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Generalized Descriptive Geometry

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Abstract. Generalized Descriptive Geometry (GDG) denotes all techniques of imaging abstract objects and their relations, which are constructive in the same sense as the methods of classical descriptive geometry, i.e., there have to be algorithms concerning the abstract objects (may be decision processes) so that there are translations into algorithms working with the pictures instead of the abstract objects. Hence GDG is more than mere visualization though the borderlines sometimes are fuzzy. The paper presents an outline of standard methods. Some of them are very old and were originally developed outside mathematics.

Key Words: Descriptive Geometry, Generalized Descriptive Geometry

MSC 2000: 51N05

1. Generalized vs. traditional descriptive geometry

The notion *Generalized Descriptive Geometry* (shortly GDG, originally in German: *Darstellende Geometrie im weiteren Sinne*) was seemingly created by myself in 1985 [18]. Since then it has proved for myself as a framework for many different, old and new, techniques of imaging and, moreover, as a spring of inspiration within the methodology of mathematics and a good basis for the discussion with all kinds of practitioners about some fundamentals of mathematics. So I'm very grateful for the invitation to explain my ideas in this journal.

1.1. Traditional descriptive geometry

Let us start with a short description of traditional (or classical) descriptive geometry. In my opinion it consists of all principles of making planar (Euclidean) pictures of spatial objects or ensembles of objects by exact rules and in such a manner that a welldefined set of algorithmic processes designed for these spatial objects can be translated into equivalent algorithms working with the plane pictures. These processes may be constructions, e.g., from the plane and elevation of two different points construct the plane and elevation of the connecting straight line, or they may be decisions, e.g., from the plane and elevation of two straight lines decide whether they are parallel or not. It is also noteworthy that within descriptive geometry (and so also in GDG) we have processes for [17]

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a) producing a picture from a given picture or completing a given picture by additional components,

b) producing a nongeometric result from a picture (e.g., 'yes' or 'no' if the process is a decision) or a number if we use the picture for counting the vertices, edges, or faces of a polyhedron,

c) producing a picture from nongeometric given objects (e.g., a formula or a verbal description),

but only processes of type a) and b) use pictures as input objects. Processes of type c) mostly produce only visualizations though their outputs may serve as inputs for further algorithms.

There are further important remarks:

1. According to this definition the methods of traditional descriptive geometry are special cases of the more general notion of *coding* and brings them in analogy to such methods as

   – working with the coordinates of geometric objects instead of the objects,

   – calculating by the use of numbering systems, i.e., with strings of symbols instead of the (abstract) numbers,

   – executing such procedures as differential and integral calculus by use of formulas denoting functions instead of with the functions themselves.

2. Producing (more or less) nice planar pictures from spatial objects is a very old art. One of the most important merits of Gaspard Monge is just that he accentuated the coding nature of descriptive geometry, the scetched constructive aspect, and hence its analogy to coordinate geometry.

3. Equal to other coding methods, the methods of descriptive geometry guarantee the one-to-one correspondence between some kinds of encoded objects and their code objects only by a more or less complex system of contextual knowledge. The more information you want to encode, the more knowledge about the coding technique the 'reader' must have. Moreover, each enlargement of the encoded kinds of objects or information is to pay by a loss of evidence of the pictures.
   An Example: For the exact and unique description of *all* kinds of straight lines in plane and elevation, the simple pair of projections is insufficient. In the well known special cases you must use a third plane or a more symbolic representation of these straight lines, and for the 'reader' of such pictures more knowledge about the used code is necessary. Often, uniqueness is forced by labelling the depicted objects. The code objects then are elements of cartesian products of one, two, or three planes and further sets (e.g., of numbers, reals, or words). Moreover, in principle the cartesian coordinate method consists of projections of a point onto three axes of points or reals instead of two planes. So, from an advanced point of view, the borderline between the methods of descriptive geometry and the coordinate methods is somewhat fuzzy.

By the way, the inspection of lots of historical technical drawings shows that methods with a small amount of one-to-one correspondence and a high grade of evidence (as central perspective or especially parallel projection) mostly were preferred for the presentation (and particularly for marketing) of existing objects against people without deeper knowledge whereas the design of not yet existing objects requires a higher amount of uniqueness of the pictures and hence a higher grade of abstractness and a higher amount of knowledge about the imaging technique both from the designer and from those people producing the objects on the basis of the drawing.

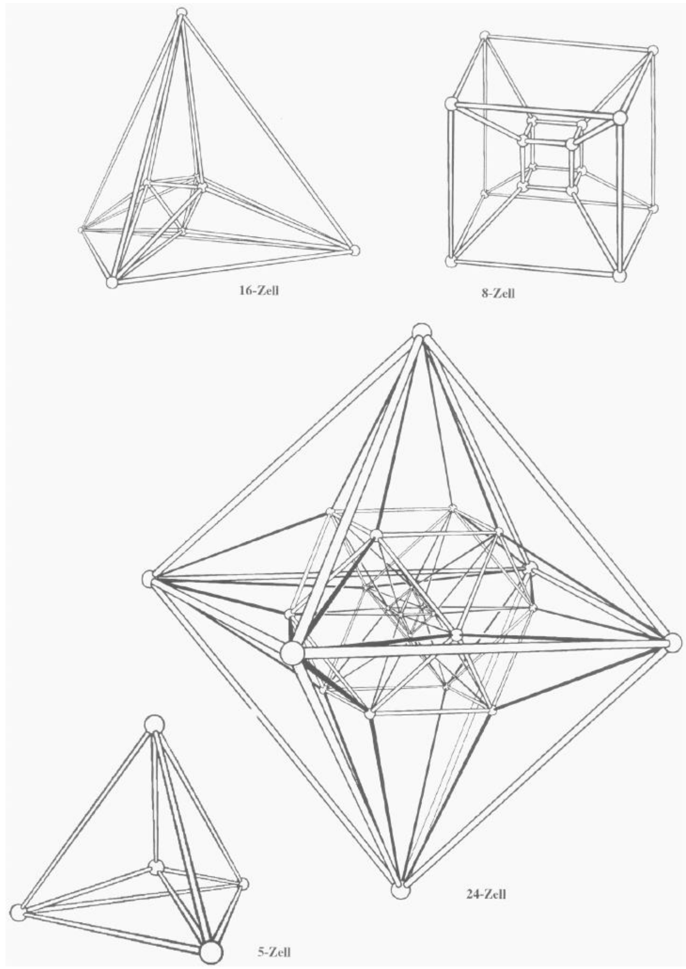


Figure 1: Projections of four of the four-dimensional regular polytopes
(c) Springer-Verlag; reproduced from [5], Figs. 169–172

1.2. Generalized descriptive geometry

Now we shall proceed to generalized descriptive geometry. By this notion we denote all methods of imaging objects and/or relations between objects which are

(a) Euclidean but of higher dimension than 3,

(b) geometric but not Euclidean,

(c) 'abstract' in other manners,

if these methods are exact defined in the same sense as scetched above for the classical methods. Particularly there must be a class of algorithms which may be translated into algorithms working with the planar pictures of the abstract objects. Because the aim of generalized descriptive geometry is visible representation of the abstract we should permit not only plane images but also three-dimensional models (or else four-dimensional models with the time as a fourth dimension, see below).

In recent times there is a growing presence of the term *visualization* [3]. In some sense it has a similar meaning, but we prefer the term generalized descriptive geometry because it stresses the constructive aspect of what we mean. Our pictures are not merely for looking and nice appeal. You should use them really as tools in certain procedures. There is an analogy with what in the arts is named *allegory* and sometimes the borderlines between visualization, generalized descriptive geometry, and allegory are somewhat fuzzy, particularly in historical examples. This analogy causes my interest in cooperation with historians of arts, whereas the notion of generalized descriptive geometry may become a bridge over the present deep gap between the exact sciences and the humanities. On the one hand, most methods of generalized descriptive geometry work also in the humanities, some of them came from there but are hardly reflected up to now. On the other hand, a popular explanation of the theory of generalized descriptive geometry is a good tool to bring a minimum of sympathy and knowledge about the specific mathematical mode of thinking into the minds of philosophers, historians, artists, linguists, ... Last but not least let us mention that according to the present state of neurobiology also the human brain internally is working with the concept of GDG. Not only what we see, but also what we hear, touch, smell, or taste is in the first instance imaged 'topographically' into the cortex [2], particularly 202 ff.

2. Examples

Let us turn to details and examples.

2.1. The fourth dimension

A good matter for explaining the fourth dimension to non-mathematicians are the analogues of the platonic solids. (There are exactly six in the four-dimensional space, four of them are drawn in Fig. 1). Describing the procedure of extending an interval to a quadrangle and a quadrangle to a cube, it is easily seen how a hypercube grows from a cube by translation in a fourth dimension (may be the time). Looking how parallel or central projection from the space onto a plane is working, everybody may easily understand (this is my experience with many visitors in an exhibition on mathematics and arts) the three-dimensional model of the four-dimensional hypercube and then also its two-dimensional picture. Next, we recall the relation between a cube and its plane net and generalize it. So we see that the solid in the painting *'Christus hypercubus'* (1954 by Salvador Dali, Fig. 2) is really the three-dimensional development of a four-dimensional hypercube.

But until now this all is only 'visualization'. Why is it generalized descriptive geometry? One simple example: The procedure of constructing a regular tetrahedron by a special choice of diagonals of the faces of a cube is easily generalized to the four-dimensional case. By constructing this with the model or with the picture of the model we may learn which of the other regular 4-polytopes is produced and that it is not the 4-dimensional simplex!

2.2. Cartography

The most popular (and perhaps the most important) case of constructive modelling of a non-Euclidean structure in the Euclidean plane is the domain of cartography. This example shows very well that different kinds of desired procedures require different kinds of modelling the same structure (here the surface of a sphere). If one is interested in the shortest or in the loxodromical route between two given points, other maps are needed than if one is looking for the area of a country (Fig. 3). The stereographic projection (i.e., central projection from a point on the surface of the sphere onto the tangential plane in the diametral point) is particularly

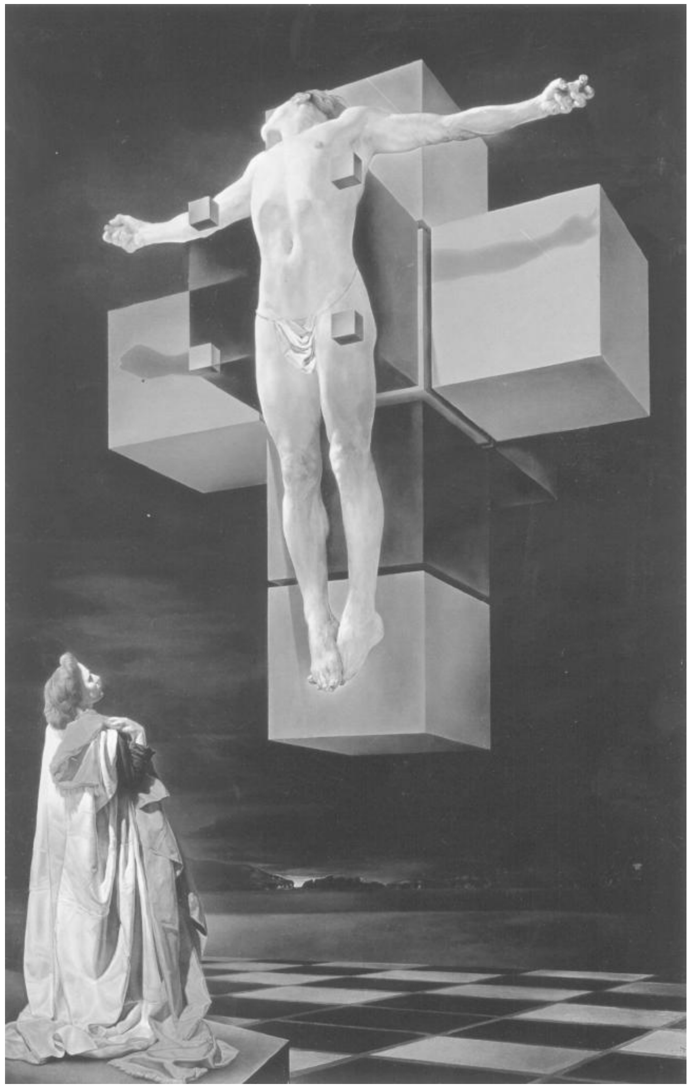


Figure 2: Salvador Dali (1904–1989): *Corpus hypercubus* (1954)
Metropolitan Museum of Art, New York, (cf. [15])

interesting from the point of view of generalized descriptive geometry: Each construction on the surface of the sphere by compass and great circle instrument (that here plays the role of the plane ruler) is transformed by the stereographic projection into a construction problem solvable by Euclidean compass and ruler [16]. Otherwise, the great circles, i.e., the orthodromic courses on a sphere, are transformed into straight lines by central projection from the midpoint of the sphere, but this projection maps only the half of the sphere onto a tangential plane. To get a global model of the sphere you must interprete the plane as the double image of two hemispheres. Then each point of the plane is the image of a point of the 'northern' hemisphere and at the same time the image of a point of the 'southern' hemisphere and in which of the both roles you want to use it has to be declared by labelling. (This is in analogy with the double role of each point of the plane if one uses the plane as the model of the whole space in classical plane and elevation technique.) An elementary but interesting problem then consists of constructing the shortest route between two given points in different hemispheres. (It is insuffficient to find only the connecting great circle, you have to find also the direction

Figure 3: Peter Apian: *Tabula orbis cogniti universalior*, Ingolstadt 1530
Area-preserving map after the principle of Johann Stoeberer (Stabius)
and Johannes Werner [I. Kupčik: *Alte Landkarten*, Artia, Prague 1980]

of shortest routing and this, though the map of course is not isometric [19]).

2.3. Models of hyperbolic geometry

Inside mathematics another important problem is to build models of hyperbolic geometry, and here we can once more explain the essence of generalized descriptive geometry in comparison with mere visualization. If one only wants to prove the consistency, i.e., the logical possibility of the hyperbolic geometry, there is no qualitative difference between the models of Felix Klein and Henri Poincaré. But in Klein's plane model hyperbolic circles are transformed in general into ellipses and hence they are not constructible by compass and ruler. Moreover, their intersection points are not constructible by compass and ruler. On the other hand it has been proved [16, 20] (in analogy to the mentioned result on the stereographic projection) that each hyperbolic construction by compass and ruler is transformed by the Poincaré model into a construction problem that is solvable by Euclidean compass and ruler. So one can really use this model for executing hyperbolic construction algorithms. M.C. Escher knew this.

His famous *'Kreislimits'* are models of the hyperbolic plane with archimidean tesselations (Fig. 4) and of course he used the Poincaré model really in the sense of generalized descriptive geometry. I think nobody has experience on how difficult it would have been in the Klein model.

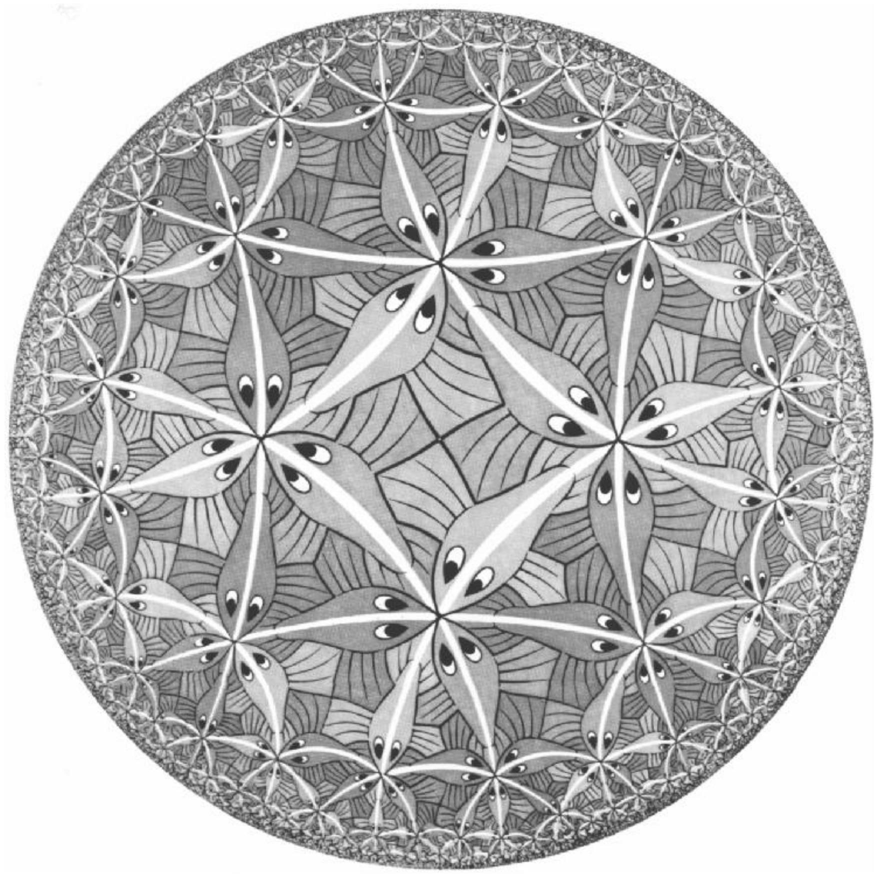


Figure 4: M.C. Escher (1898–1972): *Circle Limit III* (1959)

www.mcescher.com

Having a look on some other models of structures of non-Euclidean nature but embedded, eventually by the help of self-intersection, into the Euclidean space (Figs. 5–7) we see that a whole department of the arts (usually denoted 'minimal art' is very near to generalized descriptive geometry. At least they visualize some structures builded by exact mathematical rules and everybody may use them for executing some procedures, for instance searching what happens if we try to fill Klein's bottle with red wine.

If in any abstract structure the time is involved, always two points of view are possible:

(A) If time is accepted as a geometric dimension alike the spatial dimensions then visualizations involving time may be of

type a: if we assume Newtonian physics (because of ordinary circumstances), or of

type b: if we assume Einsteinian physics.

We illustrate the second case by the historical visualization (Fig. 8) of special relativity presented in Felix Klein's book [9]. Once more the question arises whether this is more than visualization. Indeed, one could decide graphically within the model whether a given event is reachable from another.

Figure 5: Moebius band

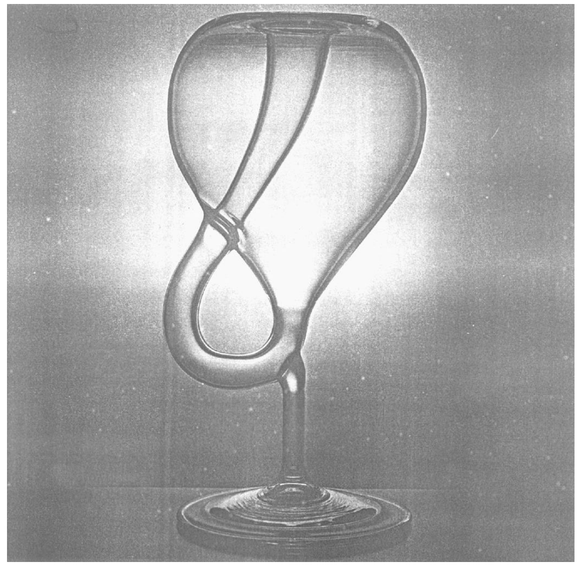


Figure 6: A glass model of
"KLEIN's bottle"

Figure 7: Model of the BOY surface[1]
made from steel bands,
Mathematical Institute Oberwolfach

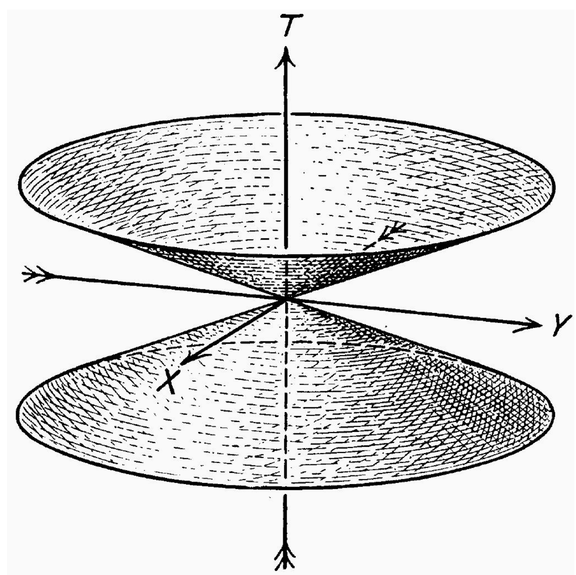


Figure 8: Drawing of the realitivistic
space-time [9]

(B) If time is seen as a (continuous or discrete) abstract quantity like probability, energy, or profit, then models involving time are of type (c).

Accepting time as a regular geometric dimension has an unexpected consequence. We then should accept it also as a legitime dimension for our models, i.e., moving pictures (film) then are possible three-dimensional visualizations, moving three-dimensional models are four-dimensional models. In visualization practice the possible exchange of time-like and space-

[1]The BOY surface is topologically equivalent to the real projective plane. For more information see [8].

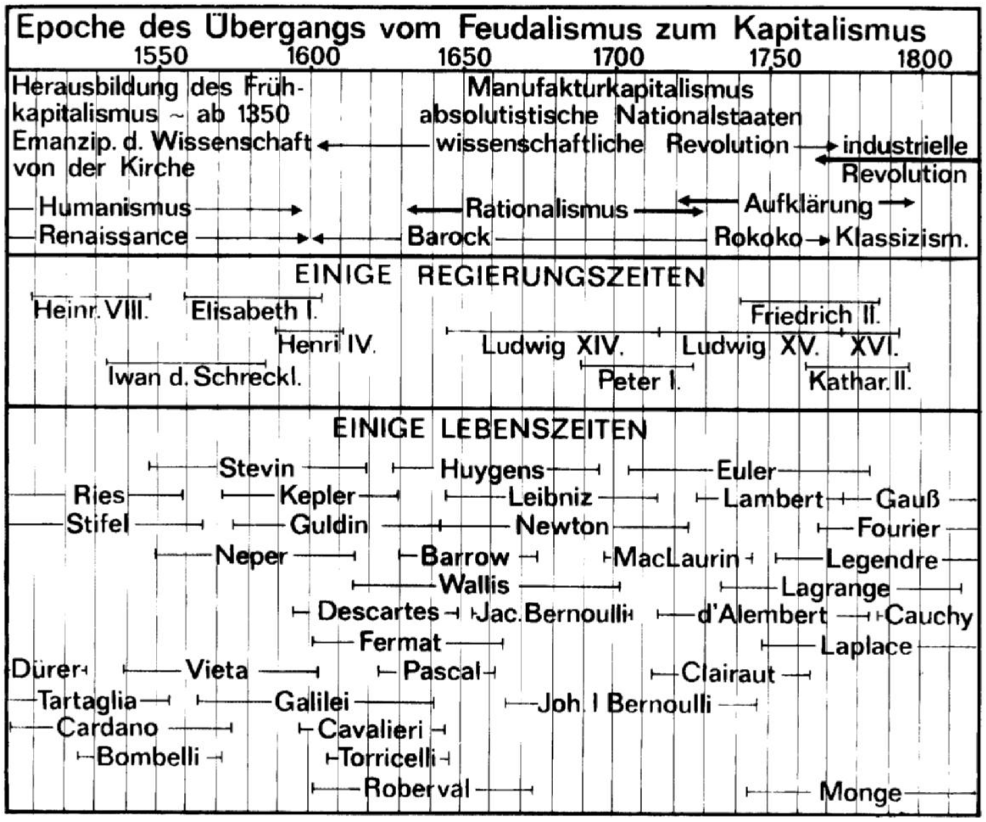


Figure 9: Synchronoptical diagram for a lecture in history of mathematics

like dimensions is very important. Almost all graphical representations of functions from science or economics use the x-axis as the t-axis in a quite self-evident way. Especially the 'synchronoptical' diagrams, frequently used by historians (Fig. 9), are of that kind: The y-axis then may be totally abstract, i.e., some of their points may represent the elements of any set. Conversely, as we saw in Section 2.1, four-dimensional Euclidean objects may be represented by objects within the cartesian product of space and time (then actually time and not the geometric visualization of a time axis).

Let us turn now to examples of type 'c'. This is a very rich field and indeed the examples of types 'a' and 'b' are merely simple particular and relatively new cases of it. As we shall see, some kinds of GDG are much older than the mathematical notion of 'model' or of the non-Euclidean geometries. Particularly some of the most abstract techniques as e.g. graph-theoretic descriptions, were invented in the Middle Ages. This period with its tendency to think about very abstract notions provided much symbolism in picturing (Fig. 10) and so, in some respects, it was nearer to modern science than the following periods (Renaissance, 17th, 18th centuries) when human interest turned towards material beings and their many details.

2.4. Relations in the field of numbers

Let us begin with the technique (usually ascribed to the Pythagoreans but apparently in use as early as in Babylonian culture) to find and to understand relations in the field of numbers by patterns of points or pebbles. To see this in the right way, recall that numbers themselves are quite abstract objects which may be represented only in principle by finite sets. 'Only in principle' means that really only relatively small numbers may be represented so, but

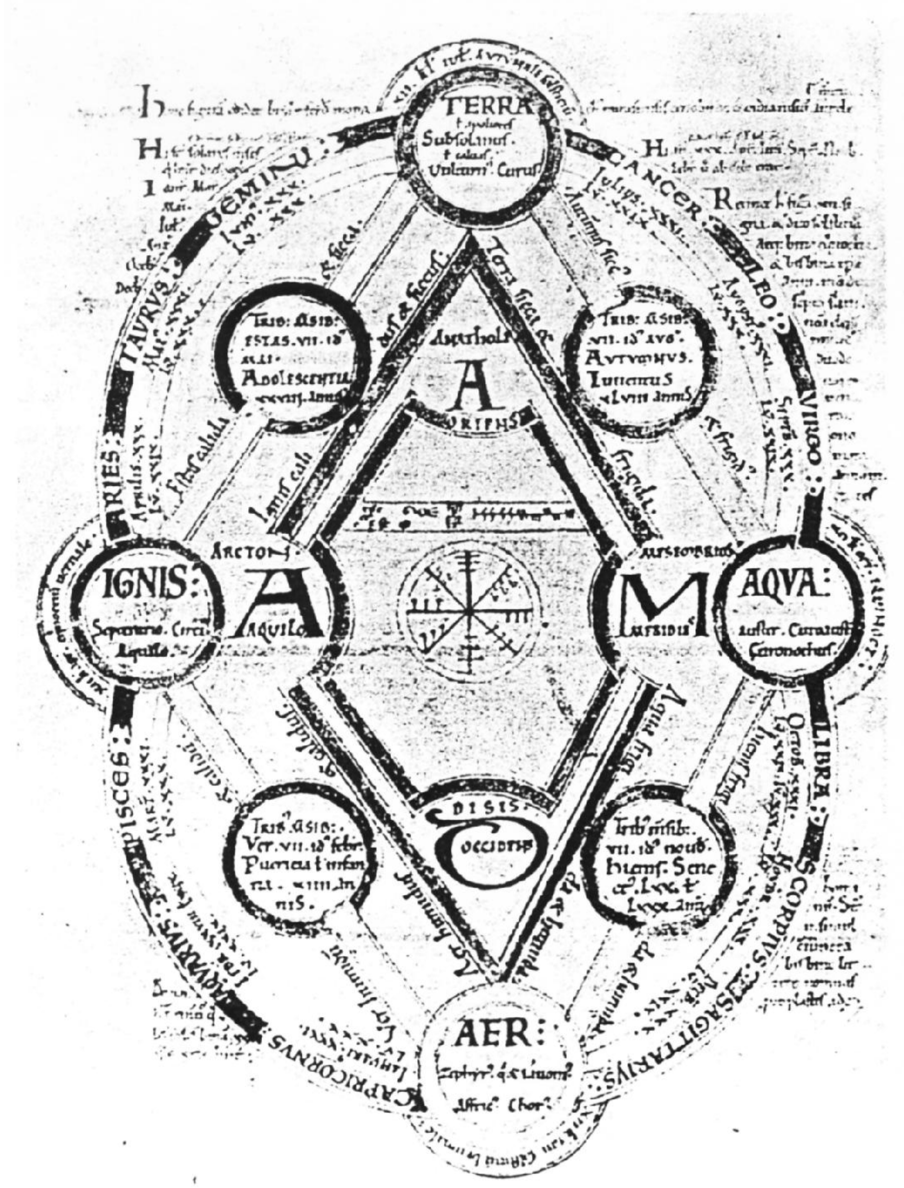


Figure 10: Complex diagram from the Middle Ages, showing connections between the four elements, the zodiac, the temperaments, the continents etc. [13]
By permission of the President and Scholars of Saint John Baptist College in the University of Oxford

particularly in the field of numbers and of the smallest kind of infinity, a ‘geometrical and so on’ plays a very important role. I should like to state that our mind would be unable to understand the countable infinite without this visible ‘and so on’. The Pythagoreans started mathematics by seeing the difference between even and odd and by seeing that

$$\text{even} + \text{even} = \text{odd} + \text{odd} = \text{even}, \ \textit{but} \ \text{even} + \text{odd} = \text{odd}$$

(Fig. 11,a). Then they proceeded to

$$n^2 + 2n + 1 = (n+1)^2 \ \text{(Fig. 11,b) and } \ 1 + 2 + \ldots + n = (n+1)n/2 \ \text{(Fig. 11,c).}$$

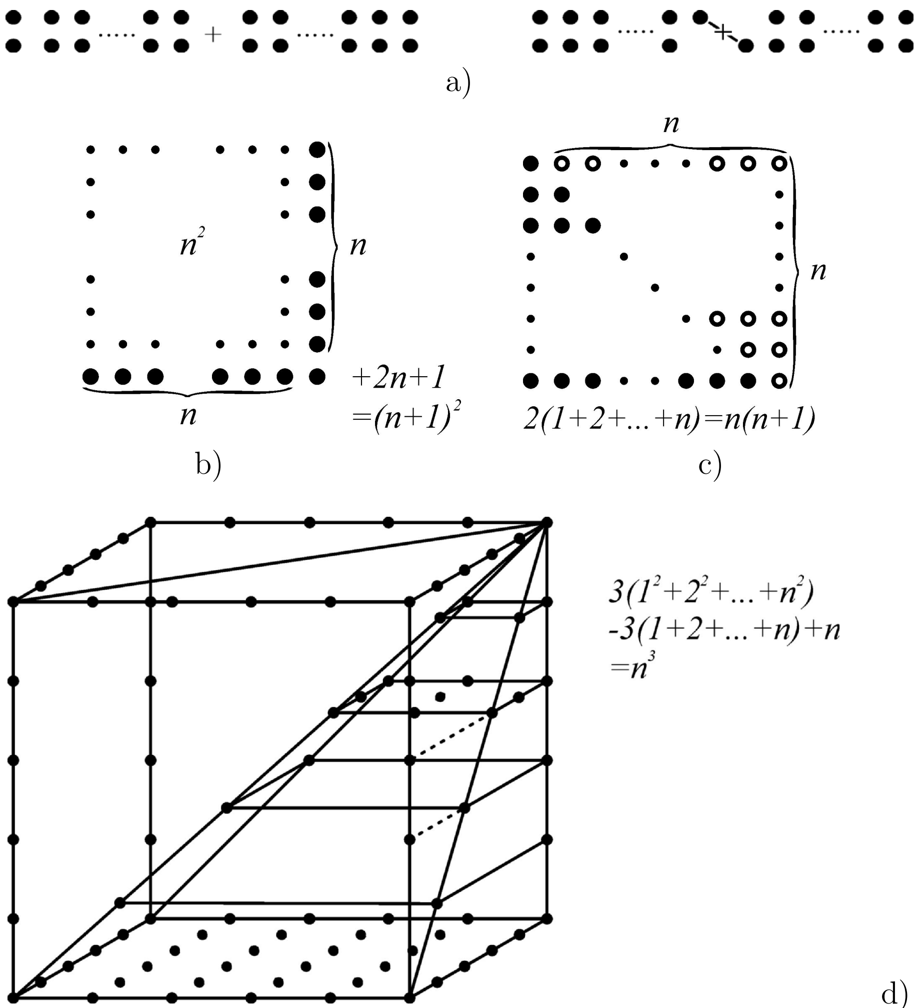


Figure 11: Number theory by pebble patterns

By the study how a cube of n^3 pebbles may be generated by three pyramids, each of $1+2^2+\ldots+n^2$ pebbles, and how many pebbles on the borders of these pyramids then are taken two times or three times (Fig. 11,d) one may get a formula for the sum of the first n squares. We do not want to repeat here all Pythagorean mathematics but to point to the fact that Cantor's famous first diagonal process for the proof that 'countable infinity times countable infinity is countable infinity' rests on this 'pebble method'. The visiblity of this idea enabled the artist G. Geyer when shaping a relief memorial honouring Cantor, to visualize at least a little part of Cantor's otherwise invisible work (Fig. 12).

The field of use of patterns of points has many features. When calculating determinants 'by hand' one wants to have a simple memory rule which of the products gets the sign '+' and which gets '−'. For $n=3$ the well known Sarrus rule solves the problem. There is an analogue for $n=4$ based on the fact that the occurring patterns of factors have a geometric appeal and their signs are invariant under congruence transformations. It is GDG because the point pattern stands for another object (a product of elements of a matrix); we decide the sign of the product by perceiving the geometric kind of the pattern. We close this chapter with two pictures (Fig. 13) illustrating the generation of the integers from the natural numbers and the rationals from the integers, each by creating the cartesian product of the set of the

Figure 12: Relief in Halle-Neustadt, honoring Georg Cantor (1845–1918).
Sculptor Gerhard Geyer (1907–1989) created it 1972.

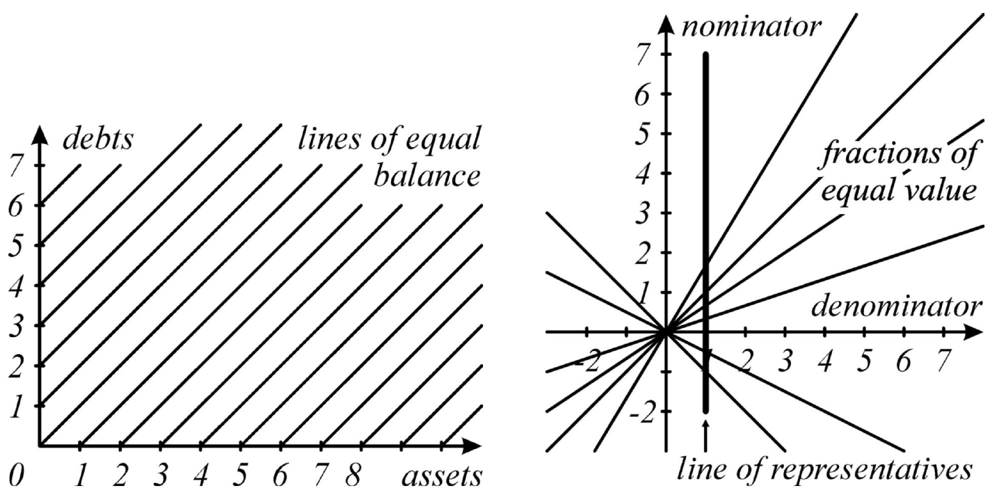


Figure 13: Geometric display of the generation of the
a) integers from numbers, b) rationals from integers

numbers with itself, defining an equivalence relation within this product and specifying a suitable representative from each class.

2.5. Continuous quantities

The next step in the historical development of GDG techniques is the representation of (in modern terminology) continuous quantities by line segments, areas, or volumina, as the Greeks did after the discovery of *incommensurability*. It is well known that the so founded algebra is impeded by the two conditions of 'homogenity' (only quantities of the same dimension can be added or subtracted) and the dimensions being less than four, but within these boundaries the 'geometric algebra' (so denoted by the Danish historian of mathematics H.G. Zeuthen) was

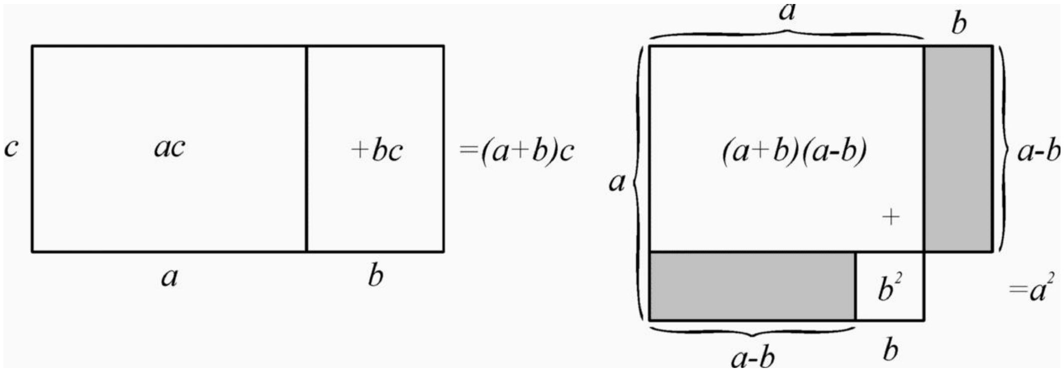
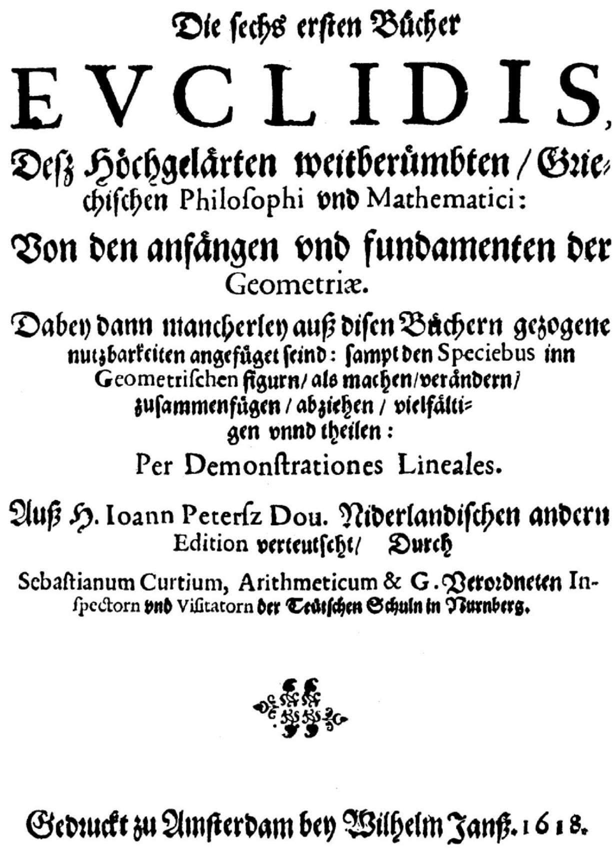


Figure 14: Examples of Greek diagrams explaining algebraic identities



Die sechs ersten Bücher
EVCLIDIS,
Deß Hochgelärten weitberümbten / Grie-
chischen Philosophi vnd Mathematici:
Von den anfängen vnd fundamenten der
Geometriæ.
Dabey dann mancherley auß disen Büchern gezogene
nutzbarkeiten angefüget seind: sampt den Speciebus inn
Geometrischen figurn / als machen / verändern /
zusammenfügen / abziehen / vielfälti-
gen vnnd theilen:
Per Demonstrationes Lineales.
Auß H. Ioann Petersz Dou. Niderlandischen andern
Edition verteutscht / Durch
Sebastianum Curtium, Arithmeticum & G. Verordneten In-
spectorn vnd Visitatorn der Teutschen Schuln in Nürnberg.

Gedruckt zu Amsterdam bey Wilhelm Janß. 1618.

Figure 15: Title-page of an edition of Euclid's *"Elements"* (1618), that recommends the execution of algebraic operations by geometric representatives
[Steck, Folkerts: Bibliographia Euclideana, Gerstenberg-Verlag, Hildesheim 1981]

able to lay exact foundations to algebraic identities (Fig. 14). Moreover, following the intention of the present paper, in the 16th and 17th centuries, many popular editions of Euclid's *'Elements'* in vernacular languages recommended the execution of algebraic operations by construction with geometric representatives of the involved quantities (Fig. 15). Descartes then, by use of an arbitrary unit length e, could overcome both limitations: if the rectangles ab and ce have equal areas then c is a line representative of the two-dimensional quantity ab. So one may transform the product of arbitrary many line factors into a line quantity and at the same time force homogenity of all quantities. The 'Cartesian' (or another) co-

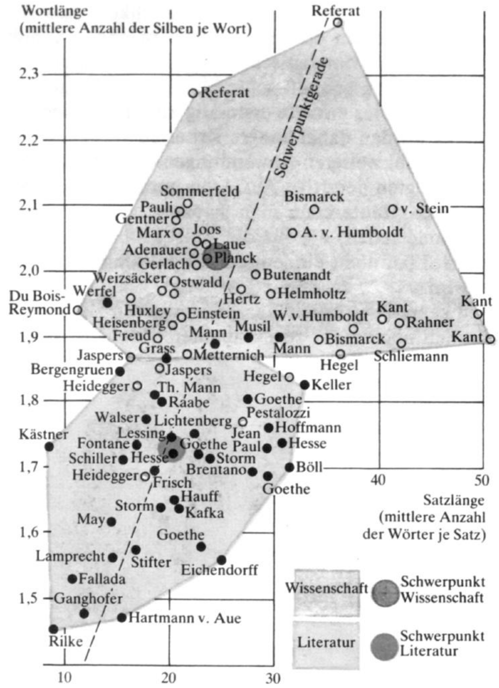


Figure 16: Visualizing the styles of authors by mean length of words and sentences [24]

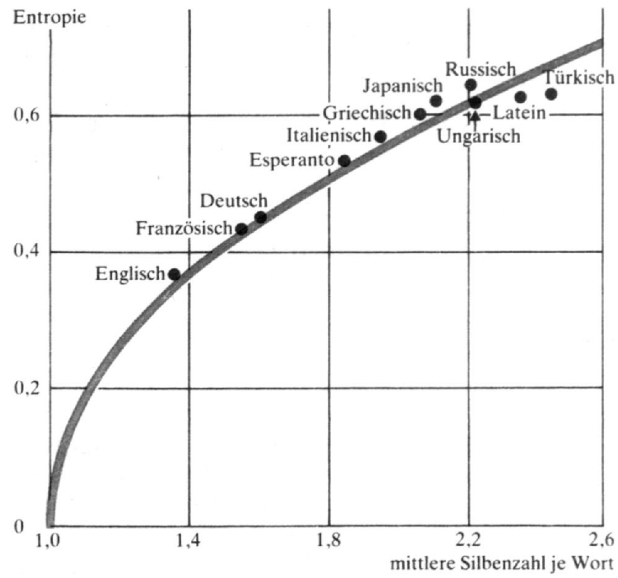


Figure 17: Visualizing characteristics of languages by the mean number of syllables per word and by the entropy [24]

ordinate method from beginning had two very different features: It allows the treatment of geometric problems by algebra, but it makes visible also algebraic 'objects' and their facts. In some periods of the historical development such visual insights were the first step towards new algebraic knowledge. The overall use of the coordinate methods has made us oblivious to the fact that the majority of the visualized relations or functions concern quantities from physics, biology, economy … Only the not so commonplace use of Cartesian visualization in the humanities (Figs. 16, 17) may remember this once more.

The method of representing quantities by line segments or areas has particular importance also for the presentation of statistical data. From the image of a function produced by the Cartesian coordinate method (or sometimes another, e.g. polar coordinates) we can get information about this function much faster and much more effectively than by a table of the values, though such a table would be nearer to the set-theoretic notion of functions (sets of ordered pairs). But there is also the possibility for mistakes and frauds. Look at Fig. 18, left [10]: The numbers of unemployed people resp. of emigrants are represented only by the height of the figures, but because the figures should have similar shape the areas are growing with the square of the height — and our eye and our mind perceives the area first! Look at Fig. 18, right [10]: Here, because the strips all have the same width, the areas are proportional to the heights (this is Descartes in action!), but the designer perhaps wanted to describe the continuity of the process — however, with the connecting lines between the columns they are perceived as a central projection of a folded wall, and so, whether we want or not, something from the impression of growing is lost by the eye's experience that nearer objects appear larger.

It was surprising for myself that the begin of the theory of geometric probabilities (as founded by Buffon's needle problem) in principle is founded on GDG: If we want to measure

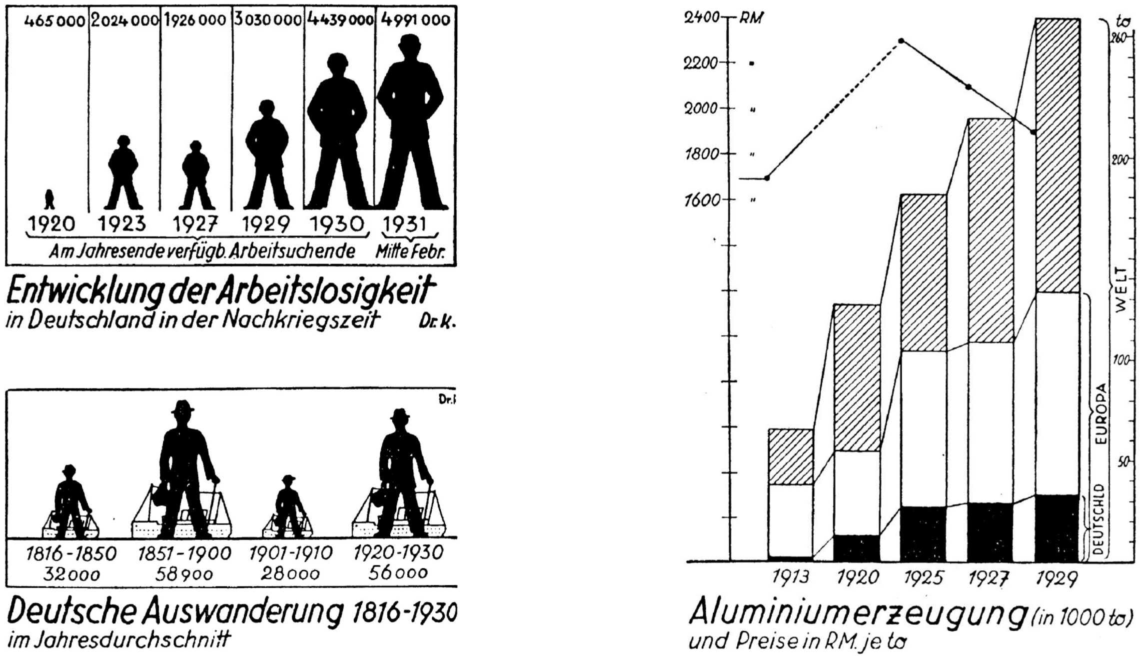


Figure 18: Statistical diagrams from *Knaurs Konversationslexikon* [10]

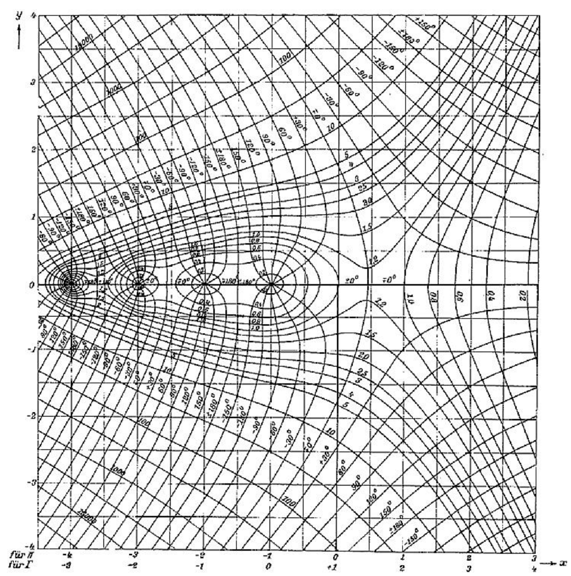


Figure 19: Display of the factorial function in the complex domain by isolines [7]

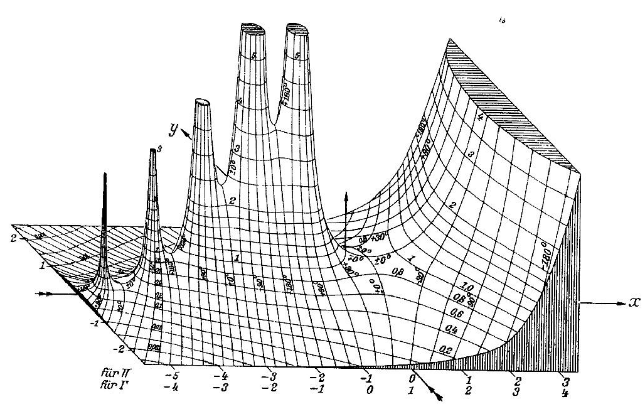


Figure 20: Display of the factorial function by a "landscape" in parallel projection [7]

a set M of geometric objects other than points we have to describe these objects by an appropriate system of k real coordinates and then to measure in $\mathbb{R}^k$ the set of the coordinate tuples of the elements of M. Particularly if k is < 4, as e.g. in the famous Bertrand Paradoxon, than one can draw pictures from the involved point sets representing sets of straight lines, circles, ... ([21], pp. 446–448, 459–461).

Let us have a look at the visualization of functions of two variables. There are two main possibilities: the map with level lines or isohypses that clearly came into mathematics from cartography (Fig. 19), and the parallel perspective of a relief ('landscape', Fig. 20). Historical search confirms what logic says: Because it is much ea sier and much more important to

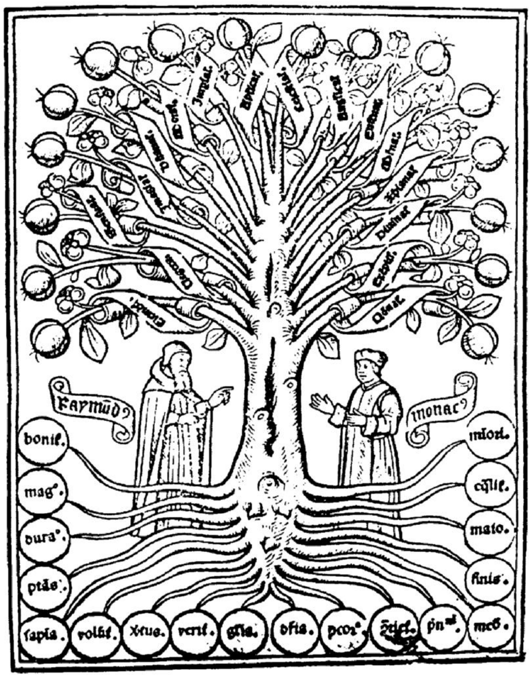
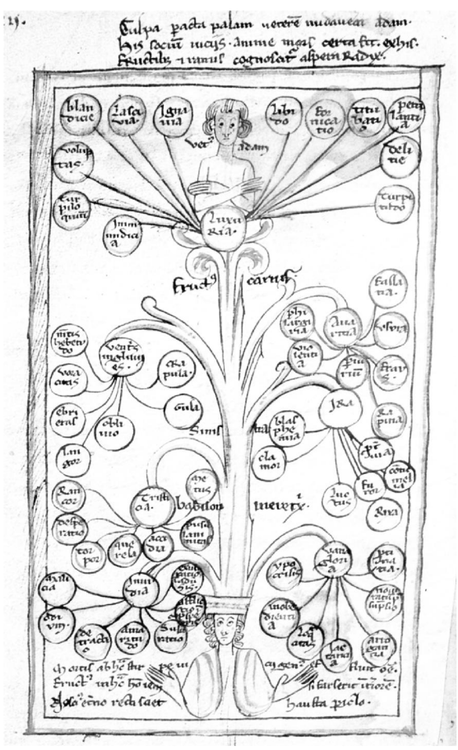


Figure 21: Raimundus Lullus (1234–1315): *The tree of the sciences* from "Arbor scientiae" (1295)



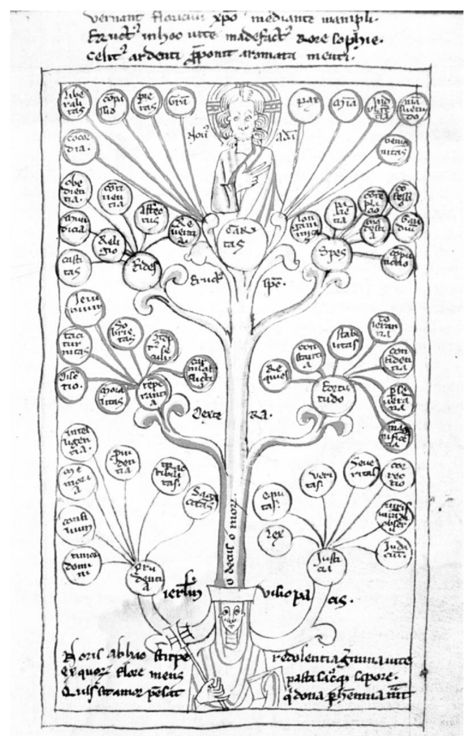


Figure 22: The tree of the vices

Figure 23: The tree of virtues

Both from a manuscript of the 14th century
[Courtesy: Technische Universität Darmstadt, Handschriftenabteilung]

measure the depth of a water than the hight in a landscape, there were first maps with isobathes (from about 1700 onwards) and the first map with isohypses (1771) was from an

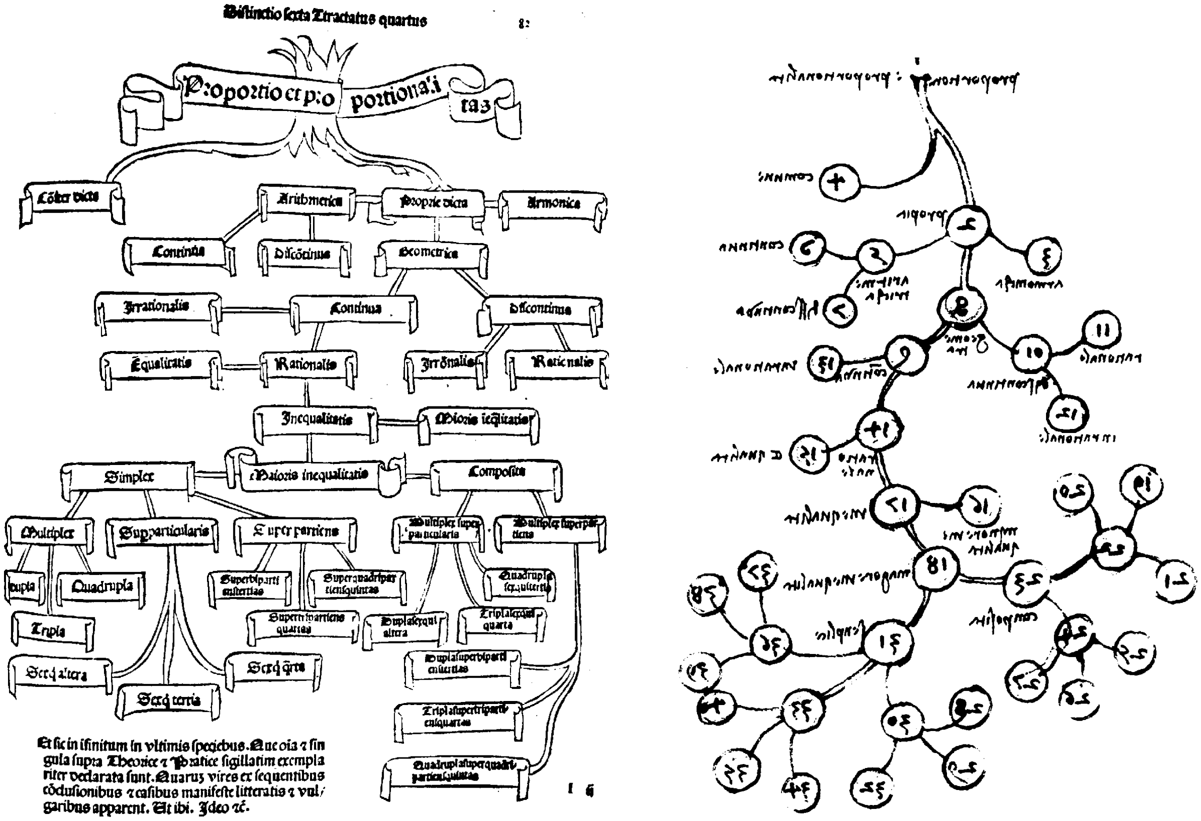


Figure 24: a) Luca Pacioli (1445?–1517): *The tree of proportions* from "Summa de arithmetica, geometria, proportioni et proportionalita (1494), [12]; b) Incomplete redrawing by Leonardo da Vinci, [12]; [These drawings are reproduced by permission of EMB-Service for Publishers, Lucerne, Switzerland.]

'imaginary isle' — only illustrating the principle [21], p. 334. But once more we have to return to our matter. Depicting landscapes or roofs is traditional descriptive geometry, the use of the methods as in Figs. 19, 20 is visualization of abstracts, if one uses it to solve a problem about these abstracts then it is generalized descriptive geometry.

In the practice of geosciences the borderlines between the pure geometric image of the (approximately sphere-shaped) earth, the description of its relief, the description of geophysical functions by isolines, and thematic cartography in the stronger sense are fuzzy. All of them of course use labelling, but also try to describe movements and tendencies by appropriate graphical means. An interesting but open problem is the adequate visualization of ternary relations in the case at least one of the involved sets is topographically distributed.

2.6. Graphs

The true paradise of GDG are the (directed or undirected) graphs because there is mostly no doubt that the depicted objects and their relations nothing have to do with geometry, whereas the difference between a continuous quantity as a geometric object (e.g. the length of a line segment) or as an abstract object (e.g., an amount of energy or financial resources) is sometimes fuzzy. As mentioned above, the display of abstract relations between non-geometric objects came up in the Middle Ages. Particularly there are lots of 'trees', the tree of the arts and sciences (Fig. 21), the trees of the vices and of the virtues (Figs. 22, 23 in [22]), the tree of the spreading of the Societas Jesu in space and time. A tree of the kinds of proportions from Pacioli's *'Summa de arithmetica, geometria, proportioni et proportionalita'* (1494) and its rescetching by Leonardo da Vinci (Figs. 24,a),b) [12]) tells us about Leonardo's intuitive clearness that in such diagrams not the shape of the nodes nor the shape or length of the edges but only their connectivity is important. From a contemporal mathematical point of view all such trees are mere pictures of data structures and the only processes one may do with them is ordering, shaping and altering their structure. From other points of view of course there is more to discuss, e.g., the fact that the virtues are upright on their tree but the vices are inclined, or that in general there are 8 kinds of virtues or vices on each main branch and a one-to-one correspondence between virtues and vices. I mention this because the neglect of the mathematically uninteresting aspects of pictures (or other realities) is dangerous in team work with non-mathematicians. It is one of the many causes why mathematicians often are called halfwitted.

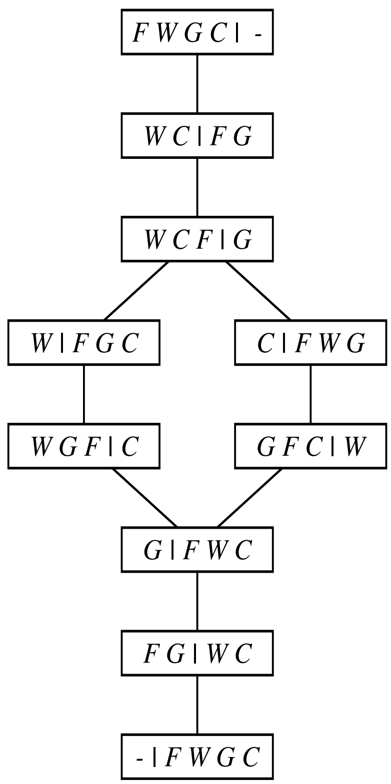


Figure 25: Graph of the problem "Wolf–goat–cabbage"

For the more useful work with graphs I give two simple examples: Problems of the kind 'wolf-goat-cabbage' are to solve most effectively if one draws the graph of all possible situations and all possible changes beween two situations and looks whether there is a (shortest) path from the given begin to the wanted finish. The special wolf-goat-cabbage-graph (Fig. 25) shows even more: its left-right symmetry corresponds to the duality of devouring and being devoured and so to the interchangeability of cabbage and wolf. The top-down-symmetry corresponds to the duality between the left and the right shore of the river. Many other combinatorial problems, if graphically represented, unveil such surprising symmetries, otherwise perhaps unexplored.

Our second example is the display of strategic two-player-games alike 'wolf and sheep'.

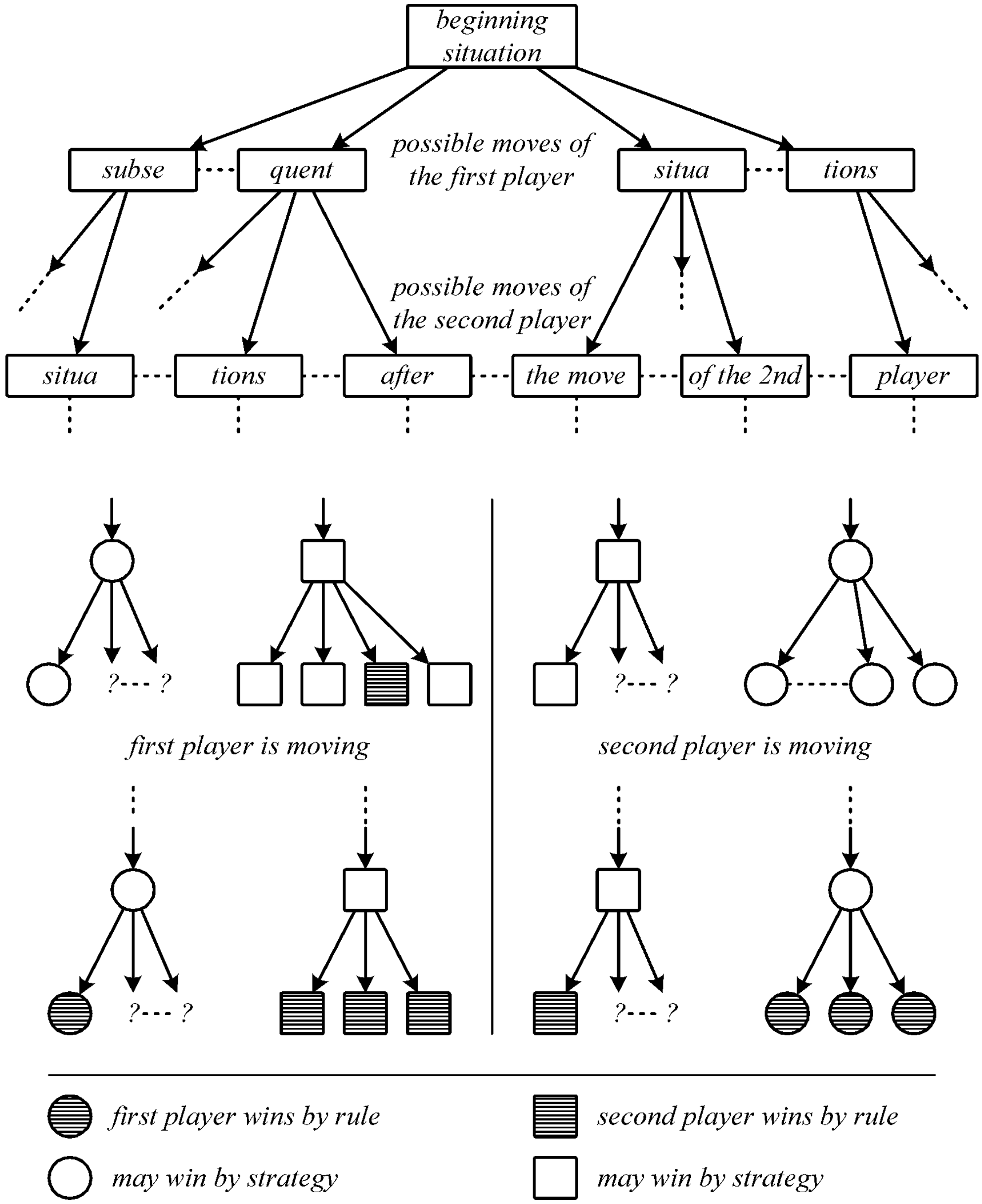


Figure 26: Principle of the recursive computing of a winning strategy

The graph, in this case with directed edges, consists of all situations as nodes. The edges correspond to all 'moves' of the first or the second player transfering the game from a situation into the next one. If the so generated graph is a rooted tree (to exclude cyclic paths sometimes there must be a rule banning infinite repeating of moves) and for each of its leaves a rule fixes who wins the game in this situation, then in principle for each other situation the question of who of the two players has a winning strategy may be decided recursively (Fig. 26). This afterwards 'simple' proposition was presented by E. Zermelo at the Fifth International Congress of Mathematicians (Cambridge 1912) when graph theory was in statu nascendi (and hence the title of Zermelo's talk tells about 'an application of *set* theory onto the theory of chess' [11], p. 114). Of course, for all nontrivial games the drawing of the complete graph is practically impossible and sometimes (especially for chess) the calculation of a winning strategy is impossible even for the most powerful computers, but my opinion is, that finding and understanding the idea of the fact is almost impossible without the optical impression of such a graph. For small games one may really graphically mark 'from bottom to top' the winning player in each node.

2.7. Pictures supporting reasoning

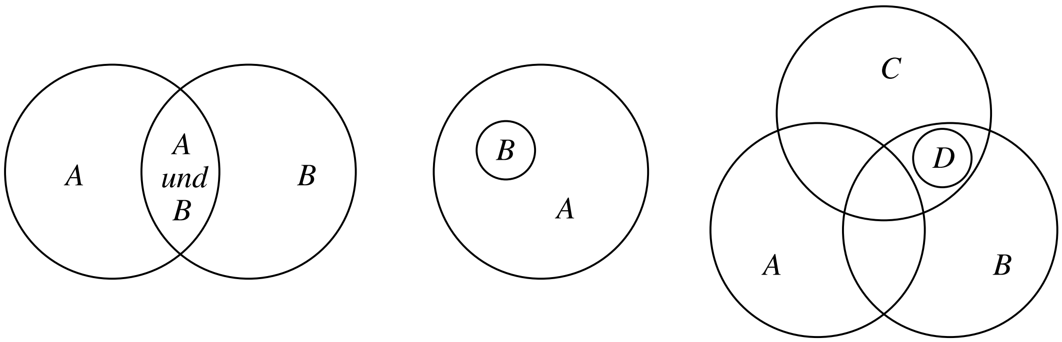


Figure 27: EULERian circles

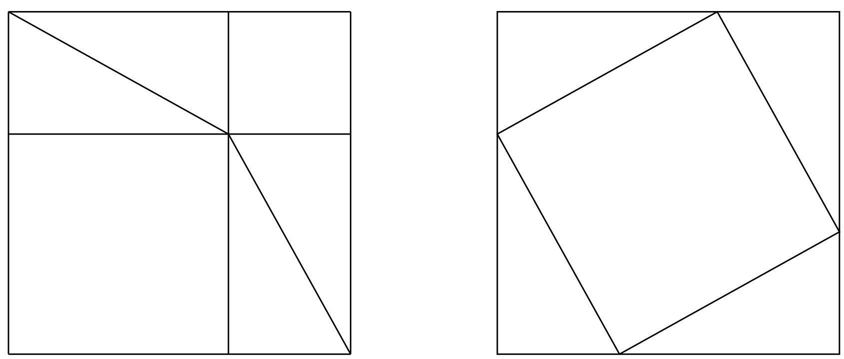


Figure 28: The "Chinese proof" of the PYTHAGORAS theorem

The role that pictures may play in finding and presenting (mathematical) truths led us to the next chapter of this paper. Quite recently some new discussion about this subject is growing up [1, 14] and not at all only in mathematics. We have to clarify whether and in which manner finding and proving 'without words' has anything to do with the concept of GDG. To simplify the matter let us assume that logical reasoning is an algorithmic process like calculation. The first example shall be the 'Eulerian circles' from [4], Fig. 27. They stand as representatives for entities of arbitrary things having a common quality. Our brain perceives such a picture and produces a set of non-geometric conclusions, e.g., 'If all D are B and C, but not A, then obviously all objects which are A and C, are not D'. J. VENN in 1880 extended this method to arrangements of sets such that each intersection of some of the sets and the complements of the others is not empty, and tried to mechanize Boolean reasoning by a kind of tiling [23].

Look at Fig. 28, the well known 'Chinese' proof of the PYTHAGORAS theorem. Our brain internally transforms the left arrangement of the six tiles into the right arrangement, deletes then from each of the two arrangements the four triangles and acknowledges by logic (EUCLID, Axiom 3: If equals are subtracted from equals then the remainders are equal) that the remaining two quadrangles at the left are equal to the remaining quadrangle at the right.

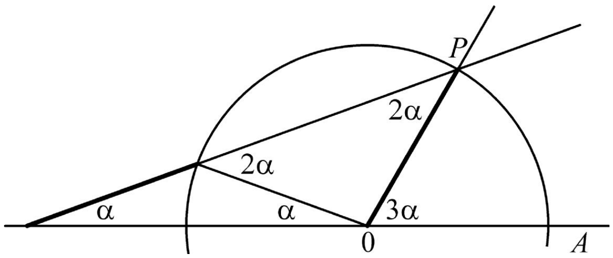


Figure 29: Description and proof for the trisection of angles by ARCHIMEDES

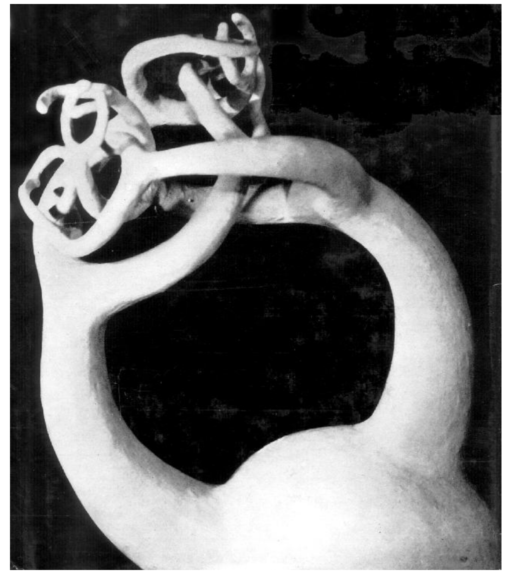


Figure 30: ALEXANDER's horned sphere[2]

(Indeed, this reasoning has to be completed by the proof that the remaining right figure has equal angles. This is not self-evident because it employs the fact about the sum of the angles of a triangle). We could do this proof by a sequence of diagrams and then it would really become a process working with geometric representatives of the treated matter. In my opinion it is not so important for the quality of the process as a kind of GDG that in such simple cases only one of the pictures is on the paper and the others are only in our mind.

It is a rare kind of elegance and effectivity in mathematics if one can communicate a solution of a construction problem and the complete proof why it works without any words in a single figure. Just to prove this I do not further explain the famous trisection of a given (acute) angle AOP by ARCHIMEDES (Fig. 29). It also works by a sequence of figures from which only the last one is on paper but the steps towards it are in our mind.

We complete the sightseeing tour by a model of a famous topological monster: ALEXANDER's horned sphere (Fig. 30). Like Figs. 4, 11, 12, 13, it appeals to our imagination of the infinite from a finite begin. I repeat my thesis that we were unable to understand and to *handle* the infinite without many such 'geometric and so on'. 'Handle' here means such results as 'countable + countable is countable, countable times countable is countable'. We may see also by pictures of some beginning steps how and why V. KOCH's continuous but not differentiable curve has these qualities, or that there is a one-to-one correspondence between intervals of different length. In all such cases we are convinced before and independently of any written formal proof. May be the principal difficulty in catching unambigously more about the higher infinite than just that the power set of M is always greater than M and some trivialities, is connected with the impossibility of visualizing such higher infinites. The human brain causes people to think in pictures, also about seemingly very abstract matters. So perhaps we can't understand what is invisible in that wider sense because we can't make a picture from it.

[2]The model was created by a student of the department of mathematics at the University of Greifswald and lateron used for the cover of W. RINOW's *"Lehrbuch der Topologie"*, Deutscher Verlag der Wissenschaften, Berlin 1975

At the very end, I want to point to the facts that the above listing of instances of GDG of course is incomplete and open and that there are many overlappings and connections between the listed cases, e.g., between proofs without words and calculating by pebbles or by geometric algebra. Really working with concrete drawings and pictures is so joyful. To make a theory about it would be a dry matter if there were not so many open questions and doubts.

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