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A New Generalization of the Butterfly Theorem

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Abstract. The butterfly theorem and some of its generalizations deal with a specific point related to a quadrangle inscribed into a circle. By use of the Sturm-Desargues involution theorem it is proved that with any such quadrangle an infinite number of butterfly points is associated which are located on an equilateral hyperbola. Finally an infinite number of quadrangles sharing the same butterfly curve is presented .

Key Words: Butterfly theorem, Desargues' involution theorem, pencil of conics

MSC 2000: 51M04

1. Introduction

We start recalling two generalizations of the butterfly theorem [2]:

Theorem 1 *Let the quadrangle $MNPQ$ be inscribed into the circle k. Let the line o connect the center O of k with one of the three diagonal points of the quadrangle, say G. Then the line s through G perpendicular to $o = GO$ intersects the pairs of opposite sides of the quadrangle at points A, A' and B, B', respectively, symmetric with respect to G (see Fig. 1, cf. [4, 5]).*[1]

Theorem 2 *Let the quadrangle $MNPQ$ be inscribed into the circle k with center O. On any given line s let S denote the pedal point of O. If S is the midpoint of the two points $A := s \cap NQ$ and $A' := s \cap MP$, then is S also the midpoint of the pairs ($B := s \cap NP$, $B' := s \cap MQ$) and ($C := s \cap MN$, $C' := s \cap PQ$) (see Fig. 2, cf. [6]).*

In order to prove the mentioned theorems as well as the other generalizations the cited authors used different methods, but not a synthetic one. However, all butterfly theorems can be proved synthetically in a uniform way:

[1]When G is specified as the diagonal point which lies in interior of k like in Fig. 1, then four sides of the quadrangle form a quadrilateral of *butterfly shape* with G as the point of self-intersection. This is the reason for the name of this theorem.

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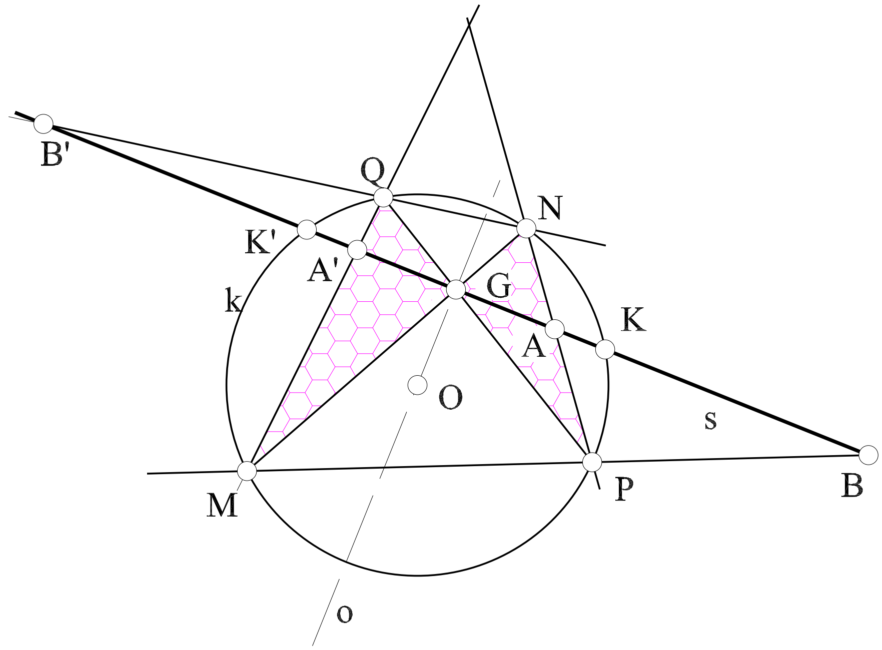

Figure 1: The Butterfly Theorem

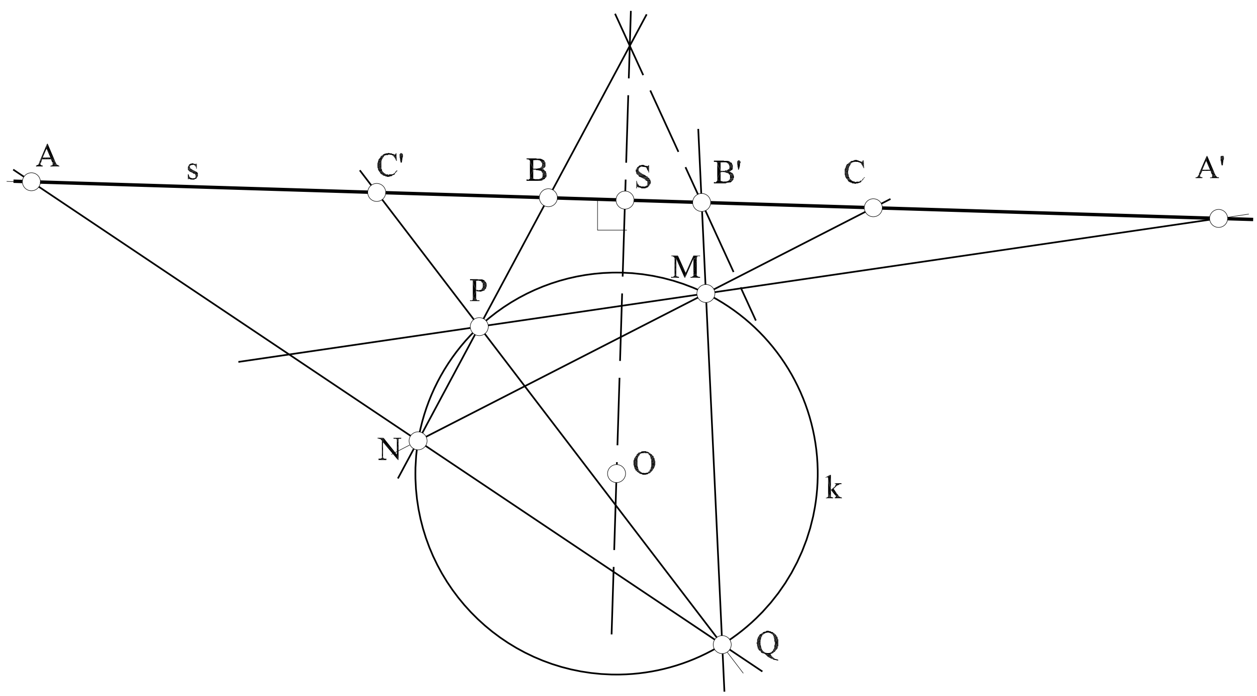

Figure 2: Statement of Theorem 2

The given points M, N, P, and Q should be regarded as the basic points of a pencil of conics which contains the circle k. In such case the pairs of opposite sides of the given quadrangle are three degenerated conics of the pencil. Due to the STURM-DESARGUES theorem the conics of this pencil intersect any line (not passing through M, N, P, Q) in pairs of points of an involution [1]. The *butterfly point* is one of the two point fixed points; the other is a point at infinity. There are two conics of the pencil tangent to s. One touches at the butterfly point, the other is always a hyperbola.

Note that in Theorem 1 the butterfly point is one of the diagonal points of the quadrangle inscribed into the circle k. So, only three choices are possible. On the other hand, Theorem 2 says nothing about the existence or uniqueness of the so-called butterfly point S and about its constructive determination. It will be shown, that the construction of the butterfly point gives rise to a new generalization of the butterfly theorem.

2. Generalizations of the Butterfly Theorem

Theorem 3 [**Butterfly Curve Theorem**]
For a given quadrangle $MNPQ$ inscribed into the circle k there is an infinite number of points S_i (so-called butterfly points) which obey the conditions of Theorem 2. All these points S_i are located on the equilateral hyperbola which passes through the three diagonal points E, F, G of the quadrangle and through the center O of k (see Fig. 3).

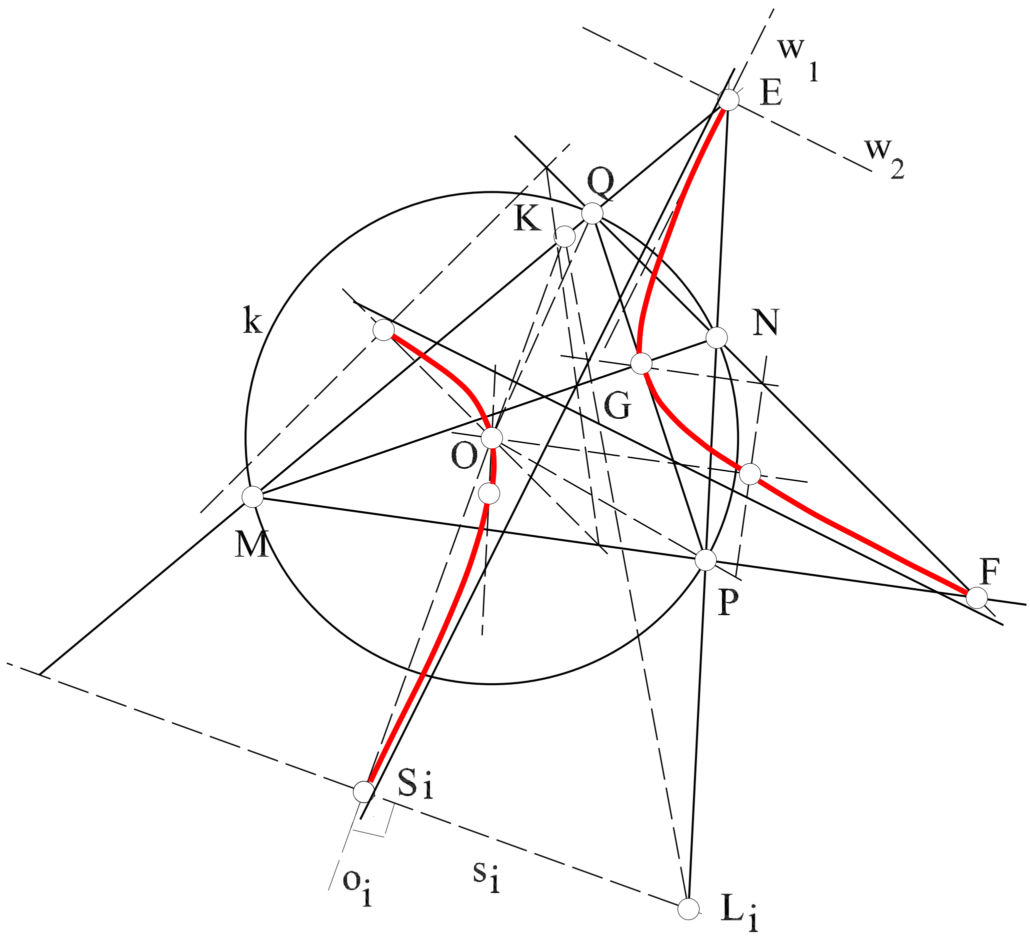

Figure 3: The Butterfly Curve Theorem

Proof. Let o_i be any line through the center O of k. Now we reflect any side of the quadrangle, e.g. MQ, in o_i. Through the point L_i of intersection between the mirror image and the opposite side NP we draw the line s_i perpendicular to o_i. Then the point $S_i = s_i \cap o_i$ of intersection has obviously the property of a butterfly point on the straight line s_i (Fig. 3) as k and the pair $MQ \cup NP$ intersect s_i at points symmetric with respect to S_i. As there is a butterfly point on each line o_i of the pencil (O), there is an infinite number of butterfly points which trace a curve.

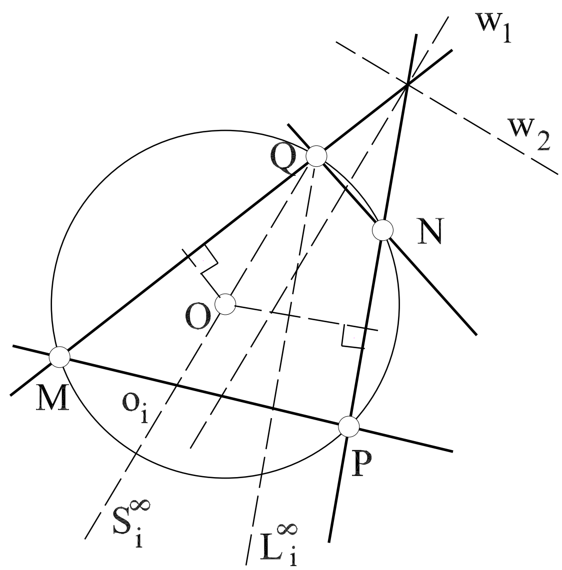

Figure 4: Particular elements of the butterfly curve

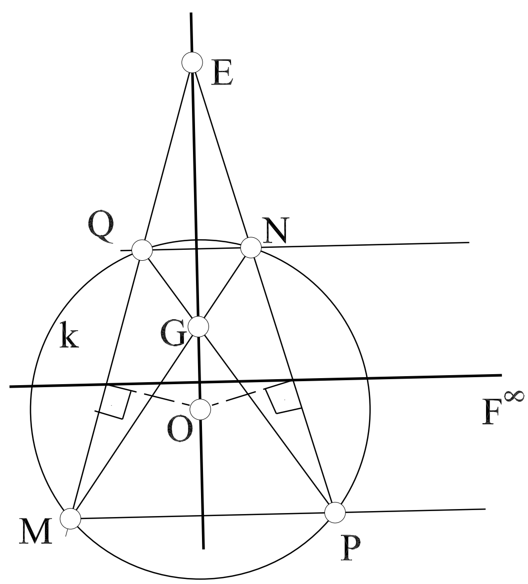

Figure 5: Case with a degenerate butterfly curve

In the pencil (O) there exists one line on which the point O plays the role of the butterfly point. Therefore, it could be concluded that the so-called *butterfly curve* is a conic.[2] It contains the diagonal points E, F, G, and the center O. The construction of S_i implies that the butterfly curve intersects each side of the quadrangle at the pedal point with respect to O.

[2]An alternative proof is added after Corollary 2.

On the other hand, EFG is a polar triangle for all conics of the pencil, in particular for k. Hence point O is the orthocenter of the triangle EFG. So it follows that the butterfly curve is an *equilateral hyperbola*. The construction of the butterfly points reveals that the infinite points of the butterfly hyperbola are lying on the bisectors w_1, w_2 of the angle formed by the opposite sides MQ and NP of the given quadrangle (Fig. 4). The perpendicularity between these lines of symmetry proves again that the hyperbola must be equilateral. In case the straight line s_i is at infinity, the infinite points of the butterfly hyperbola are the touching points of the two parabolas included in the pencil of conics.

Remark: When the center of the circle k is located on a side of the diagonal triangle EFG of the given quadrangle $MNPQ$, then the butterfly hyperbola degenerates into two perpendicular lines (Fig. 5).

We should notice that the construction of the butterfly points could be carried out by any pair of opposite sides of the given quadrangle $MNPQ$. All these implies:

Corollary 1 *The lines of symmetry of the angles made by opposite sides of the quadrangle inscribed into the circle constitute two triples of parallel lines.*

Corollary 2 *If a pencil of conics contains a circle, then the infinite points of the two included parabolas are lying in two perpendicular directions.*

We add an alternative proof for the fact that the butterfly curve is a conic: On the line s_i (see Fig. 3) the pedal point S_i of O and the point S_i' at infinity are harmonic with respect to the points of intersection with any pair of opposite sides, e.g. $MN \cup PQ$. After connecting S_i and S_i' with $G = MN \cap PQ$ we obtain a harmonic quadrupel of lines. So, for any given o_i the line GS_i is the fourth harmonic conjugate to GS_i' — perpendicular to o_i — with respect to MN and PQ. This reveals that the butterfly curve can be generated by a projectivity between the pencils (O) and (G) of lines.

S_i and S_i' are conjugate with respect to any conic passing through $MNPQ$. The polar lines of S_i with respect to these conics form the pencil S_i'. There must be any conic in this pencil for which the polar line of S_i is the line at infinity; this conic is centered at the butterfly point S_i.

Conversely, let S_i be the center of a conic l passing through $MNPQ$ and $l \neq k$. Let s_i denote the line through S_i perpendicular to $o_i = S_iO$. Then on s_i the point S_i and the point S_i' at infinity are conjugate with respect to k and l and therefore with respect to all conics of the pencil. Hence, S_i is a butterfly point. We summarize in

Theorem 4 *The butterfly curve for the quadrangle $MNPQ$ inscribed into the circle k is the locus of centers of all conics passing through $MNPQ$.*

Remark 2: This center curve of a pencil of conics is also called *nine-point-conic* of the quadrangle $MNPQ$ as it passes through the diagonal points EFG and the midpoints of the six sides (see e.g. [3], p. 93).

3. The Butterfly Swarm Theorem

So far it has been proved that it is possible to associate an unique butterfly curve to a quadrangle inscribed into the circle. We could ask the opposite question: Is it possible to find other quadrangles associated with the same butterfly hyperbola? The answer is formulated in the following theorem:

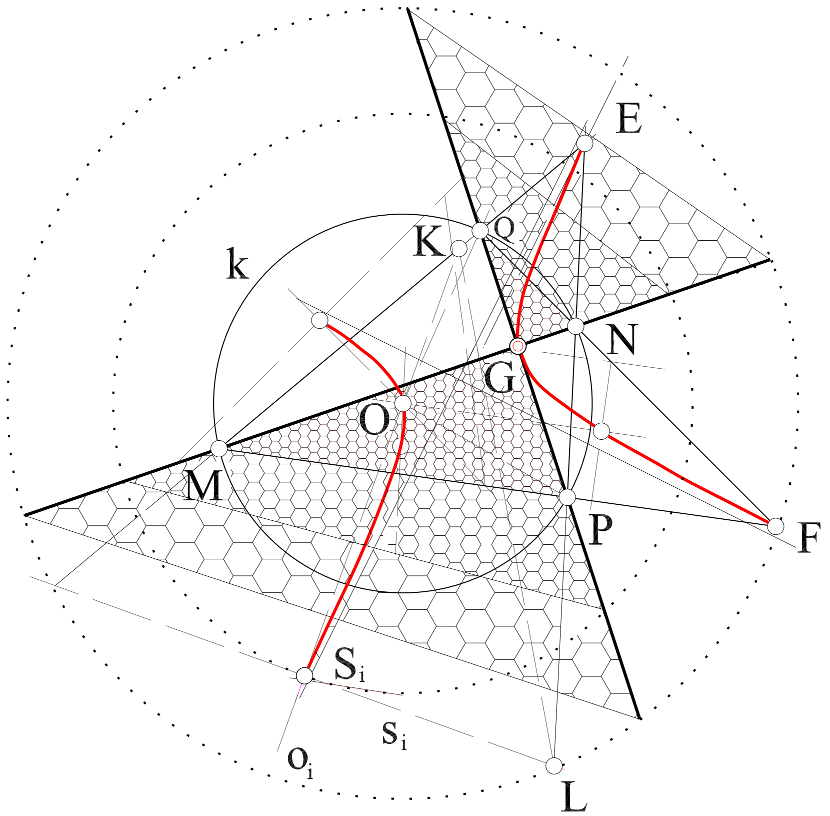

Figure 6: The Butterfly Swarm Theorem

Theorem 5 [Butterfly Swarm Theorem]
There is an infinite number of quadrilateral ("butterflies") sharing a common hyperbola as their butterfly curve. All these quadrilaterals can be classified into six families (swarms).

Proof. If two opposite sides of the given quadrangle, e.g. MN and PQ, remain fixed and the radius of the given circle varies, then a pencil of concentric circles and an associated pencil of inscribed quadrangles is created. There belongs a butterfly curve to each quadrangle. All of these butterfly curves are passing through the common center O of the circles, through the common diagonal point G and through the common pedal points of O on the common sides MN and PQ. Beside these four common points all of these butterfly curves share the infinite points, too. So it must be the same hyperbola (Fig. 6).

Each fixed pair of opposite sides of the basic quadrangle determines two butterfly (i.e. quadrilateral) families. It follows that there exist six butterfly families *(swarms)*. The figures 6, 7 and 8 show three such families. Each of them is associated to one pair of opposite sides of the basic quadrangle.

Summary

The well-known butterfly theorem and some of its generalizations deal with a specific point related to a specific quadrangle inscribed into a circle. For all these theorems a uniform synthetical proof is given. Besides it is shown that with any quadrangle inscribed into a circle an infinite number of butterfly points is associated. All of these points are located on an equilateral hyperbola called the butterfly curve. At the end it is proved, that an

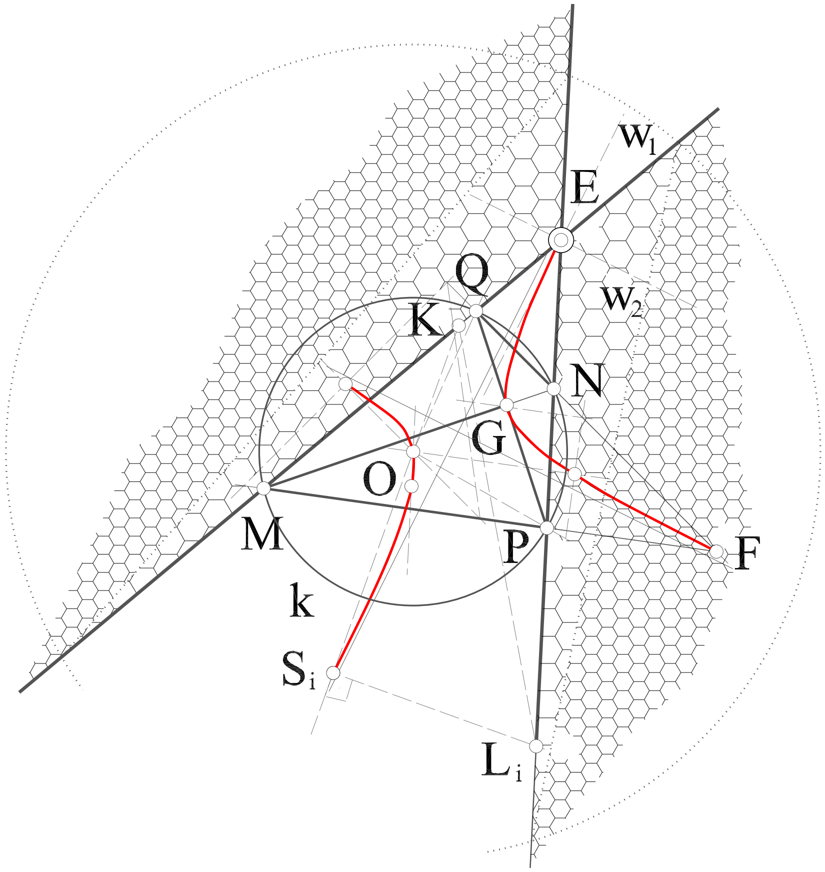

Figure 7: Second family of a butterfly swarm due to Theorem 5

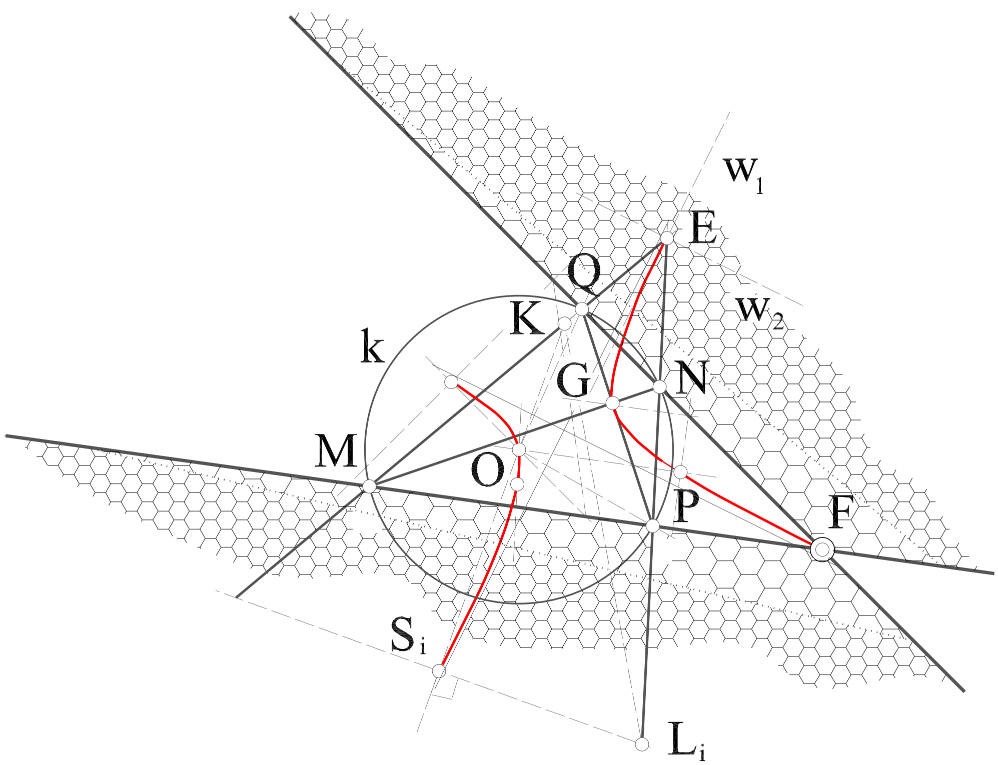

Figure 8: Third family of a butterfly swarm due to Theorem 5

infinite number of quadrangles (butterflies) classified into six families can be associated to any butterfly curve. Again all proofs are carried out synthetically and constructively.

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