/SO(2,1)$ is 3-dimensional, each two-parametric motion can be represented as a surface in this space and properties of such surfaces are the same as properties of two-parametric motions.

3. Two-parametric motions in L_2

Let $g(\mathrm{X})$ be a two-parametric motion in L_2. We may identify the elements of $SO(2,1) \times SO(2,1)$ with pairs of orthonormal frames by $\mathcal{R} = \mathcal{R}_0 g_1, \bar{\mathcal{R}} = \bar{\mathcal{R}}_0 g_2$, so that the pair (g_1, g_2) is identified with the pair $(\mathcal{R}, \bar{\mathcal{R}})$. Let $(\mathcal{R}, \bar{\mathcal{R}})$ be a *lift* of g, by definition a pair of orthonormal frames such that $g\bar{\mathcal{R}} = \mathcal{R}$. From the following formulas

$$\mathrm{d}\mathcal{R} = \mathcal{R} g_1^{-1} \mathrm{d}g_1, \quad \mathrm{d}\bar{\mathcal{R}} = \bar{\mathcal{R}} g_2^{-1} \mathrm{d}g_2$$

we obtain the left invariant form (φ, ψ) of the lift (g_1, g_2), where $\varphi = g_1^{-1}\mathrm{d}g_1$ and $\psi = g_2^{-1}\mathrm{d}g_2$. Let's define new forms $\omega = (\varphi - \psi)/2$ and $\eta = (\varphi + \psi)/2$. Forms φ, ψ, ω and η have values in $\mathfrak{so}(2,1)$. Then $\mathrm{d}\mathcal{R} = \mathcal{R}\varphi$, $\mathrm{d}\bar{\mathcal{R}} = \bar{\mathcal{R}}\psi$ and

$$\mathrm{d}\varphi + \varphi \wedge \varphi = 0, \quad \mathrm{d}\psi + \psi \wedge \psi = 0, \tag{1}$$

or, by using ω and η

$$\mathrm{d}\omega + \eta \wedge \omega + \omega \wedge \eta = 0, \quad \mathrm{d}\eta + \omega \wedge \omega + \eta \wedge \eta = 0,$$

where (1) are Maurer-Cartan relations obtained by exterior differentiation of φ and ψ. This gives the integrability conditions in the form

$$\begin{array}{ll} \mathrm{d}\omega_1 = \omega_2 \wedge \eta_3 + \eta_2 \wedge \omega_3 & \mathrm{d}\eta_1 = \eta_2 \wedge \eta_3 + \omega_2 \wedge \omega_3 \\ \mathrm{d}\omega_2 = \omega_3 \wedge \eta_1 + \eta_3 \wedge \omega_1 & \mathrm{d}\eta_2 = \eta_3 \wedge \eta_1 + \omega_3 \wedge \omega_1 \\ \mathrm{d}\omega_3 = \omega_2 \wedge \eta_1 + \eta_2 \wedge \omega_1 & \mathrm{d}\eta_3 = \eta_2 \wedge \eta_1 + \omega_2 \wedge \omega_1. \end{array} \tag{2}$$

If we choose another lift of g, say $(g_1 h, g_2 h)$, where $h \in SO(2,1)$, then for the new forms $\tilde{\omega}$ and $\tilde{\eta}$ we get

$$\begin{aligned} \tilde{\omega} &= h^{-1}\omega h \\ \tilde{\eta} &= h^{-1}\eta h + h^{-1}\mathrm{d}h. \end{aligned}$$

Due to the isomorphism between $\mathfrak{so}(2,1) \times \mathfrak{so}(2,1)$ and $\mathfrak{so}(2,2)$ for an adapted frame $\mathcal{R} = \{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ of $g(\mathrm{X})$ (an *adapted frame* is any orthonormal frame with respect to the invariant quadratical form $x_0^2 - x_1^2 - x_2^2 + x_3^2$ such that e_0 is the point of induced surface) we obtain $\mathrm{d}\mathcal{R} = \mathcal{R}M$, where

$$M = \begin{pmatrix} 0 & \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & 0 & \eta_3 & \eta_2 \\ \omega_2 & -\eta_3 & 0 & \eta_1 \\ -\omega_3 & \eta_2 & \eta_1 & 0 \end{pmatrix}.$$

According to the Killing form $\mathrm{K}(\omega) = \omega_1^2 + \omega_2^2 - \omega_3^2$ we have to distinguish three cases: $\mathrm{K}(\omega)$ is positive definite (has signature (2,0)), $\mathrm{K}(\omega)$ is indefinite (has signature (1,1)) and $\mathrm{K}(\omega)$ is singular. In this paper we leave out the last case because it is not interesting.

3.1. $\mathrm{K}(\omega)$ is positive definite

We may always find a lift of g such that $\omega_3 = 0$. An adapted frame is then any orthonormal frame such that e_0 is the point of induced surface, $\mathrm{e}_1, \mathrm{e}_2$ determine the tangent plane of surface and e_3 is normal to surface. Then $\omega_1 \wedge \omega_2 \neq 0$ and the corresponding isotropy group is

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The integrability conditions are given in (2). Using Cartan's Lemma we get

$$\begin{aligned} \eta_1 &= \beta\omega_1 + \gamma\omega_2 \\ \eta_2 &= \alpha\omega_1 + \beta\omega_2 \quad . \end{aligned}$$

From the new forms $\tilde{\omega}, \tilde{\eta}$ given by the change of frame h we obtain two invariants:

$$H_0 = \frac{1}{2}(\alpha + \gamma) \tag{3}$$

$$K_0 = \beta^2 - \alpha\gamma, \tag{4}$$

where H_0 is the *mean curvature* and K_0 is the *Gauss curvature* of the surface induced by two-parametric motion.

There always exists a lift such that $\beta = 0$. α and γ are then called the *main curvatures* of the surface. In the case $\alpha = \gamma = 0$ we get a planar point and if this condition is satisfied on a neighbourhood, we get a part of a plane.

Let us denote $\eta_3 = \kappa\omega_1 + \lambda\omega_2$, $\mathrm{d}\alpha = \alpha_1\omega_1 + \alpha_2\omega_2$, $\mathrm{d}\gamma = \gamma_1\omega_1 + \gamma_2\omega_2$, $\mathrm{d}\kappa = \kappa_1\omega_1 + \kappa_2\omega_2$ and $\mathrm{d}\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$. Computation gives:

$$\begin{aligned} \gamma_1 &= (\gamma - \alpha)\lambda \\ \alpha_2 &= (\gamma - \alpha)\kappa \end{aligned} \tag{5}$$

and

$$\kappa_2 + \kappa^2 - \lambda_1 + \lambda^2 - 1 - \alpha\gamma = 0. \tag{6}$$

These equations (5) and (6) are analogous to Codazzi and Gauss equations from the surface theory in $\mathbb{E}_3$.

3.1.1. Surfaces with constant main curvatures

In the case $\alpha = \gamma \neq 0$ we get analogical results as in the theory of two-parametric motions in $O(3)$ (see [6]). Then $\mathrm{d}(\mathrm{e}_0 - \mathrm{e}_3/\alpha) = 0$ and computation gives the equation of the surface

$$x^2 + y^2 - z^2 = -\frac{1}{\alpha^2}, \tag{7}$$

which is a double-sheet hyperboloid in dehomogenized coordinates.

In the following figures we present given surfaces in grey shade together with the absolute one-sheet hyperboloid, similarly as in Fig. 1.

Let $\alpha \neq \gamma$ be constants, then from (5) $\gamma_1 = \alpha_2 = 0$ and $\kappa = \lambda = 0$. Then $\eta_3 = 0$ and $\alpha\gamma = -1$ (due to (6)). According to integrability conditions $\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0$ we get $\omega_1 = \mathrm{d}u, \omega_2 = \mathrm{d}v$. The Frenet formulas (applied in the same sense as in the theory of curves) for the surface are

$$\begin{aligned} \mathrm{de}_0 &= \mathrm{d}u\,\mathrm{e}_1 + \mathrm{d}v\,\mathrm{e}_2 \\ \mathrm{de}_1 &= \mathrm{d}u\,\mathrm{e}_0 + \alpha\,\mathrm{d}u\,\mathrm{e}_3 \\ \mathrm{de}_2 &= \mathrm{d}v\,\mathrm{e}_0 - 1/\alpha\,\mathrm{d}v\,\mathrm{e}_3 \\ \mathrm{de}_3 &= \alpha\,\mathrm{d}u\,\mathrm{e}_1 - 1/\alpha\,\mathrm{d}v\,\mathrm{e}_2. \end{aligned}$$

Integration gives

$$\begin{aligned} \mathrm{e}_1 &= f_0\, e^{\sqrt{1+\alpha^2}\,u} + f_1\, e^{-\sqrt{1+\alpha^2}\,u} \\ \mathrm{e}_2 &= f_2\, e^{\frac{\sqrt{1+\alpha^2}\,v}{\alpha}} + f_3\, e^{\frac{-\sqrt{1+\alpha^2}\,v}{\alpha}}, \end{aligned}$$

where $\{f_0, f_1, f_2, f_3\}$ is a fixed orthonormal base. Computation yields

$$\mathrm{e}_0 = \frac{1}{1+\alpha^2}\Big(\frac{\partial \mathrm{e}_1}{\partial u} + \alpha^2 \frac{\partial \mathrm{e}_2}{\partial v}\Big)$$

and the equation of the surface is

$$\alpha^2 x = yz. \tag{8}$$

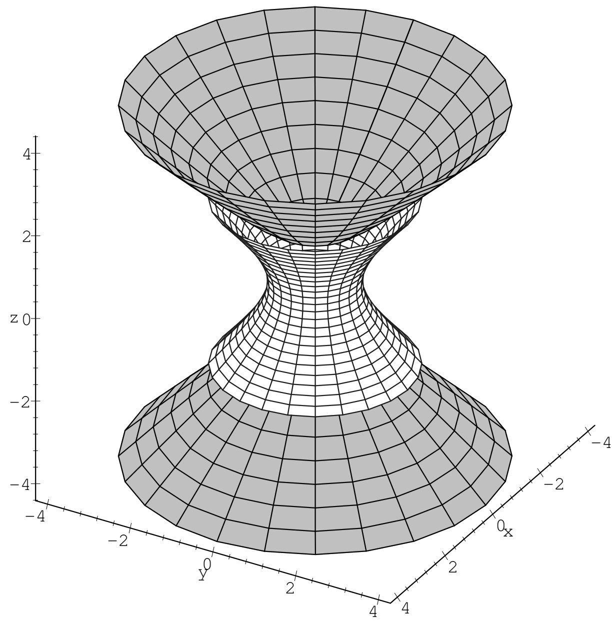

Figure 1: The surface (7) for choice $\alpha = 1$

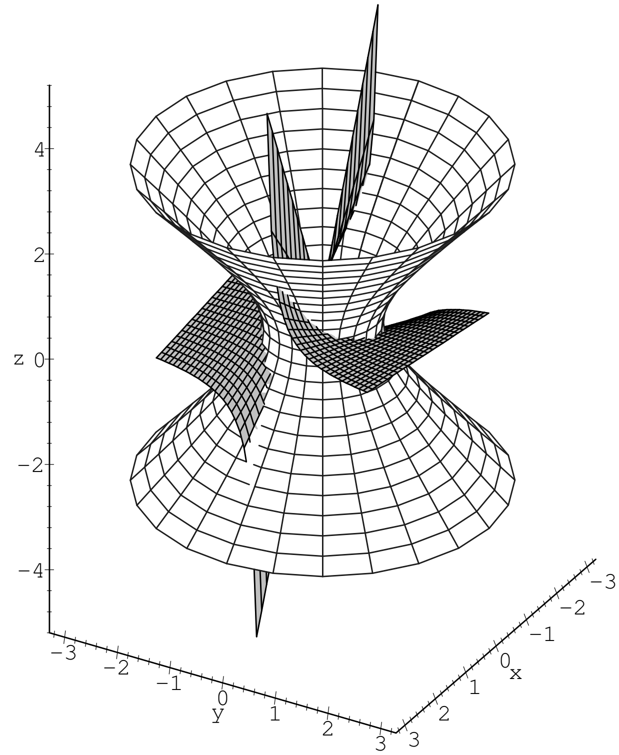

Figure 2: The surface (8) for $\alpha = 1/2$

3.2. $\mathrm{K}(\omega)$ is indefinite

If this condition is satisfied, we can take such a lift for which $\omega_1 = 0$. Then $\omega_2 \wedge \omega_3 \neq 0$ and the isotropy group is

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\phi & \sinh\phi \\ 0 & 0 & \sinh\phi & \cosh\phi \end{pmatrix}.$$

Using Cartan's Lemma we get $\eta_2 = \beta\omega_2 + \gamma\omega_3$, $\eta_3 = \alpha\omega_2 - \beta\omega_3$. The mean and Gauss curvatures are

$$H_0 = \frac{1}{2}(\alpha + \gamma), \quad K_0 = \beta^2 + \alpha\gamma.$$

Let us denote $\eta_1 = \kappa\omega_2 + \lambda\omega_3$, $\mathrm{d}\alpha = \alpha_2\omega_2 + \alpha_3\omega_3$, $\mathrm{d}\gamma = \gamma_2\omega_2 + \gamma_3\omega_3$, $\mathrm{d}\kappa = \kappa_2\omega_2 + \kappa_3\omega_3$ and $\mathrm{d}\lambda = \lambda_2\omega_2 + \lambda_3\omega_3$. For a choosen frame such that $\beta = 0$ (possibility $\alpha = \gamma = 0$ leads to the same conclusion as in **3.1.**) we obtain

$$\begin{aligned} \gamma_2 &= (\gamma - \alpha)\lambda \\ \alpha_3 &= (\alpha - \gamma)\kappa \end{aligned} \tag{9}$$

and

$$\kappa_3 - \kappa^2 - \lambda_2 + \lambda^2 - 1 + \alpha\gamma = 0. \tag{10}$$

3.2.1. Surfaces with constant main curvatures

In the case $\alpha = \gamma$ we get $\mathrm{d}(\mathrm{e}_0 + \mathrm{e}_1/\alpha) = 0$ and computation gives the equation of the surface in the form

$$x^2 - y^2 - z^2 = -\frac{1}{\alpha^2}, \tag{11}$$

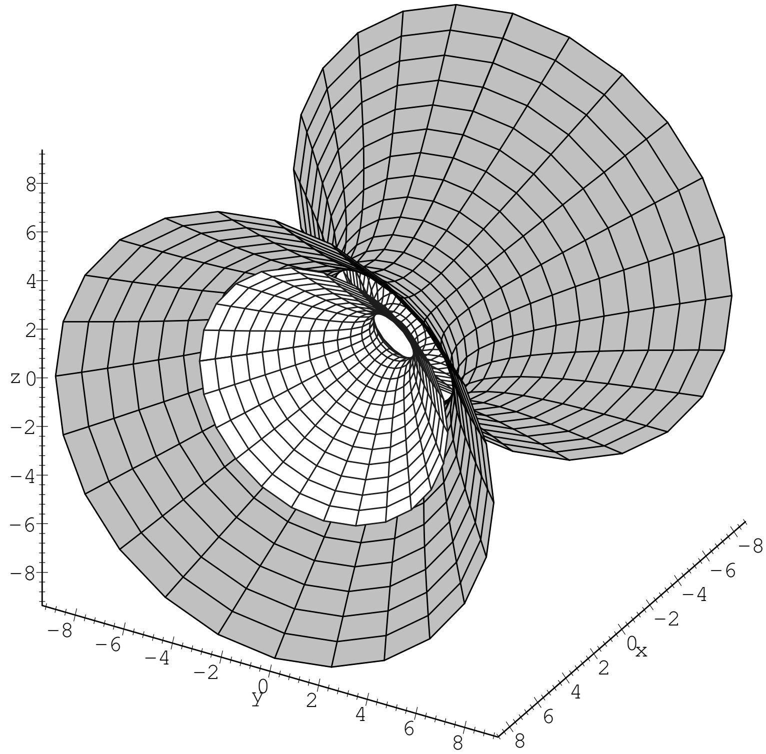

Figure 3: The surface (11) for $\alpha = 1/3$

which is a one-sheet hyperboloid (Fig. 3).

Let $\alpha \neq \gamma$ be constants. Then $\kappa = \lambda = 0$ and $\gamma = 1/\alpha$. From (2) we get $\omega_2 = \mathrm{d}u, \omega_3 = \mathrm{d}v$ and from the Frenet formulas we obtain the equation of the following surface:

$$\alpha^2 x = yz, \tag{12}$$

already mentioned in Section 3.1.1, Fig. 2.

4. Developable surfaces

A surface is called *developable*, if the Gauss curvature $K_0 = 0$. For the first type of motion (Section 3.1) we obtain $\alpha\gamma = 0$. Let us suppose $\alpha \neq 0$ and $\gamma = 0$. Then $\lambda = 0$, $\alpha_2 = -\alpha\kappa$ and $\kappa_2 + \kappa^2 - 1 = 0$ from (5) and (6). The corresponding matrix is

$$M = \begin{pmatrix} 0 & \omega_1 & \omega_2 & 0 \\ \omega_1 & 0 & \kappa\omega_1 & \alpha\omega_1 \\ \omega_2 & -\kappa\omega_1 & 0 & 0 \\ 0 & \alpha\omega_1 & 0 & 0 \end{pmatrix}.$$

From the integrability conditions we get $\mathrm{d}\omega_2 = 0$ and $\mathrm{d}\eta_2 = \mathrm{d}(\alpha\omega_1) = 0$. So let us denote $\omega_2 = \mathrm{d}u$, $\omega_1 = \mathrm{d}v/\alpha$. Integration gives $\kappa = \tanh(u - g(v))$, $\alpha = -h(v)/\cosh(u - g(v))$, where $g(u), h(u)$ are arbitrary functions. Computation gives

$$\mathrm{e}_0 = f_0(v)e^u + f_1(v)e^{-u}, \tag{13}$$

where f_0, f_1 are orthonormal vectors depending on one variable v.

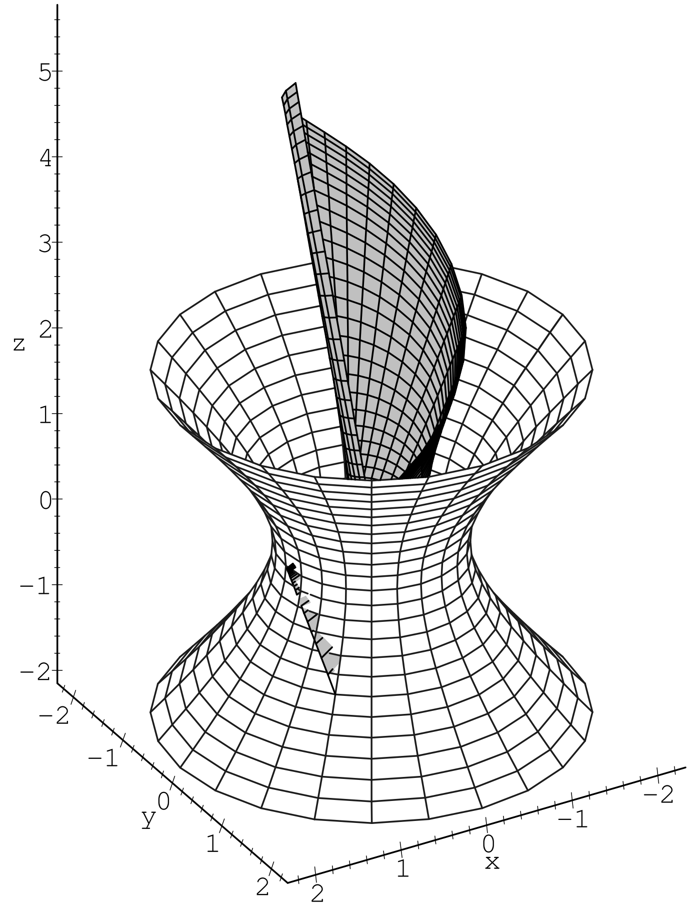

Figure 4: Developable surface (13)

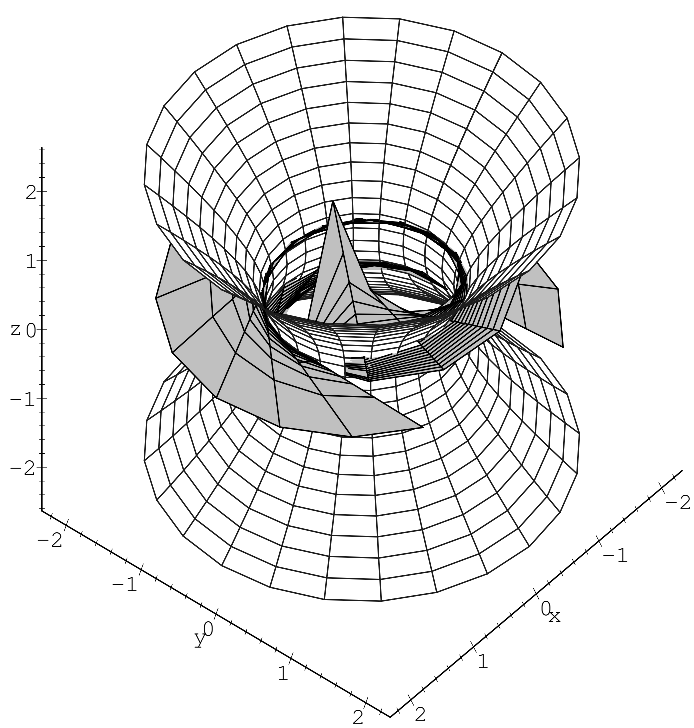

Figure 5: Developable surface (14)

For example we choose

$$f_0 = \Big(\frac{1}{v}, \frac{\cos v}{v}, \frac{\sin v}{v}, 1\Big), \ f_1 = \Big(1, -\frac{\sin v}{v}, \frac{\cos v}{v}, -\frac{1}{v}\Big).$$

The corresponding surface is displayed in Fig. 4.

In the second case (Section 3.2) the properties are similar. We obtain $\alpha\gamma = 0$ and then $\lambda = 0$ and $\alpha_3 = \alpha\kappa$, $\kappa_3 - \kappa^2 - 1 = 0$ (due to (9),(10)). The matrix

$$M = \begin{pmatrix} 0 & 0 & \omega_2 & \omega_3 \\ 0 & 0 & \alpha\omega_2 & 0 \\ \omega_2 & -\alpha\omega_2 & 0 & \kappa\omega_2 \\ -\omega_3 & 0 & \kappa\omega_2 & 0 \end{pmatrix}$$

and due to $\mathrm{d}\omega_3 = 0$ and $\mathrm{d}\eta_3 = 0 = \mathrm{d}(\alpha\omega_2)$ we can denote $\omega_3 = \mathrm{d}u$, $\omega_2 = \mathrm{d}v/\alpha$. Then we get $\kappa = \tan(v - g(u))$ and $\alpha = \cos(g(u))h(u)/\cos(v - g(u))$, where $g(u), h(u)$ are arbitrary function again. Integration yields

$$\mathrm{e}_0 = f_0(v)\cos u + f_1(v)\sin u, \tag{14}$$

where f_0, f_1 mean the same as in the previous case.

For the choice

$$f_0 = \Big(v, v\cos v, v\sin v, 1\Big), \ f_1 = \Big(\frac{1}{v}, \frac{\cos v}{v} - \sin v, \frac{\sin v}{v} + \cos v, 0\Big)$$

we get the example of a developable surface (14) presented in Fig. 5.

5. Conclusion

The results we have presented show that the theory of two-parametric motion in the hyperbolic plane is similar to the theory of the same kind of motion of unit sphere S_2 in the Euclidean space (see [1]), closely connected with the elliptic surface theory. It yields a natural interpretation of the group of projective space transformations preserving a one-sheet hyperboloid.

All computations and figures were obtained by using MAPLE software.

References

[1] W. Blaschke: *Nicht-Euklidische Geometrie und Mechanik*. Hamb. Math. Einzelschriften, 34. Heft, 1942.

[2] W. Blaschke: *Kinematik und Quaternionen*. Deutscher Verlag der Wiss., Berlin 1960.

[3] S. Helgason: *Differential Geometry and Symmetric Spaces*. Academic Press, New York 1962.

[4] V. Hlavatý: *Úvod do neeuklidovské geometrie*. JČSMF 1949.

[5] M. Husty, P. Nagy: *Dreidimensionale Lie Gruppen und ebene Kinematik*. Math. Pannonica **1**, 1–15 (1990).

[6] A. Karger: *Two-Parametric Motions in* E_3. Aplikace Matematiky **2/32**, 96–119 (1987).

[7] A. Karger: *Kinematic geometry of regular motions in homogeneous space.* Czech. Math. J. **28** (103), 327–338 (1978).

[8] A. Karger, J. Novák: *Prostorová kinematika a Lieovy grupy.* SNTL 1978. English translation: *Space Kinematics and Lie Groups.* Gordon & Breach Science Publishers, Montreux 1985.

[9] J.-P. Serre: *Lie algebras and Lie groups.* Springer-Verlag 1992.

Received May 7, 2002; final form July 13, 2002