Journal for Geometry and Graphics
Volume 6 (2002), No. 2, 111–119.

Special Quartics with Triple Points

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Abstract. This paper deals with a special class of 4th-order surfaces in the 3-dimensional Euclidean space. The surfaces of this class contain the absolute conic, a double straight line and triple points. It is shown that such surfaces may contain on the double line at least two real triple points which are classified according to the type of their tangent cones. The selected examples of the surfaces are displayed using the program Mathematica 4.1.

Keywords: Algebraic surfaces of 4th order, triple point

MSC 2000: 51N35

1. Introduction

In this paper we use the term *"quartic"* for the 4th-order surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$. In homogeneous Cartesian coordinates

$$(x:y:z:w), \quad (x,y,z) \in \mathbb{R}^4,\ w \in \{0,1\}, \quad (x,y,z,w) \neq (0,0,0,0),$$

a quartic is given by the homogeneous equation $F_4(x,y,z,w) = 0$ of degree 4. In the 19th and at the beginning of the 20th century their properties were studied intensively in a number of geometric books and papers. We present here the basic classification of those quartics which contain singular lines [7, vol. II, pp. 200–252], [6, pp. 1537–1787], i.e.,

- quartics with a triple straight line (this class contains only ruled quartics);
- quartics with a double twisted cubic (this class contains only ruled quartics);
- quartics with a double conic section (this class contains cyclides);
- quartics with a double conic section and a double line (this class contains only ruled quartics);
- quartics with three double lines (this class contains Steiner's quartics);
- quartics with two double straight lines (this class contains only ruled quartics);
- quartics with one double straight line (this class contains ruled quartics, the pedal surfaces of (1,2)-congruences [2] and the surfaces which will be considered in this paper).

ISSN 1433-8157/$ 2.50

2. Tangent cones at singular points

The point $T(x_0 : y_0 : z_0 : w_0)$ is singular for the surface given by equation $F_n(x, y, z, w) = 0$ if and only if

$$\frac{\partial F_n}{\partial x}(T) = \frac{\partial F_n}{\partial y}(T) = \frac{\partial F_n}{\partial z}(T) = \frac{\partial F_n}{\partial w}(T) = 0.$$

At such a point the tangents of the surface form a tangent cone of order equal to its multiplicity. We recall here some analytical properties of tangent cones, according to [7, Vol. I, p. 56] and [5, p. 251], which will be used in the proofs presented in the following section.

Proposition 1 *Each homogeneous equation $H_n(x, y, z) = 0$ of degree n represents in $\mathbb{E}^3$ a cone $\mathcal{C}^n_O$ with the vertex $(0 : 0 : 0 : 1)$. If the polynomial $H_n(x, y, z)$ is irreducible over the field $\mathbb{R}$, the cone $\mathcal{C}^n_O$ is a proper cone of order n. If*

$$H_n(x, y, z) = H_{n_1}(x, y, z) \cdot H_{n_2}(x, y, z) \cdots H_{n_k}(x, y, z), \quad n_1 + n_2 + \ldots + n_k = n$$

and $k \leqslant n$, the cone $\mathcal{C}^n(O)$ splits into the cones $\mathcal{C}^{n_1}(O)$, $\mathcal{C}^{n_2}(O)$, $\ldots$, $\mathcal{C}^{n_k}(O)$.

Proposition 2 *Each homogeneous equation $H_2(x, y) = 0$ of degree 2 represents in $\mathbb{E}^3$ a pair of planes through the axis z. If $H^2(x, y)$ is reducible over the field $\mathbb{R}$ the planes are real. They coincide if $H^2(x, y)$ is a total square, and they are a pair of imaginary planes if $H^2(x, y)$ is irreducible over $\mathbb{R}$.*

Proposition 3 *If $X \subset \mathbb{R}^n$ is a hypersurface given as the zero set of the polynomial $F(\mathbf{x})$, $\mathbf{x} := (x_1, \ldots, x_n)$, and we write*

$$F(\mathbf{x}) = H_m(\mathbf{x}) + H_{m+1}(\mathbf{x}) + \ \ldots \ + H_d(\mathbf{x}),$$

where $H_k(\mathbf{x})$ is homogeneous of degree k in $x_1, \ldots, x_n$ for $k = m, \ldots, d$, then the tangent cone at the origin $(0, \ldots, 0)$ will be a cone of order m given by the polynomial H_m.

3. Triple points on non-ruled quartics through the absolute conic and with a double straight line

In homogeneous Cartesian coordinates each quartic $\mathcal{F}$ with the double line $z\colon\ x = y = 0$ which contains the absolute conic $\omega\colon\ x^2 + y^2 + z^2 = w = 0$ can be presented by the following equation:

$$F(x, y, z, w) = (x^2 + y^2 + z^2)\mathbf{a}_2(x, y) + w\mathbf{d}_3(x, y) + zw\mathbf{b}_2(x, y) + w^2\mathbf{c}_2(x, y) = 0, \qquad (1)$$

where $\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2, \mathbf{d}_3$ are homogeneous polynomials in x, y of degrees 2, 2, 2 and 3, respectively, with real cofficients; at least one coefficient in the polynomial $\mathbf{a}_2$ is different from 0, i.e., $\mathbf{a}_2 \neq \mathbf{0}$.

The proof is a direct consequence of the equation of the quartic with the double line z [7, vol. II, p. 217] and the fact that the section of such quartic and the plane at infinity is the absolute conic ω and the pair of lines through the point $(0 : 0 : 1 : 0)$ given by the equations $\mathbf{a}_2 = w = 0$. Such proof can be found in [3, p. 136].

Theorem 1 *T is a triple point on the surface $\mathcal{F}$ given by equation (1), if and only if*

$$T(0 : 0 : t : 1), \quad t \in \mathbb{R}, \qquad (2)$$

$$t^2\mathbf{a}_2 + t\mathbf{b}_2 + \mathbf{c}_2 = 0, \quad \mathbf{d}_3 + \tilde{z}(2t\mathbf{a}_2 + \mathbf{b}_2) \neq \mathbf{0} \ \ \textit{for} \ \ \tilde{z} := z - t. \qquad (3)$$

Proof: The triple point of the surface $\mathcal{F}$ must be located on the double line z. (Otherwise the line joining the triple point with any point on the double line would intersect the surface at five points which is impossible for a non-ruled quartic.) Furthermore, the point $(0:0:1:0)$ cannot be a triple point of the surface $\mathcal{F}$. (If $(0:0:1:0)$ was a triple point, it would be a triple point of any planar section through it, but the plane at infinity cuts the surface $\mathcal{F}$ along the degenerated 4th-order curve $(x^2+y^2+z^2)\mathbf{a}_2 = w = 0$ with a double point $(0:0:1:0)$.) Therefore, the coordinates of triple points are always $(0:0:t:1)$.

If we translate the Cartesian coordinate system $O(x,y,z)$ into the system $T(\tilde{x},\tilde{y},\tilde{z})$, where

$$\tilde{x} := x, \quad \tilde{y} := y, \quad \tilde{z} := z - t,$$

we obtain the equation

$$F(\tilde{x},\tilde{y},\tilde{z},1) = (\tilde{x}^2+\tilde{y}^2)\mathbf{a}_2 + \mathbf{d}_3 + (\tilde{z}+t)^2\mathbf{a}_2 + (\tilde{z}+t)\mathbf{b}_2 + \mathbf{c}_2 = 0. \tag{4}$$

If T is a triple point, then according to Prop. 3 the minimal degree of the homogeneous polynomials in (4) is 3 and it follows that $t^2\mathbf{a}_2 + t\mathbf{b}_2 + \mathbf{c}_2 = 0$ and $\mathbf{d}_3 + \tilde{z}(2t\mathbf{a}_2 + \mathbf{b}_2) \neq \mathbf{0}$.

On the other hand, if $T(0:0:t:1)$, $t^2\mathbf{a}_2 + t\mathbf{b}_2 + \mathbf{c}_2 = 0$ and $\mathbf{d}_3 + \tilde{z}(2t\mathbf{a}_2 + \mathbf{b}_2) \neq \mathbf{0}$, then the tangent cone at the point $T(0:0:t:1)$ is a cone of third order, i.e., T is a triple point of the surface $\mathcal{F}$. □

Corollary 1 *If there are two different triple points on the double line of the surface $\mathcal{F}$, then the tangent planes at other points of the double line are given by the equation $\mathbf{a}_2 = 0$.*

Proof: Without loss of generality we assume that $T_1(0:0:0:1)$ and $T_2(0:0:t:1)$, $t \in \mathbb{R}\setminus\{0\}$, are triple points of the surface $\mathcal{F}$. Then from Theorem 1 we conclude

$$\mathbf{c}_2 = \mathbf{0} \quad \text{and} \quad \mathbf{b}_2 = -t\,\mathbf{a}_2. \tag{5}$$

Let $Z_0(0:0:z_0:1)$ be a point on the double line different from T_1 and T_2, i.e., $z_0 \neq 0$ and $z_0 \neq t$. In the coordinate system $Z_0(x',y',z')$,

$$x' := x, \quad y' := y, \quad z' := z - z_0,$$

the surface $\mathcal{F}$ is given by the equation

$$F(x',y',z',1) = (x'^2+y'^2+z'^2)\mathbf{a}_2 + \mathbf{d}_3 + z'(2z_0-t)\mathbf{a}_2 + z_0(z_0-t)\mathbf{a}_2 = 0. \tag{6}$$

According to Prop. 3 the equation of the tangent cone at the binode Z_0 is $z_0(z_0-t)\mathbf{a}_2 = 0$. From $z_0(z_0-t) \neq 0$ follows that $\mathbf{a}_2 = 0$ is the equation of the tangent cone at any point different from T_1 and T_2 on the double line (see Figs. 1–3). □

Theorem 2 *If $T(0:0:t:1)$ is a triple point of the surface $\mathcal{F}$ given by eq. (1), then for the tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ of the surface $\mathcal{F}$ at point T one of the following statements is valid:*

1. *$\mathcal{TC}_{(T,\mathcal{F})}$ is a proper (non degenerate) cone of third order if and only if $\mathbf{d}_3 + 2\tilde{z}t\mathbf{a}_2 + \tilde{z}\mathbf{b}_2$ is irreducible over the field $\mathbb{R}$.*
2. *$\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into the real 2nd-order cone and a plane if and only if the polynomials $\mathbf{d}_3$ and $2t\mathbf{a}_2 + \mathbf{b}_2$ have a common linear factor in x and y.*
3. *$\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three (independent) planes intersecting at point T if and only if either $2t\mathbf{a}_2 + \mathbf{b}_2$ is a factor of $\mathbf{d}_3$ or $\mathbf{d}_3 = \mathbf{0}$.*

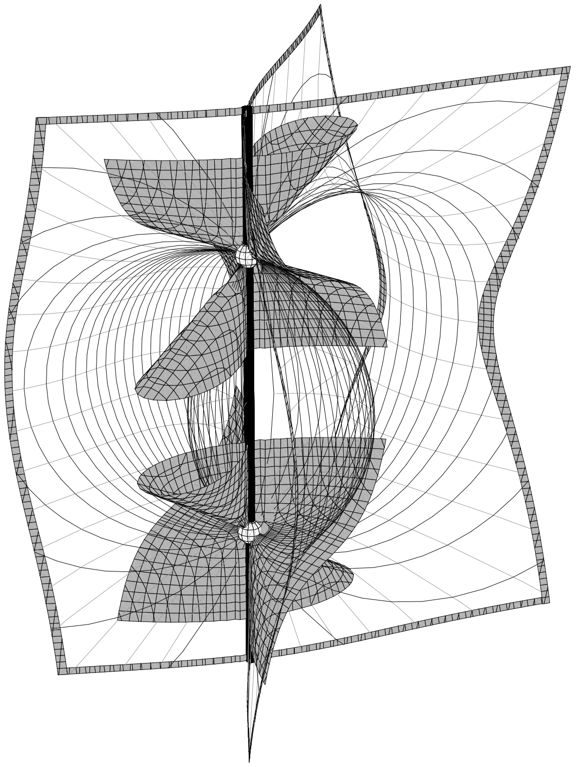

Figure 1: $(x^2+y^2+z^2)(x^2-2y^2)-2x^3-$
$-2y^2x+3(2y^2-x^2)z=0$,
2 triple points (proper tangent cones) and ordinary binodes on the double line

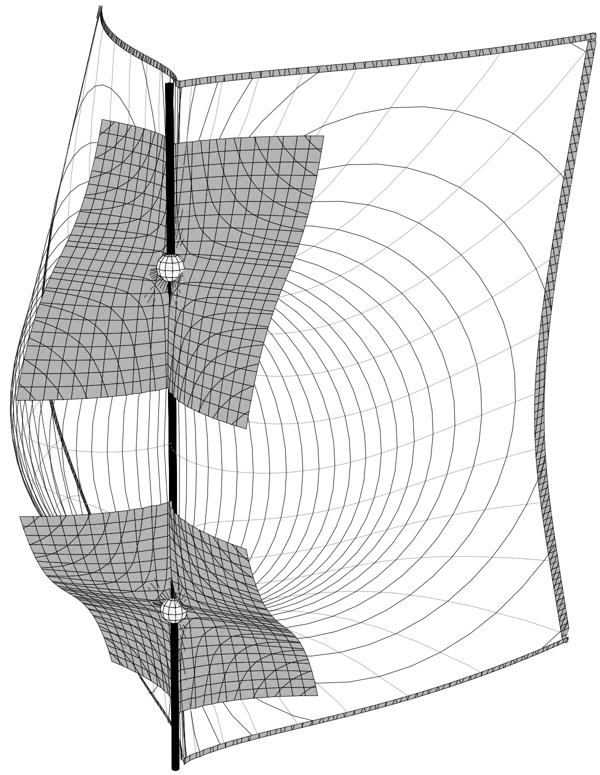

Figure 2: $(x^2+y^2+z^2)x^2-2x^2y-$
$-2y^3-3x^2z=0$,
2 triple points (proper tangent cones) and pinch-points (unodes) on the cuspidal line

4. *$\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes through the double line z if and only if $2t\mathbf{a}_2+\mathbf{b}_2=0$ and $\mathbf{d}_3\neq\mathbf{0}$.*

Proof: According to Theorem 1, (4) and Prop. 3, with respect to the coordinate system $T(\tilde{x},\tilde{y},\tilde{z})$ the tangent cone of the surface $\mathcal{F}$ at the triple point $T(0:0:t:1)$ is given by

$$\mathbf{d}_3+\tilde{z}(2t\mathbf{a}_2+\mathbf{b}_2)=0. \tag{7}$$

The axis $z:\ x=y=0$ is a double line of the tangent cone given by eq. (7). Namely, each third-order cone with the double line z and the vertex $(0:0:0:1)$ is given by the equation $\mathbf{u}_3+z\mathbf{u}_2=0$, where $\mathbf{u}_3$, $\mathbf{u}_2$ are homogeneous polynomials in x and y of degree 3 and 2, respectively.

If the polynomial $\mathbf{d}_3+2\tilde{z}(t\mathbf{a}_2+\mathbf{b}_2)$ is irreducible over $\mathbb{R}$, the tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ is an irreducible third-order cone with the double line z (according to Prop. 1).

If the polynomial $\mathbf{d}_3+\tilde{z}(2t\mathbf{a}_2+\mathbf{b}_2)$ is reducible consisting of a linear and an (irreducible) 2nd-order factor, we obtain three different types of tangent cones because the greatest exponent of $\tilde{z}$ is 1.

- The 2nd-order factor is homogeneous in x, y and z, while the linear factor is homogeneous in x and y. In this case $\mathbf{d}_3$ and $2t\mathbf{a}_2+\mathbf{b}_2$ have a common linear factor and the tangent cone degenerates into the plane through z and an irreducible real cone of order 2 through z (Prop. 1).

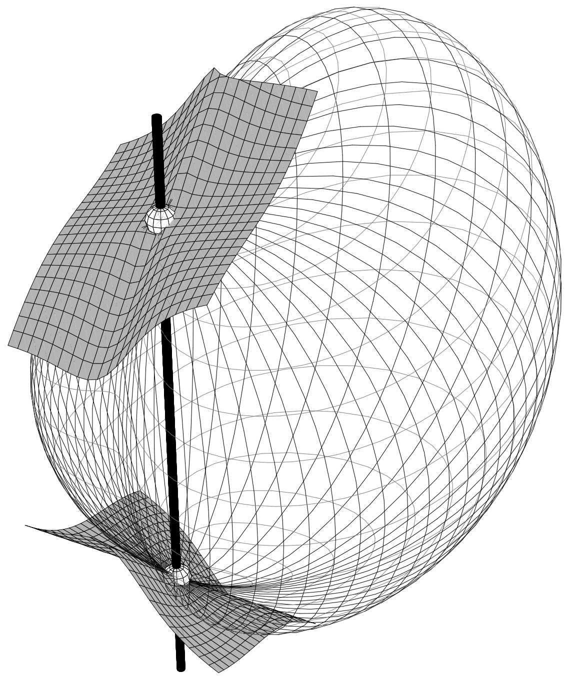

Figure 3: $(x^2+y^2+z^2)(x^2+3y^2)-2x^3+2x^2y-2xy^2+2y^3-2(x^2+3y^2)z=0$,
2 triple points (proper tangent cones) and isolated binodes on the double line

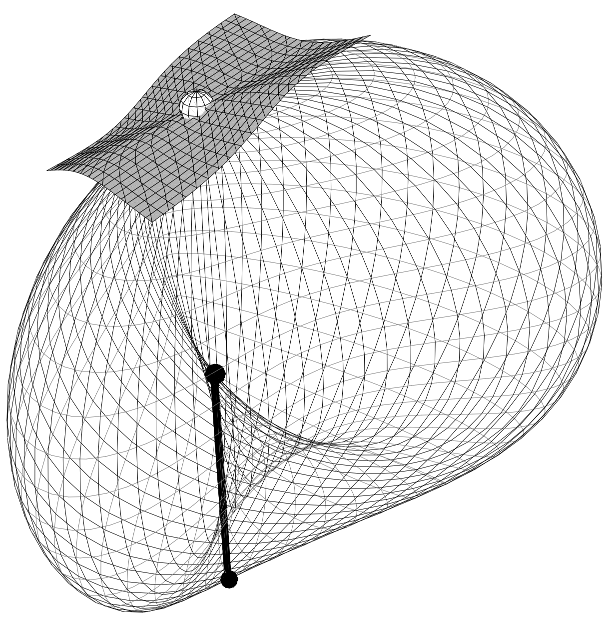

Figure 4: $(x^2+y^2+z^2)(x^2+y^2)+2x^3+4x^2y+$ $+2xy^2+4y^3+2(3x^2-xy+6y^2)z=0$,
1 triple point (a proper tangent cone),
2 pinch-points, ordinary and isolated
binodes on the double line

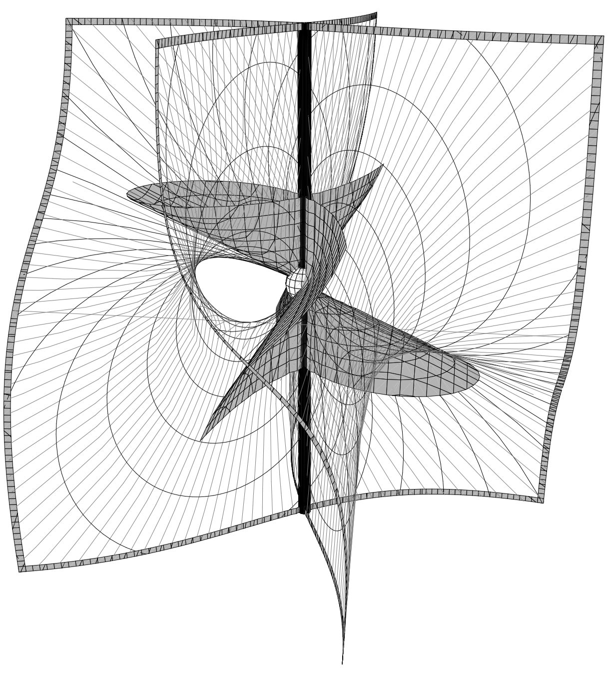

Figure 5: $(x^2+y^2+z^2)xy-x^2y-y^3+$ $+(x^2+2xy-2y^2)z=0$,
1 triple point (a proper tangent cone),
ordinary binodes on the double line

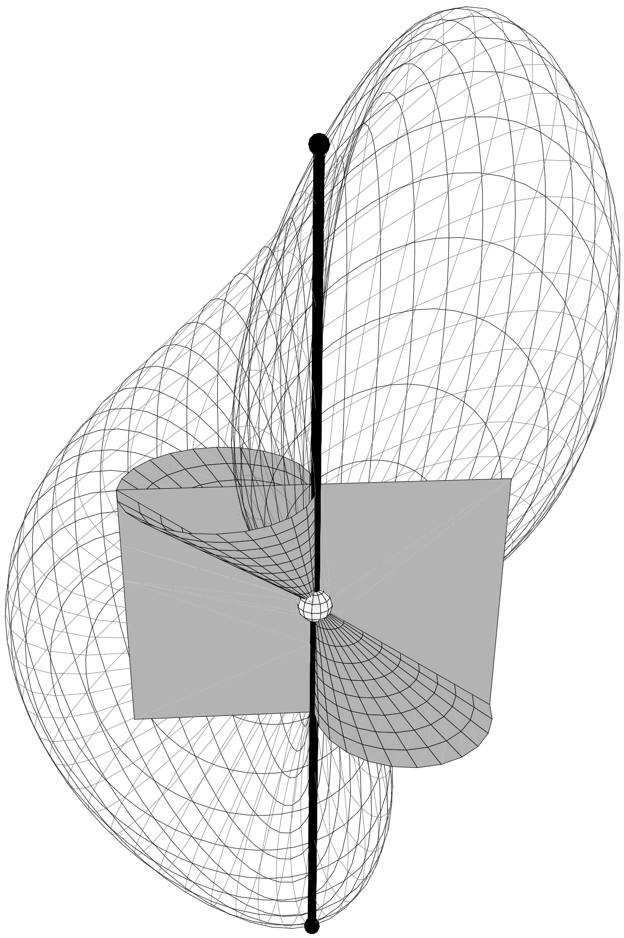

Figure 6: $(x^2+y^2+z^2)(x^2+y^2)-x^3+$
$+x^2y-xy^2+y^3+(x^2-y^2)z=0$,
1 triple point (a degenerated tangent cone: 2nd-order cone and a plane),
2 pinch-points, ordinary and isolated binodes on the double line

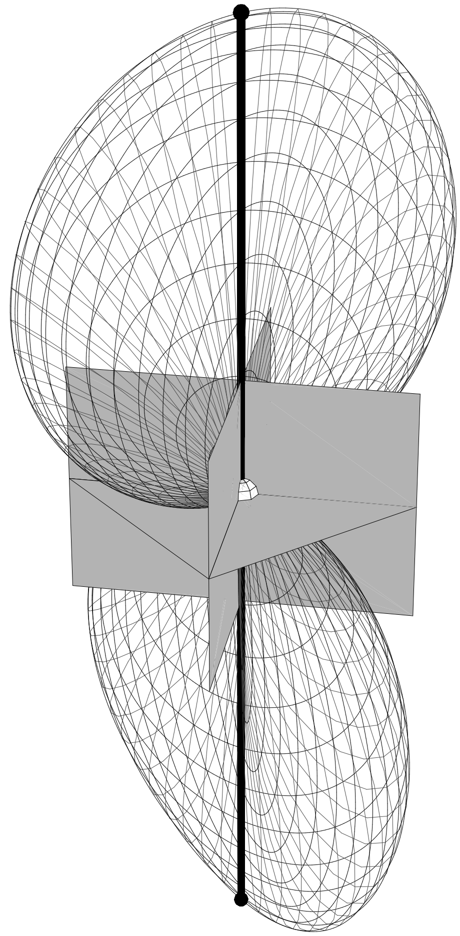

Figure 7:
$(x^2+y^2+z^2)(x^2+y^2)+2z(x^2-y^2)=0$,
1 triple point (a degenerated tangent cone: 3 real and different non collinear planes),
2 pinch-points, ordinary and isolated binodes on the double line

- The 2nd-order factor is homogeneous in x and y, while the linear factor is homogeneous in x, y and z. In this case $2t\mathbf{a}_2+\mathbf{b}_2$ is a factor of $\mathbf{d}_3$ or $\mathbf{d}_3=\mathbf{0}$ and $2t\mathbf{a}_2+\mathbf{b}_2$ is irreducible over $\mathbb{R}$. The tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes. Two of them are imaginary planes through the line z (Prop. 2), and the third is a real plane through T which does not contain the line z.
- Both, the linear and the 2nd-order factor are homogeneous in x and y. This case occurs only if $2t\mathbf{a}_2+\mathbf{b}_2=0$, $\mathbf{d}_3\neq\mathbf{0}$, with $\mathbf{d}_3$ reducible over the field $\mathbb{R}$ to a linear and irreducible factor of order 2. The cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes through the line z and two of them are imaginary.

If the polynomial $\mathbf{d}_3+\tilde{z}(2t\mathbf{a}_2+\mathbf{b}_2)$ is reducible to three linear factors, we can obtain two different types of a tangent cone.

- $2t\mathbf{a}_2+\mathbf{b}_2$ is a factor of $\mathbf{d}_3$ or $\mathbf{d}_3=\mathbf{0}$ and $2t\mathbf{a}_2+\mathbf{b}_2$ is reducible over the field $\mathbb{R}$. The tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes, two of them real (different or coinciding) through the line z (Proposition 2) and the third plane not containing z.

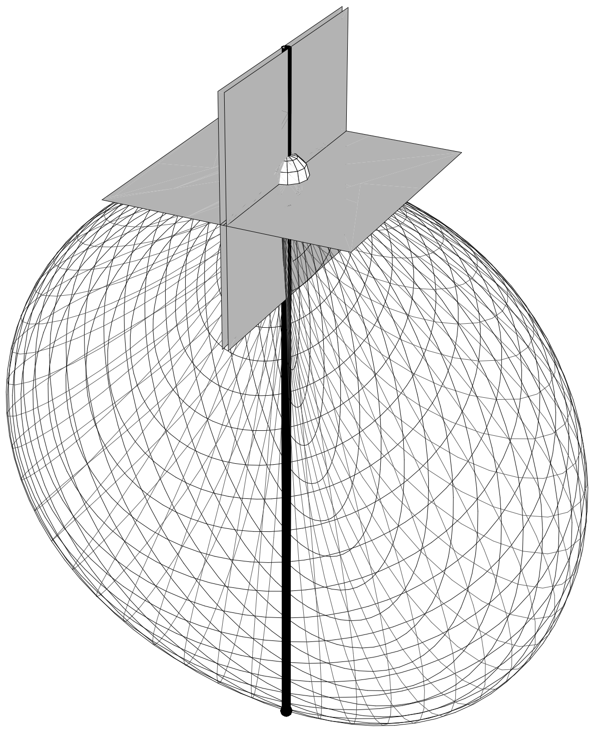

Figure 8:
$(x^2+y^2+z^2)(x^2+y^2)+zx^2=0$,
1 triple point (a degenerated tangent cone: 3 real planes, 2 coinciding), 1 pinch-point, ordinary and isolated binodes on the double line

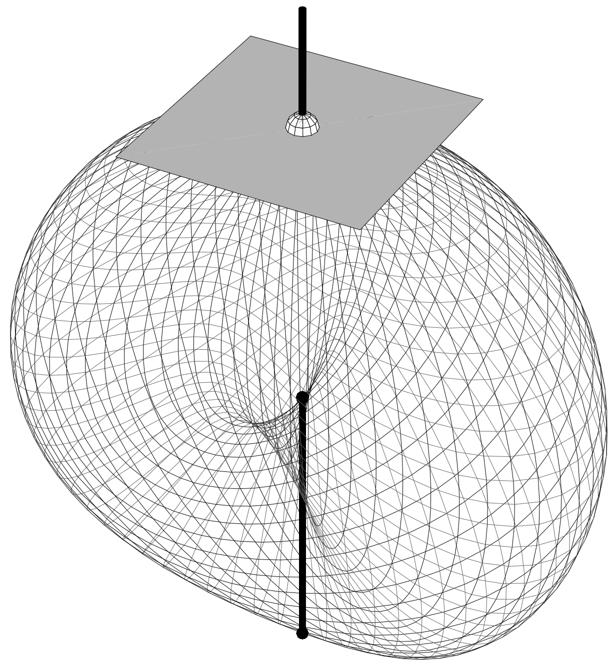

Figure 9:
$(x^2+y^2+z^2)(x^2+y^2)+z(2x^2+y^2)=0$,
1 triple point (a degenerated tangent cone: 3 non collinar planes, 2 imaginary planes through the double line), 2 pinch-points, ordinary and isolated binodes on the double line

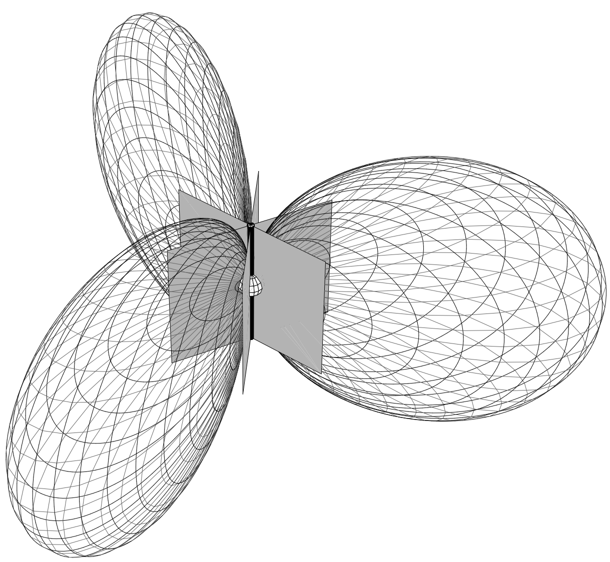

Figure 10:
$(x^2+y^2+z^2)(x^2+y^2)-3x^2y+y^3=0$

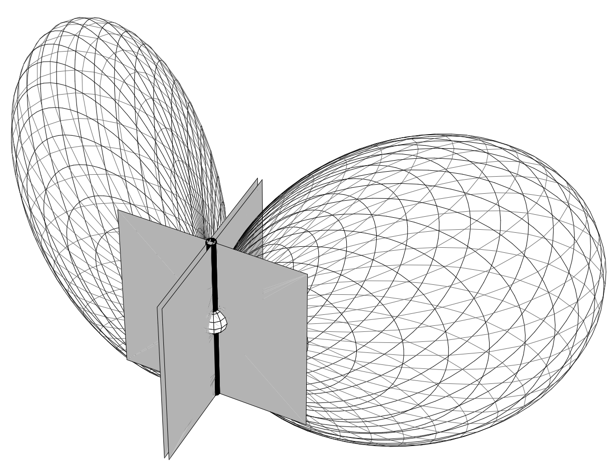

Figure 11:
$(x^2+y^2+z^2)(x^2+y^2)+x^2y=0$

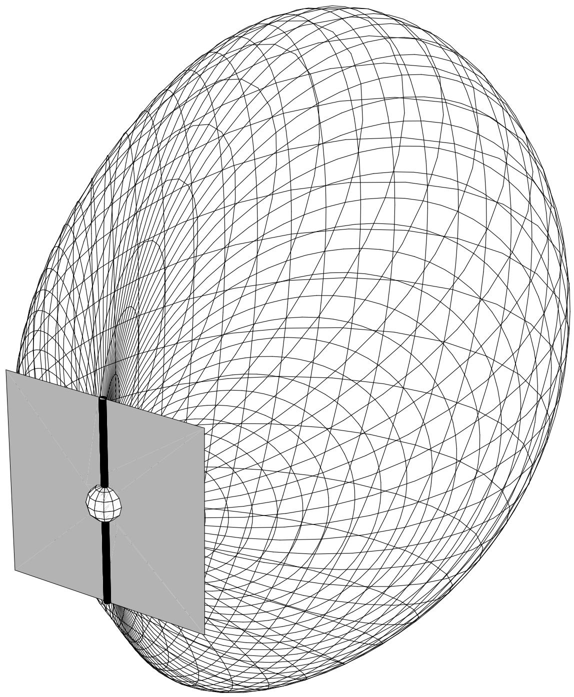

Figure 12:
$(x^2+y^2+z^2)(2x^2+y^2)-2y(x^2+y^2)=0$

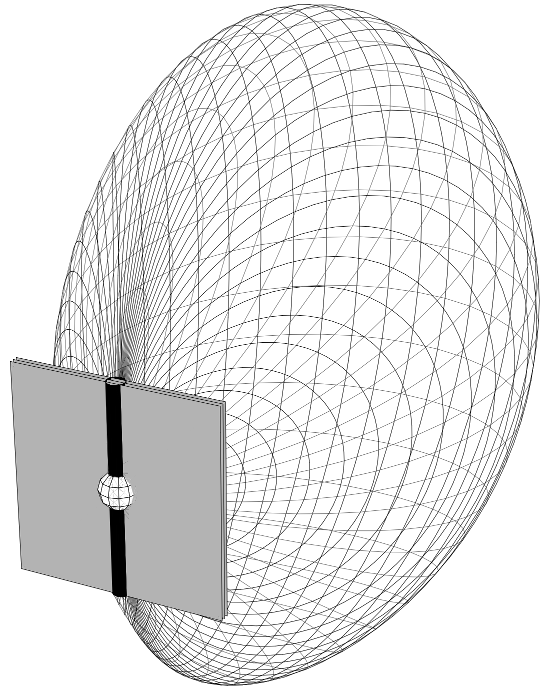

Figure 13:
$(x^2+y^2+z^2)(x^2+y^2)+x^3=0$

- $2t\mathbf{a}_2+\mathbf{b}_2=0$, $\mathbf{d}_3\neq\mathbf{0}$ and $\mathbf{d}_3$ is reducible over the field $\mathbb{R}$ to three linear factors. The cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three real planes through the line z, whereby all three planes or just two of them may coincide.

This is illustrated in Figs. 1–13. □

4. Examples

In this section we present pictures of thirteen different surfaces as considered before, their triple points and tangent cones. The pictures have been produced with Mathematica 4.1. For each surface its equation in Cartesian coordinates (x,y,z) is given.

The Figures 1–3 show three types of pedal surfaces of (1,2)-congruences with two triple points (types VI_1, II_1 and I_1 according to [2]). The directing curves of the congruences are the double line of the surface and a hyperbola, a parabola and an ellipse. The pole lies on the double line. For the surfaces and the tangent cones at their triple points which are shown in the Figs. 1–3 the double line is nodal, cuspidal and isolated, respectively.

The considered quartics can also contain only one triple point with a proper tangent cone. Then they can contain at least two pinch-points on the double line. The Figures 4 and 5 show two examples of pedal surfaces of (1,2)-congruences (types $\mathrm{I}_{3,1}$ and V_1 according to [2]).

If the tangent cone at a triple point of the considered quartic degenerates into the plane and a 2nd-order cone, the cone must be real, because it contains the double line of the surface as a real generator. This is Case 2 of Theorem 2. Fig. 6 shows the pedal suface of type $\mathrm{I}_{3,2}$ (according to [2]).

In Figures 7–9 the tangent cones at the triple points degenerate into three planes. One of the tangent planes does not contain the double line of the surface. We obtained the equations of the surfaces by summing up the polynomials of the degenerated tangent cones and the polynomial $(x^2+y^2+z^2)(x^2+y^2)$. Therefore, the surfaces in Figs. 7–9 contain the absolute conic and a pair of isotropic lines at infinity. These surfaces illustrate Case 3 of Theorem 2.

The surfaces displayed in Figures 10–13 contain triple points with degenerated tangent cones as classified in Case 4 of Theorem 2. Three tangent planes are collinear with a double line; they are real and different in Example 10, two of them coincide in Example 11, two of them are imaginary in Example 12 and all three coincide in Example 13. All other points on the double line are isolated binodes.

We obtained the equation of each surface by summing up the polynomials which represent the section at infinity and the tangent cone at a triple point.

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Received May 21, 2002; final form December 9, 2002