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Extension of the 'Villarceau-Section' to Surfaces of Revolution with a Generating Conic

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Abstract. When a surface of revolution with a conic as meridian is intersected with a double tangential plane, then the curve of intersection splits into two congruent conics. This decomposition is valid whether the surface of revolution intersects the axis of rotation or not. It holds even for imaginary surfaces of revolution. We present these curves of intersection in different cases and we also visualize imaginary curves. The arguments are based on geometrical reasoning. But we also give in special cases an analytical treatment.

Keywords: Villarceau-section, ring torus, surface of revolution with a generating conic, double tangential plane

MSC 2000: 51N05

1. Introduction

Due to Y. Villarceau the following statement it is valid (compare e.g. [1], p. 412, [3], p. 204, or [4]):
The curve of intersection between a ring torus Ψ and any double tangential plane τ splits into two congruent circles.

We assume that r is the radius of the meridian circles k of Ψ and that their centers are in the distance d, $d > r$, from the axis a of rotation. We generalize and replace k by a conic which may also intersect the axis a. Under these conditions it is still true that the intersection with a double tangential plane τ is reducible.

2. Extension of the 'Villarceau-section'

The following two theorems will be proved by standard arguments from Algebraic Geometry:

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Theorem 1 *Let τ be a double tangential plane τ of a surface Ψ of revolution with a conic k as meridian. Then τ intersects Ψ in two congruent conic sections.*

Theorem 2 *The two congruent conics of $v := \tau \cap \Psi$ according to Theorem 1 are either real and of the same type as the generating conic k, or they are imaginary.*

In the sequel the intersection curve between Ψ and τ is called a *'Villarceau-section'* or briefly a *'v-section'*.

3. Geometrical Treatment

According to MacLaurin's theorem (see e.g. [2], p. 49) an irreducible plane curve of order n can possess at most

$$d_n := \frac{(n-1)(n-2)}{2}$$

double points. A surface of revolution Ψ with a generating conic k is an algebraic surface of order four. Thus any plane section is an algebraic curve of order four which in the irreducible case can have at most $d_4 = 3$ double points. If the curve has more than 3 singularities, then it splits into at least two irreducible components.

Let k' denote the image of the generator k under reflection in the axis a of rotation. Then k and k' form a complete meridian section of the surface Ψ (see Fig. 1). A double tangential plane τ passes through a common tangent t of k and k' which is not perpendicular to a. (In Fig. 1 t is a common inner tangent.) Then τ touches the surface Ψ at the points $B \in k$ and $B' \in k'$. These points are two double points of the intersection $v = \tau \cap \Psi$.

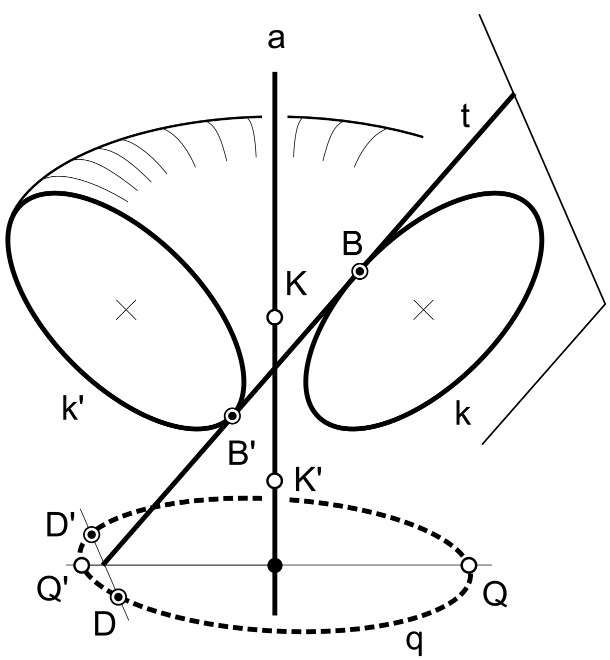

Figure 1: Double tangential plane τ of the surface Ψ of revolution

The two coplanar meridian conics k, k' intersect each other in four points.[1] Two of them,

[1] The case $k = k'$ with Ψ being a twofold covered quadric is excluded here. Furthermore we exclude the trivial case where the meridian conics touch the rotation axis. In this case the v-sections coincide with complete meridians.

the points K, K', are located on the axis a of Ψ and they can be real or conjugate imaginary. The two remaining points Q, Q' are symmetric with respect to a. Under rotation about a they trace a double circle q of Ψ. This circle is symbolically indicated in Fig. 1 as well as the points $K, K' \in a$. The circle q can be real or imaginary depending on whether the points Q, Q' are real or conjugate imaginary.

In the generic case the double tangential plane τ intersects the double circle q at two double points D, D'.[2] Thus the intersection curve $v = \tau \cap \Psi$ contains at least 4 double points B, B', D, D'. Hence v splits into two parts. Due to the symmetry of v with respect to t the two components are congruent conics c, c'. This proves Theorem 1.

A generating ellipse k has two complex conjugate points at infinity. Therefore Ψ intersects the plane at infinity along two imaginary conics. Hence any component of a v-section is either an ellipse or imaginary.

For a parabola k the surface Ψ touches the plane at infinity along a real conic. In the hyperbolic case Ψ has at infinity two real — perhaps coinciding — conics. The components c, c' of the v-section are in a certain relation to the meridians k, k': For non-intersecting meridians τ, B and B' are real; for intersecting k, k' the plane τ is imaginary and B, B' are complex conjugate. Hence also in these cases the components of a v-section are either imaginary or of the same type as k. □

The well-known Villarceau-section is the intersection of a ring torus with a double tangential plane. It consists of two congruent circles[3] according to the presented theorems (see Fig. 2). We consider a few other special examples:

- For a torus Ψ the points Q, Q' of intersection of the meridian circles k, k' are conjugate imaginary and at infinity. The double circle q of Ψ is the absolute circle. At a ring torus the double tangential plane τ touches at the real points B, B' and it intersects the absolute circle q at the points D, D'. Thus the v-section splits into two congruent circles that pass through the points B and B' of tangency and the absolute points.
- If k, k' are equilateral hyperbolas with axes parallel to a, their points Q, Q' of intersection are real and at infinity. Thus q is a real conic. When the inclination of τ is greater than 45° then the double points D, D' of v are real points at infinity; the v-section consists of two congruent hyperbolas (see Fig. 7). It turns out that otherwise v consists of imaginary curves.
- If k, k' are parabolas with axes perpendicular to a, then they share — apart from the finite points K, K' of intersection — their infinite point Q. The surface Ψ touches the plane at infinity along q. This shows that the v-section consists of two congruent parabolas with their axis perpendicular to a.

4. Analytical Treatment

Any v-section of a torus with a generating imaginary circle k is obviously imaginary. However, even an apple-shaped so-called *spindle torus* Ψ that intersects the rotation axis a has no real double tangential plane τ. Hence, the v-section consists of two imaginary circles. Such cases are noticeable as they provide the possibility of dealing with imaginary structures and their visualization. Imaginary structures can be handled well through analytical equations.

[2] Examples with the special case $Q = Q'$ are presented as Cases 7 and 8.

[3] The remarkable property that these circles are isogonal trajectories of the parallel circles on Ψ is not considered here (compare [5], vol. I, pp. 154–155).

In order to present the analytical treatment as simply as possible we choose a cartesian coordinate system $(O; x_1, x_2, x_3)$ with the axis a of rotation as x_3-axis. The generating conic k is specified in the x_2x_3-plane.

An analysis of the v-section of these surfaces of revolution shows a certain relationship among them that has become evident only after inclusion of imaginary elements. Only Case 9 shows an example with a generating ellipse.

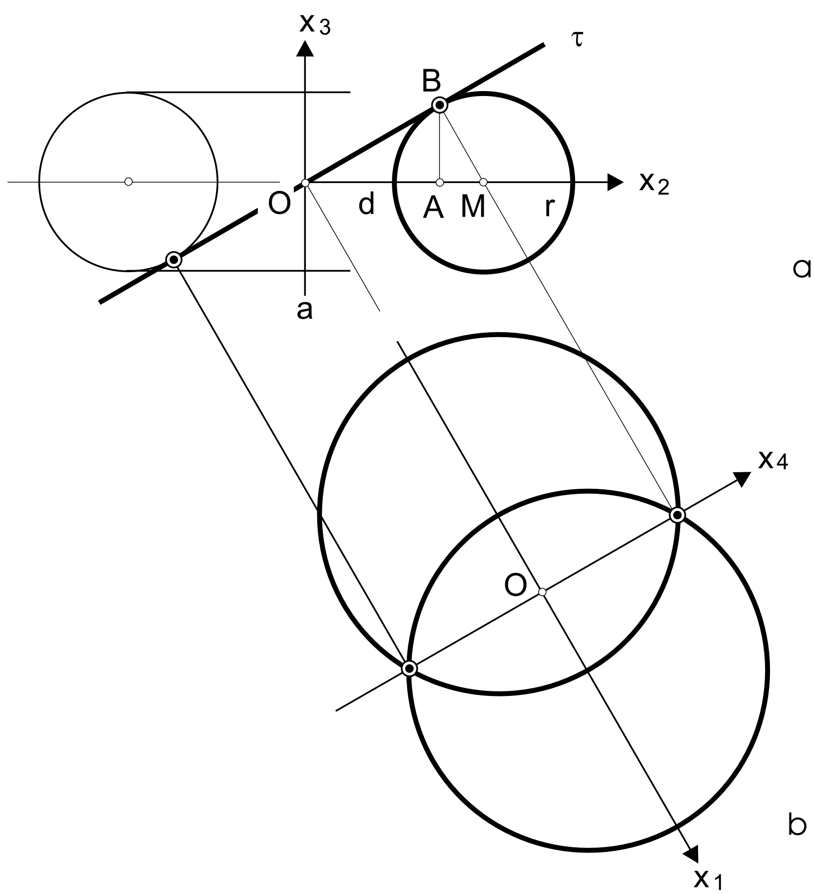

Figure 2: v-section of a ring torus (Case 1, $d : r = 2 : 1$)

Case 1: *Generating circle $k(M, r)$, $d > r$:*
In the equation of the meridian

$$k : \ (x_2 - d)^2 + x_3^2 - r^2 = 0 \tag{1}$$

we replace x_2^2 by $x_1^2 + x_2^2$ and get after separating the terms which are linear in x_2

$$\Psi : \ (x_1^2 + x_2^2 + x_3^2 + d^2 - r^2)^2 - 4d^2(x_1^2 + x_2^2) = 0. \tag{2}$$

A plane through the x_1-axis obeys the equation $x_3 = mx_2$. We substitute this in eq. (1) and obtain

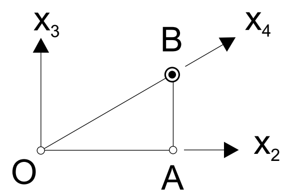

Figure 3: Position of the double tangential plane ($\triangle OAB$)

$$(m^2+1)x_2^2 - 2dx_2 + d^2 - r^2 = 0. \tag{3}$$

This plane is tangent to Ψ if the discriminant $D = d^2 - (m^2+1)(d^2-r^2)$ vanishes. This results in

$$\tau: \; x_3 = mx_2 \quad \text{with} \quad m = \frac{r}{\sqrt{d^2-r^2}}. \tag{4}$$

Due to (4) we eliminate x_3 in (2). After simplification the top view v_1 of v obeys

$$v_1: \; \left(x_1^2 + \frac{d^2}{d^2-r^2}\,x_2^2 + d^2 - r^2\right)^2 - 4d^2(x_1^2+x_2^2) = 0. \tag{5}$$

We introduce a cartesian coordinate system $(O; x_1, x_4)$ in τ (see Fig. 3 showing a front view). Then due to

$$x_4 = \sqrt{1+m^2}\,x_2$$

we obtain from (5) after some computation the equation of the v-section

$$v: \; \left[(x_1-r)^2 + x_4^2 - d^2\right]\left[(x_1+r)^2 + x_4^2 - d^2\right] = 0 \tag{6}$$

consisting of two circles with radius d (see Fig. 2).

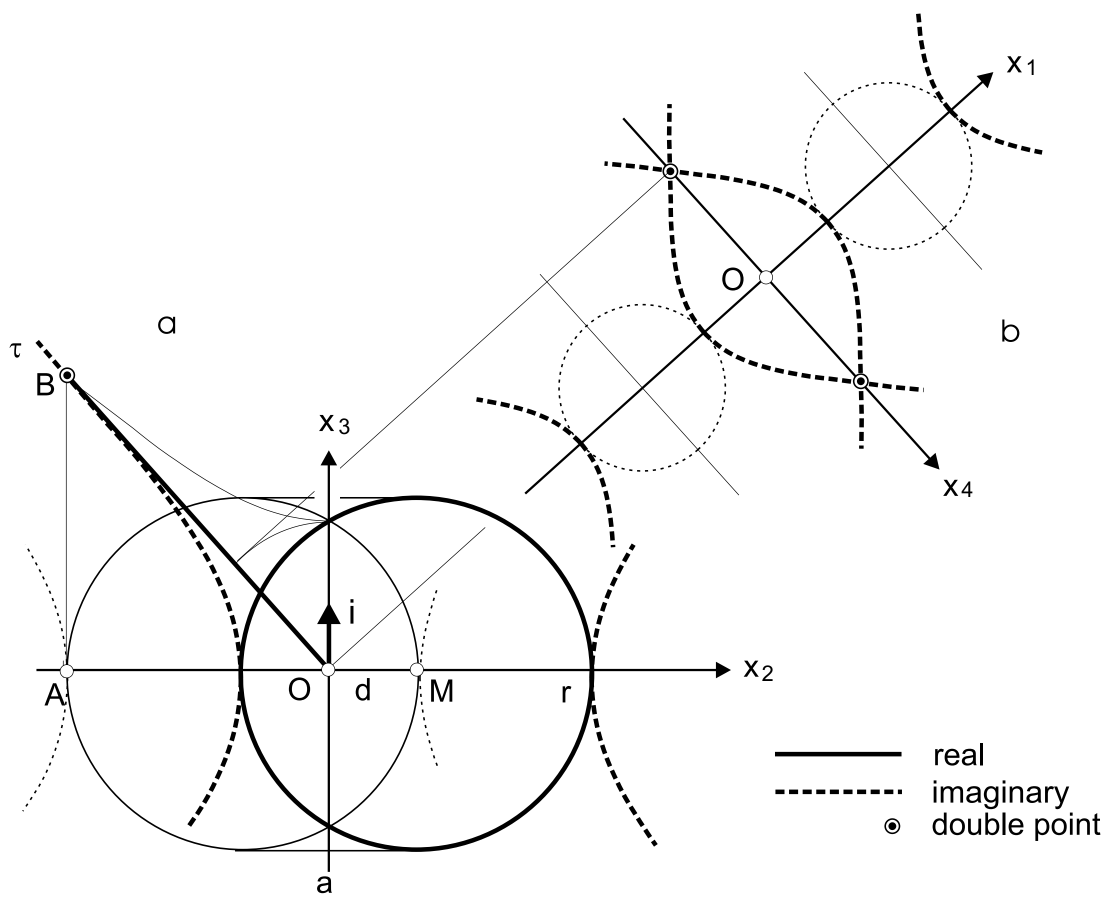

Figure 4: v-section of a dorn torus (Case 2, $d : r = 1 : 2$)

Case 2: *Generating circle* $k(M, r)$, $d < r$:
In this case the slope m of τ is imaginary according to (4). Fig. 4a shows the meridian section of Ψ in the x_2x_3-plane. But at the same time it shows also its image under the imaginary affine transformation $(x_2, x_3) \mapsto (x_2, ix_3)$ which transforms the meridian circles into equilateral hyperbolas. After this transformation τ has a real image.

Also the x_4-axis in τ is imaginary. Nevertheless the equation (6) shows two symmetric hyperbolas which are displayed in Fig. 4b as if x_4 would be a real axis. Note that this auxiliary view does not correspond to the imaginary transformed front view. Therefore there is no order line connecting the two views of the point B of contact.

Case 3: *Generating imaginary circle* $k(M, r = i\rho)$, $\rho \in \mathbb{R}$:
Again the slope m of τ is imaginary according to (4). The two circles of the v-section according to (6) have complex conjugate centers. The top view in Fig. 5a shows at the same time the image under the imaginary scaling $(x_2, x_3) \mapsto (x_2, ix_3)$. The auxiliary view in Fig. 5b shows the image under $(x_1, x_4) \mapsto (ix_1, x_4)$ which gives equilateral hyperbolas constituting the v-section.

Case 4: *Generating equilateral hyperbola* $k(M, r)$, $d > r$:
In analogy to Case 1 we compute

$$\begin{aligned} k:&\quad (x_2 - d)^2 - x_3^2 - r^2 = 0, \\ \Psi:&\quad (x_1^2 + x_2^2 - x_3^3 + d^2 - r^2)^2 - 4d^2(x_1^2 + x_2^2) = 0, \\ \tau:&\quad x_3 = mx_2, \quad m = ir/\sqrt{d^2 - r^2}. \end{aligned}$$

The top view of the v-section $\Psi \cap \tau$ obeys

$$v_1: \left(x_1^2 + \frac{d^2}{d^2 - r^2}\, x_2^2 + d^2 - r^2\right)^2 - 4d^2(x_1^2 + x_2^2) = 0.$$

We introduce in the complex plane τ a cartesian coordinate system $(O; x_1, x_5)$ (in the unitary sense). Due to

$$x_5 := \sqrt{1 - m^2}\, x_2$$

we replace x_2 in the equation of v_1 by x_5 and obtain

$$v: \left(x_1^2 + x_5^2 - d^2 + r^2\right)^2 - 4r^2x_1^2 = 0$$

which can be decomposed into

$$(x_1 \pm r)^2 + x_5^2 = d^2$$

describing two congruent circles in the imaginary plane. These circles are displayed in Fig. 6b.

Case 5: Generating hyperbola $k(M, r)$, $d < r$:
Following the computations of Case 4 we get a real plane τ (see Fig. 7a). v consists of two hyperbolas. When we set

$$x_5 := i\sqrt{1 - m^2}\, x_2 = \frac{d}{\sqrt{r^2 - d^2}}\, x_2,$$

then the projection of v into the x_1x_5-plane obeying

$$(x_1 \pm r)^2 - x_5^2 = d^2 \tag{7}$$

consists of two equilateral hyperbolas which are displayed in Fig. 7b.

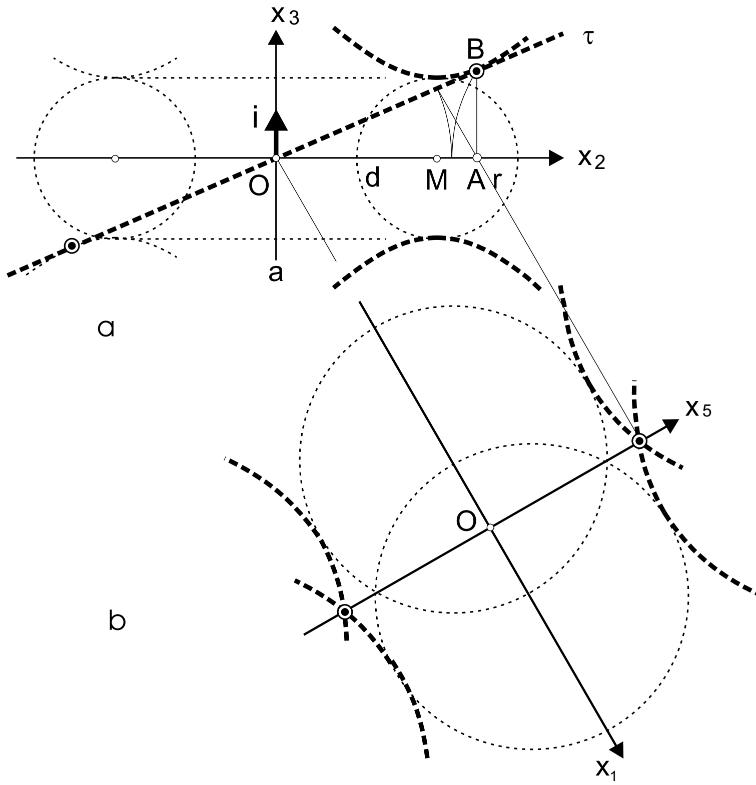

Figure 5: v-section of an imaginary torus (Case 3, $d : r = 2 : i$)

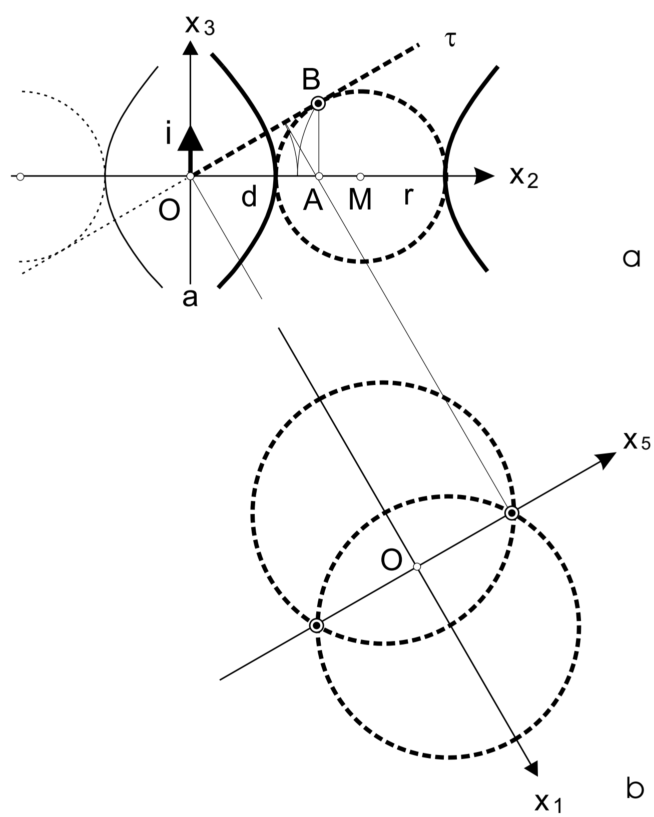

Figure 6: v-section of a surface of revolution with generating hyperbola, rotation axis is intersected (Case 4, $d : r = 2 : 1$)

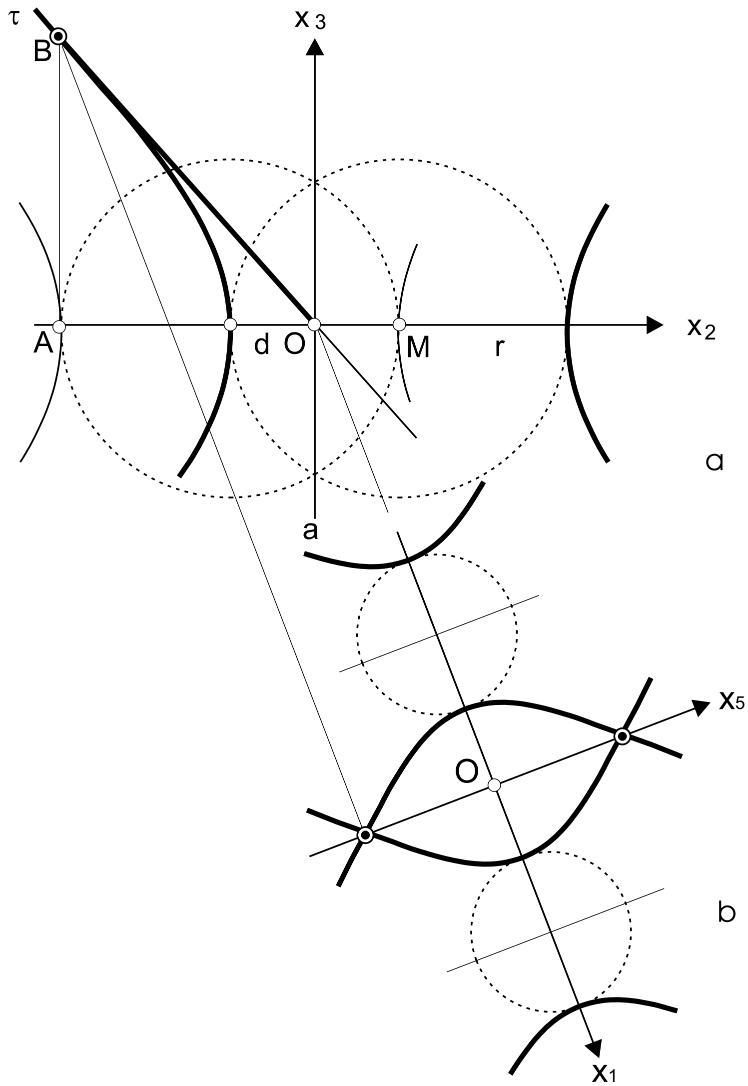

Figure 7: v-section of a surface of revolution with generating hyperbola, rotation axis is not intersected (Case 5, $d : r = 1 : 2$)

Case 6: *Generating hyperbola* $k(M, r = i\rho)$, $\rho \in \mathbb{R}$:
τ is a real plane. Eq. (7) represents two imaginary circles with complex conjugate centers. Fig. 8b shows their image under the imaginary scaling $(x_1, x_5) \mapsto (ix_1, x_5)$.

Case 7: Generating parabola k with $p, d > 0$:
In the same way as in previous cases we get

$$\begin{aligned} k:&\quad x_3^2 = 2p(x_2 - d),\\ \Psi:&\quad (x_3^2 + 2pd)^2 - 4p^2(x_1^2 + x_2^2) = 0,\\ \tau:&\quad x_3 = mx_2, \quad m = \sqrt{p/2d}. \end{aligned}$$

The top view of the intersection $\tau \cap \Psi$ obeys

$$v_1: \ \left(x_2^2 - 4d^2\right)^2 - 16d^2x_1^2 = 0$$

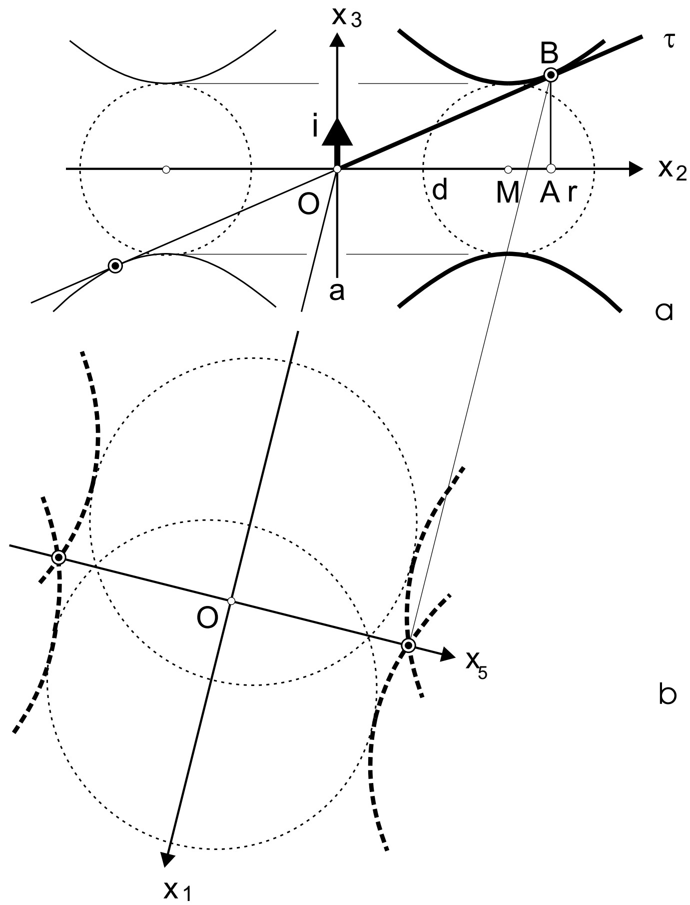

Figure 8: v-section of a surface of revolution with generating hyperbola, rotation axis is intersected (Case 6, $d : r = 2 : i$)

which can be decomposed into the equations

$$x_2^2 = \pm 4d(x_1 \pm d) \tag{8}$$

representing two parabolas (see Fig. 9) which turn out to be independent from the initial parameter p.

Case 8: *Generating parabola k with $p > 0$, $d < 0$:*
This time the double tangential plane τ is imaginary and eq. (8) represents two imaginary parabolas. Fig. 10b shows their image under an imaginary scaling.

We omit here the trivial cases where the conic k degenerates.

Remarks The eight presented cases can be arranged in pairs which are corresponding under the imaginary affine transformation

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, ix_3).$$

The pairs are 1–4, 2–5, 3–6, and 7–8. What is real in one case that is imaginary in the corresponding case. This can also be seen by comparing the corresponding Figures 2–6, 4–7, 8–5, and 9–10.

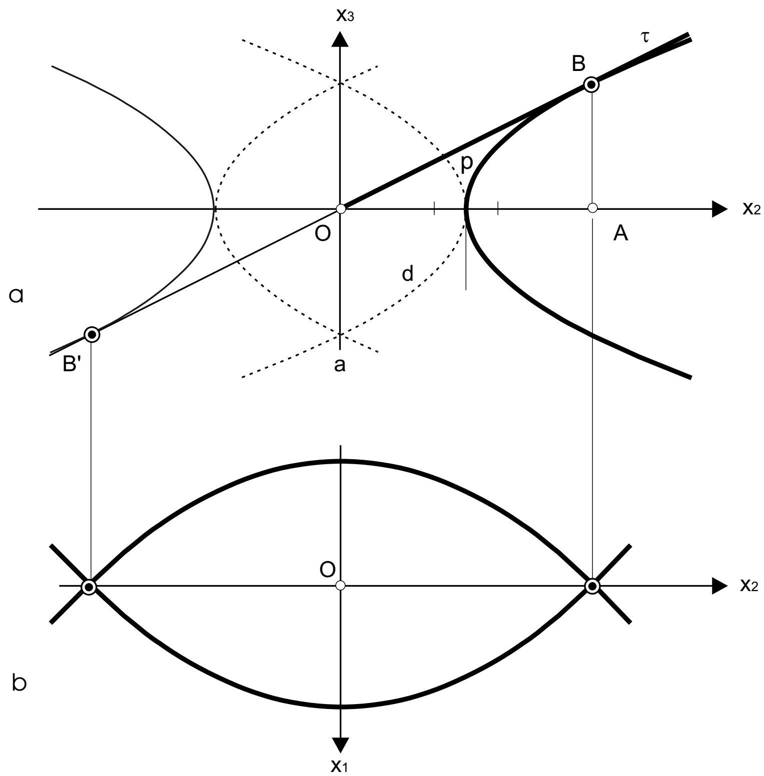

Figure 9: v-section of a surface of revolution with generating parabola, rotation axis is not intersected (Case 7, $d : p = 2 : 1$)

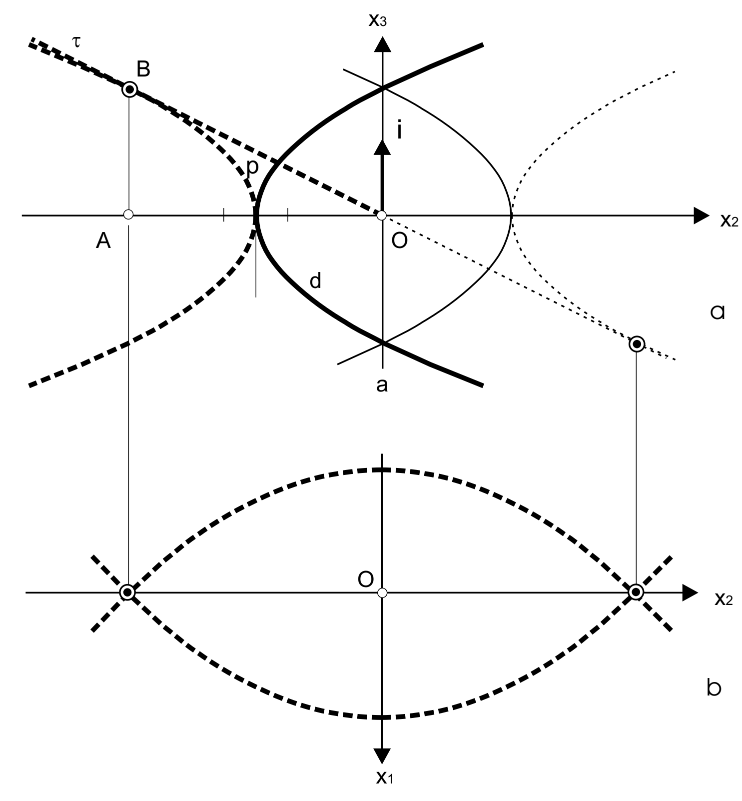

Figure 10: v-section of a surface of revolution with generating parabola, rotation axis is intersected (Case 8, $d : p = -2 : 1$)

Case 9: *Generating meridian ellipse k in general position:*

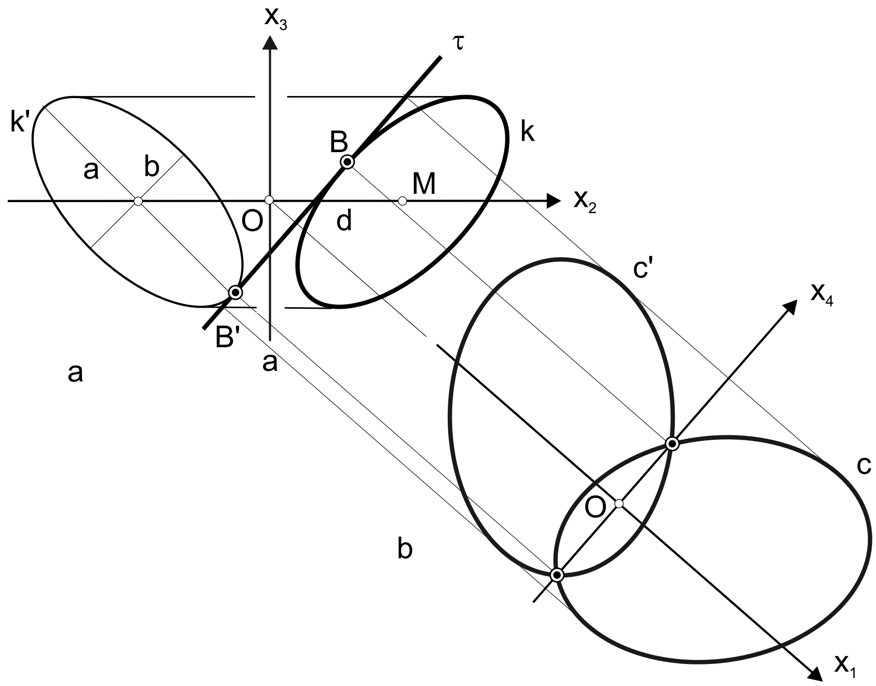

Figure 11: υ-section of Ψ with a meridian ellipse k in general position (Case 9)

As an example we specify k as an ellipse with semiaxes $a = 2$ and $b = 1$. The principal axis is rotated under $\varphi = 45°$ and the center of the ellipse has the distance $d = 2$ to the axis (see Fig. 11a). We proceed in the same way as in Case 1 and obtain the equations:

$$k: \; 5x_2^2 - 6x_2x_3 + 5x_3^2 - 20x_2 + 12x_3 + 12 = 0,$$

$$\Psi: \; \left(5x_1^2 + 5x_2^2 + 5x_3^2 + 12x_3 + 12\right)^2 - (6x_3 + 20)^2(x_1^2 + x_2^2) = 0,$$

$$\tau: \; x_3 = \sqrt{\tfrac{31}{24}}\, x_2 - 3/4 = 1.13652\, x_2 - 0.75\,.$$

The top view υ_1 of the υ-sections obeys

$$\begin{aligned}\upsilon_1: \quad & 1.0 + 1.75977\, x_2 - 5.39068\, x_1^2 - 2.39427\, x_2^2 - 4.74317\, x_1^2 x_2 - 2.78787\, x_2^3 + \\ & + 0.73997\, x_1^4 + 2.01518\, x_1^2 x_2^2 + 2.50978\, x_2^4 = 0.\end{aligned}$$

We substitute

$$x_2 = 0.66057\, x_4$$

and obtain

$$\begin{aligned}\upsilon: \quad & 1.0 + 1.1624\, x_4 - 5.39068\, x_1^2 - 1.04477\, x_4^2 - 3.13323\, x_1^2 x_4 - 0.8036\, x_4^3 + \\ & + 0.73997\, x_1^4 + 0.87935\, x_1^2 x_4^2 + 0.47789\, x_4^4 = 0\end{aligned}$$

This describes two congruent ellipses (see Fig. 11b) with equations

$$0.86021\, x_1^2 + 0.55676\, x_1 x_4 + 0.6913\, x_4^2 \pm 1.91579\, x_1 - 0.58123\, x_4 - 1.0 = 0\,.$$

5. Conclusions

In the present paper we have performed in a simple way an extension of the term 'Villarceau-section' by replacing the ring torus by surfaces of revolution with conics as meridians. There are many other ways for a generalization. It is possible, for example, to specify a generating conic section not in a meridian plane or to transform the surface of revolution into cyclides.

References

[1] R. Bereis: *Darstellende Geometrie I.* Akademie-Verlag, Berlin 1964.

[2] G. Fischer: *Plane Algebraic Curves.* American Mathematical Society, Providence 2001.

[3] K. Strubecker: *Vorlesungen über Darstellenden Geometrie.* Vandenhoeck & Ruprecht, Göttingen 1967.

[4] G. Weiss: *Villarceau-Kreise des Ringtorus, ein elementarer Zugang.* IBDG **2** (1984).

[5] W. Wunderlich: *Darstellende Geometrie I.* BI-Hochschultaschenbuch Bd. 96, Bibliographisches Institut, Mannheim 1966.

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