Journal for Geometry and Graphics
Volume 6 (2002), No. 1, 69–81.

Laguerre Voronoi Diagram on the Sphere

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Abstract. The Laguerre Voronoi diagram, also called the power diagram, is one of the important generalizations of the Voronoi diagram in the plane, in which the generating points are generalized to circles and the distance is generalized to the Laguerre distance. In this paper, an analogue of the Laguerre Voronoi diagram is introduced on the sphere. The Laguerre distance from a point to a circle on the sphere is defined as the geodesic length of the tangent line segment from the point to the circle. This distance defines a new variant of the Voronoi diagram on the sphere, and it inherits many characteristics from the Laguerre Voronoi diagram in the plane. In particular, a Voronoi edge in the new diagram is part of a great circle (i.e., the counterpart of a straight line), and the Voronoi edge is perpendicular to the great circle passing through the centers of the two generating circles. Furthermore, the construction of this diagram is reduced to the construction of a three-dimensional convex hull, and thus a worst-case optimal $O(n \log n)$ algorithm is obtained. Applications of this diagram include the computation of the union of spherical circles and related problems.

Key Words: Voronoi diagram, sphere, power diagram, Laguerre distance, convex hull.

MSC 2000: 51K99, 68W99.

1. Introduction

The *Voronoi diagram* is one of the most fundamental concepts in computational geometry, and has been generalized in many directions [5, 8, 10, 17, 19]. One of the directions is the generalization of the distance. The Euclidean distance can be replaced by a variety of distances, including the L_p distance [14], the convex distance [13], the additively and/or multiplicatively weighted distances [1] and the boat-sail distance [20].

However, many good properties disappear in such generalized Voronoi diagrams. For example, the Voronoi edges are complicated curves in the generalized Voronoi diagrams while they are portions of straight lines in the ordinary Voronoi diagram.

ISSN 1433-8157/$ 2.50

An exception is the Laguerre Voronoi diagram [11], which is also called the *power diagram* [3, 4]. In this diagram the Voronoi edges are portions of straight lines, and they are perpendicular to the line segments connecting the centers of the associated two generating circles.

In this paper we will show that an analogue of the Laguerre Voronoi diagram can be defined on the sphere. This diagram inherits good properties in that the Voronoi edges are portions of great circles (counterparts of straight lines) and they are perpendicular to the geodesic arcs connecting the associated two generators. We will also propose an algorithm for constructing this diagram via the three-dimensional convex hull; it runs in $\mathrm{O}(n \log n)$ time, where n denotes the number of generators, and this time complexity is worst-case optimal.

It should be emphasized that this diagram is different from that obtained from the planar Laguerre Voronoi diagram through the stereographic projection of the plane onto the sphere. The stereographic projection defines a correspondence between a sphere and a tangent plane using the lines that pass through the antipole of the tangent point [6]. By this projection, a circle on the plane is mapped to a circle on the sphere, but a straight line is not mapped to a great circle.

2. Brief review of the Laguerre Voronoi diagram in the plane

Let $G = \{c_1, c_2, \ldots, c_n\}$ be a set of n circles in the plane $\mathbf{R}^2$, and let P_i and r_i be the center and the radius of the circle c_i. For point P and circle c_i, let us define

$$d_{\mathrm{L}}(\mathrm{P}, c_i) = d(\mathrm{P}, \mathrm{P}_i)^2 - {r_i}^2, \tag{1}$$

where $d(\mathrm{P}, \mathrm{P}_i)$ denotes the Euclidean distance between P and P_i. We call $d_{\mathrm{L}}(\mathrm{P}, c_i)$ the *Laguerre distance* from P to c_i.

The above misuse of the term ‘distance’ is intentional. Actually $d_{\mathrm{L}}(\mathrm{P}, c_i)$ has the dimension of the square of the distance. However, we do not take the square root of this value because we want to define the ‘Laguerre distance’ even when the point P is inside the circle c_i, in which case the right-hand side of equation (1) is negative. Thus, the Laguerre distance is not a kind of the distance in a mathematical sense; it just represents a ‘degree of farness’.

We define

$$R(G; c_i) = \{\mathrm{P} \in \mathbf{R}^2 \mid d_{\mathrm{L}}(\mathrm{P}, c_i) < d_{\mathrm{L}}(\mathrm{P}, c_j),\ j \neq i\}, \tag{2}$$

and call it the *Laguerre Voronoi region* for c_i. The plane is partitioned into the regions $R(G; c_1), R(G; c_2), \ldots, R(G; c_n)$ and their boundaries. This partition is called the *Laguerre Voronoi diagram* for G, and the elements of G are called the *generating circles* (or *generators* in short).

As shown in Fig. 1, suppose that P is outside c_i, and let l be the line that passes through P and that is tangent to c_i. The Laguerre distance $d_{\mathrm{L}}(\mathrm{P}, c_i)$ is the square of the distance between P and

the point of contact of l to c_i. Hence, if generating circles are mutually disjoint, a point on an edge of the diagram has the tangent line segments with equal lengths to the associated two generating circles.

Fig. 2 shows an example of the Laguerre Voronoi diagram in the plane.

If two generating circles touch at a point, the Voronoi edge is on the common tangent line passing through the point. If two generating circles cross each other, the Voronoi edge is on the line passing through the two cross points. The region $R(G; c_i)$ may be empty for

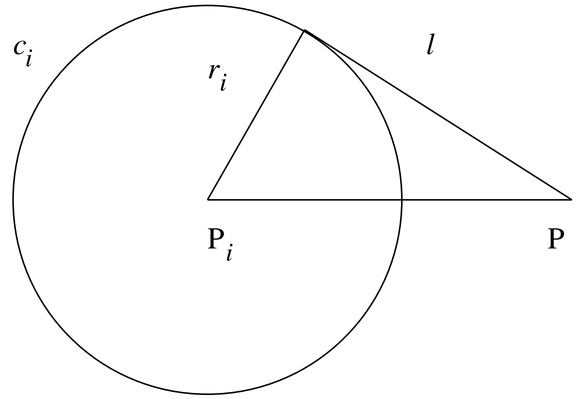

Figure 1: Geometric interpretation of the Laguerre distance

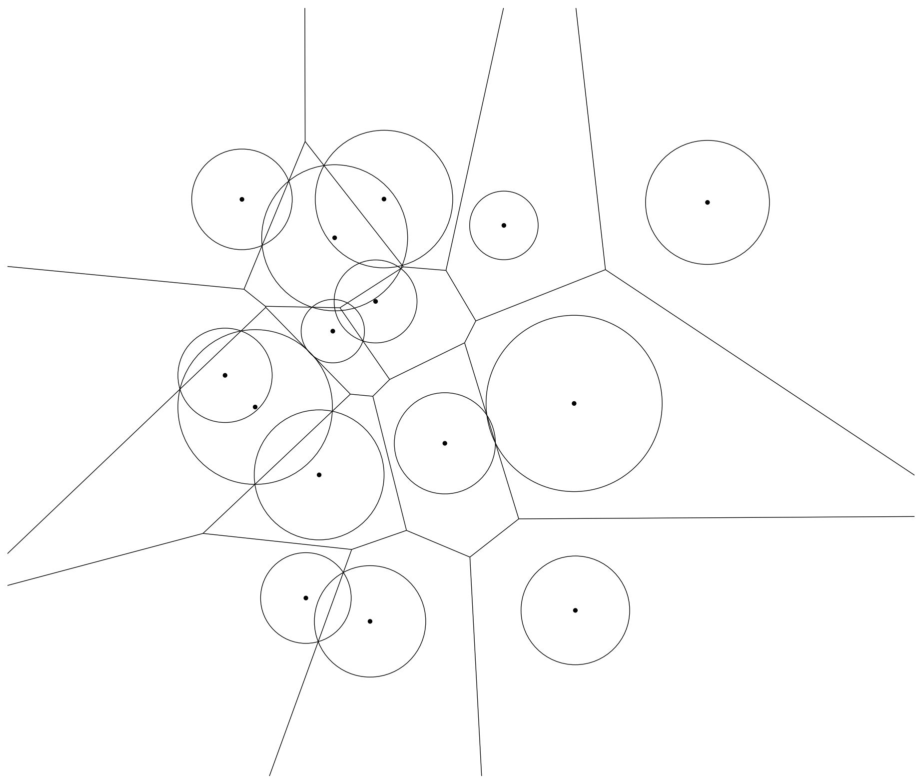

Figure 2: Laguerre Voronoi diagram in the plane

some c_i. Even if $R(G;c_i)$ is not empty, this region may not contain c_i. These properties are direct consequences of the definition of the Laguerre Voronoi diagram. We can also see these properties in the example diagram in Fig. 2.

3. Spherical Laguerre distance and the associated Voronoi diagram

The Voronoi diagram can be defined also on the sphere [7, 15], where the distance is measured along the shortest arc connecting two points. In this section we will generalize the Voronoi diagrams on the sphere in the way analogous to the generalization of the ordinary Voronoi diagram to the Laguerre Voronoi diagram in the plane.

Here we consider the geometry on a sphere. Without loosing generality we assume that the radius of the sphere is 1. Let (x, y, z) be the Cartesian coordinate system, and U be the unit sphere with the center at the origin $(0, 0, 0)$. The intersection of U and a plane containing the center is called a *great circle*. A connected portion of a great circle is called a *geodesic arc*.

Suppose that P and Q are two points on U such that the line segment connecting P and Q is not a diameter of U. Then, there is a unique great circle that passes through both P and Q. The length of the shorter geodesic arc connecting P and Q is called the *geodesic distance* of P and Q, and is denoted by $\tilde{d}(\mathrm{P}, \mathrm{Q})$. Since U is the unit sphere, $\tilde{d}(\mathrm{P}, \mathrm{Q})$ is equal to the angle subtended at the center by the geodesic arc connecting P and Q, that is, the angle formed by the two lines $\overline{\mathrm{OP}}$ and $\overline{\mathrm{OQ}}$, where O is the center of the sphere U.

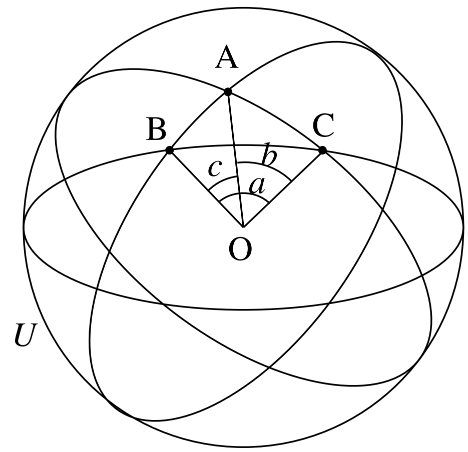

Figure 3: Spherical triangle

As shown in Fig. 3, let $\mathrm{A}, \mathrm{B}, \mathrm{C}$ be three points on U. The figure composed of the three geodesic arcs connecting A and B, B and C, and C and A is named the *spherical triangle* ABC. Let a, b, c be the geodesic length $\tilde{d}(\mathrm{B}, \mathrm{C})$, $\tilde{d}(\mathrm{C}, \mathrm{A})$, $\tilde{d}(\mathrm{A}, \mathrm{B})$, respectively. Note that a, b, c are equal to the angles subtended by the arcs at the center of U, as shown in Fig. 3.

Suppose that the angle at C is equal to $\pi/2$; in other words the two edges cross perpen-

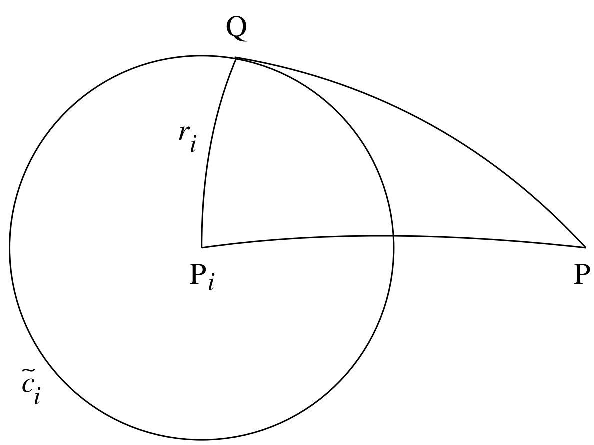

Figure 4: Interpretation of the Laguerre proximity

dicularly at C. Then

$$\cos c = \cos a . \cos b \tag{3}$$

holds [16]. This equation is the counterpart of the Pythagoras' theorem for the right-angled planar triangle in the sense that it represents the relation among the three edges of the right-angled triangle.

Let P_i be a point on U and $0 \le r_i < \pi/2$. The subset of U defined by

$$\{\mathrm{P} \in U \mid \tilde{d}(\mathrm{P}, \mathrm{P}_i) = r_i\} \tag{4}$$

is called a *circle* on U with the *center* P_i and the *radius* r_i. Throughout the paper, we denote this circle by $\tilde{c}_i$.

For any point P on U, let us define

$$\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i) = \frac{\cos \tilde{d}(\mathrm{P}, \mathrm{P}_i)}{\cos r_i}. \tag{5}$$

We call this value the *Laguerre proximity*. We use the term "Laguerre" because $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i)$ can be interpreted as the cosine of the length of the geodesic arc that emanates at P and is tangent to $\tilde{c}_i$. Actually, as shown in Fig. 4, let Q be the point on $\tilde{c}_i$ at which the geodesic arcs between P and Q and between P_i and Q cross in the right angle. Then, from equation (3) we get

$$\cos \tilde{d}(\mathrm{P}, \mathrm{P}_i) = \cos \tilde{d}(\mathrm{P}, \mathrm{Q}) \cdot \cos \tilde{d}(\mathrm{P}_i, \mathrm{Q}). \tag{6}$$

Since $\tilde{d}(\mathrm{P}_i, \mathrm{Q}) = r_i$, we get

$$\cos \tilde{d}(\mathrm{P}, \mathrm{Q}) = \frac{\cos \tilde{d}(\mathrm{P}, \mathrm{P}_i)}{\cos r_i}, \tag{7}$$

which means $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i) = \cos \tilde{d}(\mathrm{P}, \mathrm{Q})$.

We chose the term "proximity" instead of the "distance". This is because $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i)$ is monotone decreasing in the geodesic distance from the center of $\tilde{c}_i$ to P. Indeed,

$$\begin{array}{rcll} \tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i) & = & 1/\cos r_i & \text{if} \;\; \tilde{d}(\mathrm{P}, \mathrm{P}_i) = 0, \\ & = & 1 & \text{if} \;\; \tilde{d}(\mathrm{P}, \mathrm{P}_i) = r_i, \\ & = & 0 & \text{if} \;\; \tilde{d}(\mathrm{P}, \mathrm{P}_i) = \pi/2, \\ & = & -1/\cos r_i & \text{if} \;\; \tilde{d}(\mathrm{P}, \mathrm{P}_i) = \pi. \end{array}$$

Suppose that we place the center of the spherical circle $\tilde{c}_i$ at the North Pole, and that we move the point P from the North Pole to the South Pole along a longitudinal great circle. When P is at the North Pole, $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i)$ is the largest. When P crosses the circle $\tilde{c}_i$, $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i)$ crosses 1. When P crosses the equator, $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i)$ crosses 0 from positive to negative. When P reaches the South Pole, $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i)$ becomes the smallest.

For two circles $\tilde{c}_i$ and $\tilde{c}_j$, we define

$$B_{\mathrm{L}}(\tilde{c}_i, \tilde{c}_j) = \{\mathrm{P} \in U \mid \tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i) = \tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_j)\}, \tag{8}$$

and call it the *Laguerre bisector* of $\tilde{c}_i$ and $\tilde{c}_j$. The next theorem holds.

Theorem 1 *The Laguerre bisector $B_{\mathrm{L}}(\tilde{c}_i, \tilde{c}_j)$ is a great circle, and it crosses the geodesic arc connecting the two centers P_i and P_j at the right angle.*

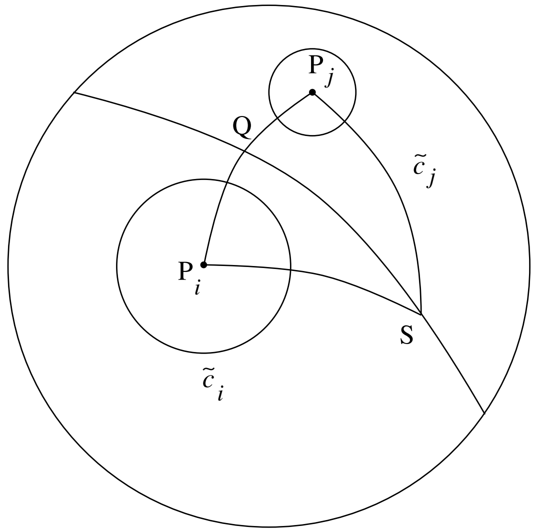

Figure 5: Laguerre bisector

Proof. As shown in Fig. 5, let $\tilde{c}_i$ and $\tilde{c}_j$ be two circles on U, and let Q be the point on the geodesic arc connecting P_i and P_j such that

$$\frac{\cos \tilde{d}(\mathrm{Q}, \mathrm{P}_i)}{\cos r_i} = \frac{\cos \tilde{d}(\mathrm{Q}, \mathrm{P}_j)}{\cos r_j} \tag{9}$$

is satisfied. From equation (5) we can see that Q is on the Laguerre bisector of $\tilde{c}_i$ and $\tilde{c}_j$.

Next let $\tilde{c}_{ij}$ be the great circle that crosses the geodesic arc connecting P_i and P_j perpendicularly at point Q, and let S be any point on $\tilde{c}_{ij}$ other than Q. Since the spherical triangle P_iQS is right-angled, we get

$$\cos \tilde{d}(\mathrm{S}, \mathrm{P}_i) = \cos \tilde{d}(\mathrm{Q}, \mathrm{P}_i). \cos \tilde{d}(\mathrm{Q}, \mathrm{S}), \tag{10}$$

and consequently we get

$$\tilde{d}_{\mathrm{L}}(\mathrm{S}, \tilde{c}_i) = \frac{\cos \tilde{d}(\mathrm{S}, \mathrm{P}_i)}{\cos r_i} = \frac{\cos \tilde{d}(\mathrm{Q}, \mathrm{P}_i). \cos \tilde{d}(\mathrm{Q}, \mathrm{S})}{\cos r_i}. \tag{11}$$

Similarly for the right-angled spherical triangle P_jQS, we get

$$\tilde{d}_{\mathrm{L}}(\mathrm{S}, \tilde{c}_j) = \frac{\cos \tilde{d}(\mathrm{S}, \mathrm{P}_j)}{\cos r_j} = \frac{\cos \tilde{d}(\mathrm{Q}, \mathrm{P}_j). \cos \tilde{d}(\mathrm{Q}, \mathrm{S})}{\cos r_j}. \tag{12}$$

Equations (9), (11), (12) altogether imply that

$$\tilde{d}_{\mathrm{L}}(\mathrm{S}, \tilde{c}_i) = \tilde{d}_{\mathrm{L}}(\mathrm{S}, \tilde{c}_j), \tag{13}$$

and hence the great circle $\tilde{c}_{ij}$ coincides with the Laguerre bisector. ∎

It might be interesting to note that Theorem 1 implies that the Laguerre bisector always divides the sphere into two regions with the same area, that is, the areas of the two regions do not depend on the sizes of the circles. When the sizes of the circles change, the location of the bisector changes, but the sizes of the resultant regions do not change. This property

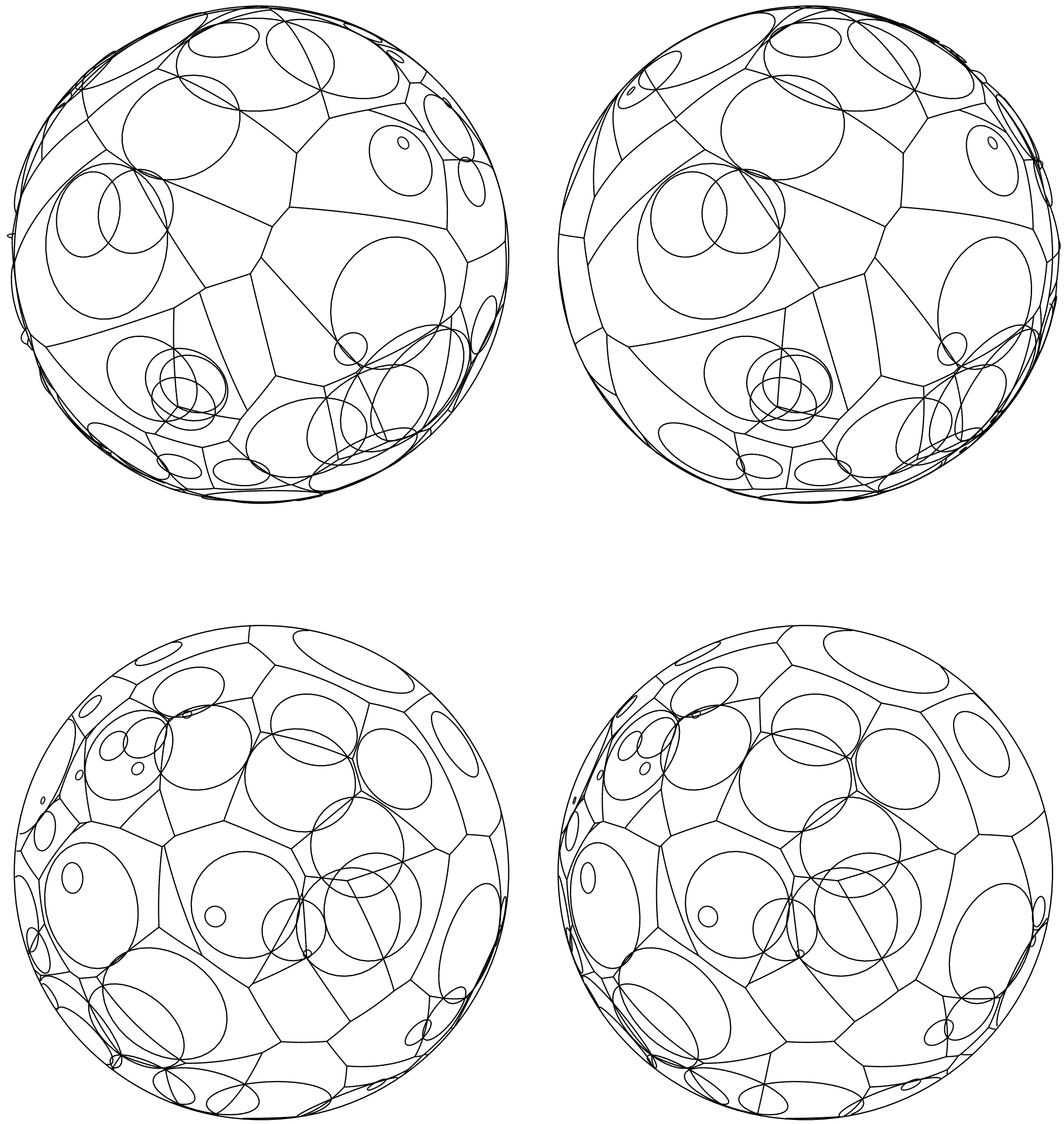

Figure 6: Laguerre Voronoi diagram on the sphere

can be interpreted intuitively in such a way that a larger circle has greater influence in its neighborhood but less influence in the other side of the sphere than a smaller circle.

Suppose that we are given a set $\tilde{G} = \{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n\}$ of n circles on U. We define

$$\tilde{R}(\tilde{G}; \tilde{c}_i) = \{\mathrm{P} \in U \mid \tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i) < \tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_j),\ j \neq i\}. \tag{14}$$

$\tilde{R}(\tilde{G}; \tilde{c}_i)$ represents the region composed of the points that are nearer to $\tilde{c}_i$ than to any other circles in $\tilde{G}$ in terms of the Laguerre proximity. The regions $\tilde{R}(\tilde{G}; \tilde{c}_1), \tilde{R}(\tilde{G}; \tilde{c}_2), \ldots, \tilde{R}(\tilde{G}; \tilde{c}_n)$ and their boundaries define the partition of U, which we call the *spherical Laguerre Voronoi diagram* for $\tilde{G}$.

Theorem 1 implies that any edge of the spherical Laguerre Voronoi diagram is a geodesic arc.

Figure 6 shows an example of the spherical Laguerre Voronoi diagram. This is a cross-type stereo pair of the diagrams; if we see the right diagram by the left eye and the left diagram by the right eye, we can perceive the three-dimensional structure. The upper pair of the diagrams represent the front hemisphere and the lower pair represent the rear hemisphere.

From this figure we can see that many properties of the Laguerre Voronoi diagram in the plane are inherited by this spherical version. If two circles contact at a point, the Laguerre bisector goes through that point and is perpendicular to the geodesic arc connecting the centers of the two circles. If two circles intersect at two points, the Laguerre bisector contains these two points. A Laguerre Voronoi region does not necessarily contain the associated generating circle. Moreover, some of the Laguerre Voronoi regions may be empty.

From the spherical Laguerre Voronoi diagram, we can construct another diagram through duality. Let us draw the geodesic arc connecting P_i and P_j if and only if the regions $R(\tilde{G};\tilde{c}_i)$ and $R(\tilde{G};\tilde{c}_j)$ share a common boundary arc. The resultant diagram is called the *spherical Laguerre Delaunay diagram* for $\tilde{G}$. The edge sets of the two diagrams admit the one-to-one correspondence through duality, and the corresponding edges are mutually orthogonal (recall Theorem 1).

4. Algorithms

For circle $\tilde{c}_i$ on U, let $\pi(\tilde{c}_i)$ be the plane containing $\tilde{c}_i$, and let $H(\tilde{c}_i)$ be the half space bounded by $\pi(\tilde{c}_i)$ and including the center O of U. Let, furthermore, l_{ij} be the line of intersection of $\pi(\tilde{c}_i)$ and $\pi(\tilde{c}_j)$, as shown in Fig. 7; this figure shows the cross section obtained when we cut the sphere U by the plane containing P_i, P_j and the center O of U.

Theorem 2 *The bisector $B_{\mathrm{L}}(\tilde{c}_i,\tilde{c}_j)$ is the intersection of U and the plane containing l_{ij} and* O.

Proof. Let π_{ij} be the plane containing $\mathrm{P}_i,\mathrm{P}_j$ and O. As shown in Fig. 7, let Q be the point of intersection of the plane π_{ij} and the line l_{ij}. Next, let S_i be a point of intersection of $\tilde{c}_i$ and π_{ij}, and S_j be the point of intersection of $\tilde{c}_j$ and π_{ij}. Furthermore, let T be the point of intersection of U and line OQ. Suppose that $\alpha_i=\tilde{d}(\mathrm{P}_i,\mathrm{T})$ and $\alpha_j=\tilde{d}(\mathrm{P}_j,\mathrm{T})$. Then we get

$$d(\mathrm{O},\mathrm{Q})\cos\alpha_i=d(\mathrm{O},\mathrm{S}_i)\cos r_i, \tag{15}$$

and hence

$$\frac{\cos\alpha_i}{\cos r_i}=\frac{d(\mathrm{O},\mathrm{S}_i)}{d(\mathrm{O},\mathrm{Q})}. \tag{16}$$

Similarly we get

$$\frac{\cos\alpha_j}{\cos r_j}=\frac{d(\mathrm{O},\mathrm{S}_j)}{d(\mathrm{O},\mathrm{Q})}. \tag{17}$$

Since $d(\mathrm{O},\mathrm{S}_i)=d(\mathrm{O},\mathrm{S}_j)$, we get from equations (15) and (16)

$$\frac{\cos\alpha_i}{\cos r_i}=\frac{\cos\alpha_j}{\cos r_j}, \tag{18}$$

which implies that

$$\tilde{d}_{\mathrm{L}}(\mathrm{T},\tilde{c}_i)=\tilde{d}_{\mathrm{L}}(\mathrm{T},\tilde{c}_j). \tag{19}$$

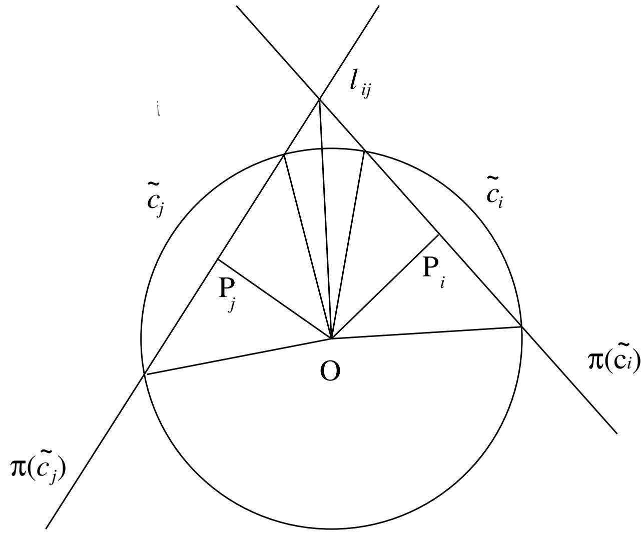

Figure 7: Planes containing generating circles

This equation means that T is on the Laguerre bisector of $\tilde{c}_i$ and $\tilde{c}_j$. Since the line l_{ij} is perpendicular to the plane π_{ij}, the Laguerre bisector $B_{\mathrm{L}}(\tilde{c}_i, \tilde{c}_j)$ is exactly the great circle defined by the plane containing O and l_{ij}. ∎

Theorem 2 immediately implies the next algorithm.

Algorithm 1 (spherical Laguerre Voronoi diagram).
Input: set $\tilde{G} = \{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n\}$ of n circles on U.
Output: spherical Laguerre Voronoi diagram for $\tilde{G}$.
Procedure:
1. Compute the intersection $I(\tilde{G})$ of all the half spaces $H(\tilde{c}_i)$, $i = 1, 2, \ldots, n$.
2. Project the edges of $I(\tilde{G})$ onto the sphere U by the central projection with the center at O. ∎

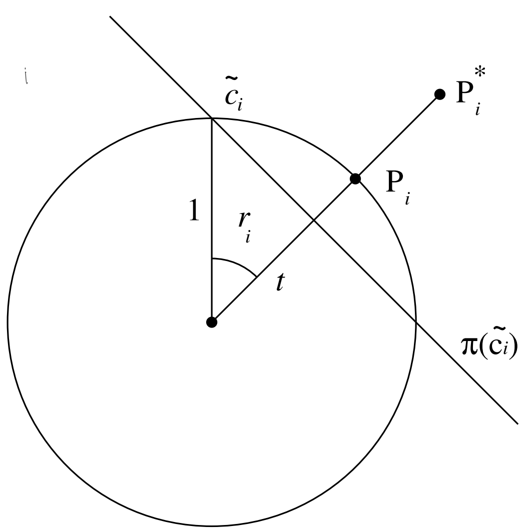

Figure 8: Dual transformation between a plane and a point with respect to a sphere

Step 1 requires $O(n \log n)$ time [19] and Step 2 requires $O(n)$ time. Hence the Laguerre Voronoi diagram on the sphere can be constructed in $O(n \log n)$ time. This time complexity is worst-case optimal because the following facts are known. First the spherical Laguerre Voronoi diagram includes the ordinary Voronoi diagram on the sphere; actually the former diagram reduces to the latter when all the generating circles have the same radius. Secondly there is a one-to-one correspondence between the ordinary Voronoi diagram on the sphere and that in the plane through the stereographic projection [6] or other transformations [2, 9, 18]. Thirdly, the construction of the ordinary Voronoi diagram in the plane requires $\Omega(n \log n)$ time [8, 19].

Next let us consider an algorithm for direct construction of the spherical Laguerre Delaunay diagram. The intersection $I(\tilde{G})$ of all the half spaces $H(\tilde{c}_i)$ is a convex polyhedron. As we have seen in Algorithm 1, the edge structure of $I(\tilde{G})$ is isomorphic to the spherical Laguerre Voronoi diagram in a graph-theoretic sense. Consequently, the edge structure of the dual of $I(\tilde{G})$ is isomorphic to the spherical Laguerre Delaunay diagram. Hence, what we have to do is to construct the dual of $I(\tilde{G})$. It is known that the dual of a convex polyhedron can be obtained as the convex hull of the 'dual points' in the following way.

As before, suppose that the center O of the sphere U is at the origin of the (x, y, z) coordinate system. Let the coordinates of the center P_i of the circle $\tilde{c}_i$ be (x_i, y_i, z_i). Let $t = \cos r_i$, where r_i is the geodesic radius of c_i. As shown in Fig. 8, t represents the (Euclidean) distance from O to the plane $\pi(\tilde{c}_i)$. We introduce new point P_i^* with the coordinates $(x_i/t, y_i/t, z_i/t)$, and define $\tilde{G}^*$ as

$$\tilde{G}^* = \{\mathrm{P}_1^*, \mathrm{P}_2^*, \ldots, \mathrm{P}_n^*\}.$$

Let $C(\tilde{G}^*)$ be the convex hull of $\tilde{G}^*$. It is known that the edge structure of $C(\tilde{G}^*)$ is the dual of the edge structure of $I(\tilde{G})$. Hence, we obtain the next algorithm.

Algorithm 2 (spherical Laguerre Delaunay diagram)**.**
Input: set $\tilde{G} = \{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n\}$ of n circles on U.
Output: spherical Laguerre Delaunay diagram.
Procedure:
1. Generate the point set $\tilde{G}^*$.
2. Construct the convex hull $C(\tilde{G}^*)$.
3. Project the edges of $C(\tilde{G}^*)$ onto the sphere U by the central projection with the center at O. ∎

This algorithm also runs in $O(n \log n)$ time, because the three-dimensional convex hull can be constructed in this order of time by the divide-and-conquer method [19].

5. Applications

The Laguerre Voronoi diagram in the plane can be applied to many problems concerning a collection of circles [11]. Similarly, the spherical Laguerre Voronoi diagram can be applied to problems concerning a collection of spherical circles. In this section we list typical applications.

Problem 1. Given n circles on the sphere, determine whether a query point P is included in their union.

Suppose that we constructed the spherical Laguerre Voronoi diagram for the set $\tilde{G} = \{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n\}$ of n circles in the preprocessing stage. Then, what we have to do to answer Problem 1 is just to find the Voronoi region $\tilde{R}(\tilde{G}; \tilde{c}_i)$ containing P and to check whether P is contained in $\tilde{c}_i$. The answer is "yes" if P is contained in $\tilde{c}_i$, and "no" otherwise. This is because, if P is not contained in $\tilde{c}_i$, then $\tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_j) \geq \tilde{d}_{\mathrm{L}}(\mathrm{P}, \tilde{c}_i) > 0$ for any j and therefore P is not in any circle.

The spherical Laguerre Voronoi diagram can be constructed in $\mathrm{O}(n \log n)$ time. To find the Voronoi region containing the query point can be done in $\mathrm{O}(\log n)$ time and $\mathrm{O}(n)$ storage with $\mathrm{O}(n \log n)$ preprocessing if we apply the planar-case technique [12] to the longitude-latitude coordinate system. Therefore we can solve Problem 1 in $\mathrm{O}(\log n)$ time and $\mathrm{O}(n)$ storage with $\mathrm{O}(n \log n)$ preprocessing.

Problem 2. Given n circles on the sphere, find the boundary of their union.

To solve this problem, we first construct the spherical Laguerre Voronoi diagram for the set $\tilde{G} = \{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n\}$ of n circles, and next collect the part of $\tilde{c}_i$ contained in $\tilde{R}(\tilde{G}, \tilde{c}_i)$ for all i. Recall that, if $\tilde{c}_i$ and $\tilde{c}_j$ intersect, the Laguerre bisector $B_{\mathrm{L}}(\tilde{c}_i, \tilde{c}_j)$ passes through both of the points of intersection. This means that the part of $\tilde{c}_i$ outside $\tilde{R}(\tilde{G}, \tilde{c}_i)$ is contained in other circles. Hence the collection of all the parts of $\tilde{c}_i$ contained in $\tilde{R}(\tilde{G}; \tilde{c}_i)$ for $i = 1, 2, \ldots, n$ constitute the boundary of the union of the circles.

A circle intersects each of its Voronoi edges at most twice, and there are only $\mathrm{O}(n)$ Voronoi edges. Therefore, once we construct the spherical Laguerre Voronoi diagram, we can collect all of the circular arcs contained in their own Voronoi regions in $\mathrm{O}(n)$ time; this also means that the boundary of the union of n circles consists of at most $\mathrm{O}(n)$ circular arcs. Thus, Problem 2 can be solved in $\mathrm{O}(n \log n)$ time through the spherical Laguerre Voronoi diagram.

Problem 3. Given n circles on the sphere, classify them into the connected components.

This problem arises in availability of mobile-telephone communication. Suppose that circle $\tilde{c}_i$ represents the region in which we can use a mobile telephone via a telephone station located at the center of $\tilde{c}_i$. We want to know whether we can move from a point inside $\tilde{c}_i$ to a point inside $\tilde{c}_j$ while keeping in touch with some other person using a mobile telephone. The answer is "yes" if and only if $\tilde{c}_i$ and $\tilde{c}_j$ belong to the same connected components.

To solve Problem 3, we construct a graph whose vertices are the given n circles and whose edges are pairs of circles such that they share a common Voronoi edge and that they intersect each other. Then the connected components of this graph give the solution of Problem 3. Since the connected components of a graph can be computed in $\mathrm{O}(n)$ time, Problem 3 can be solved in $\mathrm{O}(n \log n)$ time through the spherical Laguerre Voronoi diagram.

6. Concluding remarks

We have shown that an analogue of the Laguerre Voronoi diagram in the plane can be defined on the sphere, and that this analogue inherits many good properties possessed by the planar diagram. In particular, Laguerre Voronoi edges are geodesic arcs that are orthogonal to the corresponding Delaunay edges. This property enables us to efficiently construct the newly introduced diagrams through the intersection of half spaces or the three-dimensional convex hull of points. Because of its simplicity, this diagram can be a basic tool for geographic analysis such as facility layout and environment accessment on the earth.

It has been known that many kinds of Voronoi diagrams can be constructed through the three-dimensional convex hull. They include the ordinary Voronoi diagram in the plane or on the sphere, the farthest-point Voronoi diagram in the plane, the Laguerre Voronoi diagram in the plane and the elliptic Voronoi diagram in the plane [2, 5, 8, 9, 17, 18]. This paper adds one new diagram to this list.

Algorithm 2 was implemented in FORTRAN, and the source code is made open for public use in the web page:

http://www.simplex.t.u-tokyo.ac.jp/~sugihara/

Actually Fig. 6 was drawn by this program.

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Received March 4, 2002