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Golden Hexagons

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Dedicated to Prof. Hellmuth Stachel on the occasion of his 60$^{\text{th}}$ birthday

Abstract. A "golden hexagon" is a set of six points, which is projectively equivalent to the vertices of a regular pentagon together with its center. Such a geometric figure generalizes in some sense the classical one-dimensional golden section to two dimensions.
This paper deals with some remarkable properties of golden hexagons and with special Euclidean representatives as well as with further generalizations.

Keywords: golden section, golden ratio, golden cross ratio, geometrically defined iterative processes, Desargues' Theorem, polarity with respect to a conic, Moebius circle geometry, bio-geometry, regular polyhedra

MSC 2000: 51M04, 51M05, 51M20

1. An iterative process based on the polarity with respect to a conic

In the real projective plane π let a set $\mathcal{P}$ of six arbitrarily chosen points P_j be given such that each selected quintuple $\mathcal{P} \setminus \{P_i\}$, $i = 1, \ldots, 6$, defines a regular conic c_i, and not all six points belong to the same conic. We proceed as follows:

- In a first step we polarize P_i at c_i for $i = 1, \ldots, 6$. This gives a set $\mathcal{L}$ of six lines $l_1, \ldots, l_6$ in π. In general, each quintuple of these lines again defines a (dual) conic and its polarity.
- Another polarizing step — dual to the first step — produces again a hexagon $\mathcal{P}'$.

Obviously, step one and its dual start an *iteration process*

$$\mathcal{P} \mapsto \mathcal{P}' \mapsto \mathcal{P}'' \mapsto \ldots$$

in π which is connected to the first hexagon $\mathcal{P}$ in a projectively invariant way.

There the question arises: To what extent does an eventually existing *limit hexagon* $\mathcal{P}^\infty$ depend on the initial figure $\mathcal{P}$, and do there exist additional characterizing properties of $\mathcal{P}^\infty$?

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An analytical treatment can be based on projective coordinates [5], [4]

$$\mathbb{R}(x_0^{(i)}, x_1^{(i)}, x_2^{(i)}) \text{ of } P_i \in \pi, \quad (y_0^{(i)}, y_1^{(i)}, y_2^{(i)})\mathbb{R} \text{ of } l_i \subset \pi, \quad i = 1, \ldots, 6. \tag{1}$$

E.g., the equation

$$a_{00}^{(6)} x_0^2 + a_{11}^{(6)} x_1^2 + a_{22}^{(6)} x_2^2 + 2a_{01}^{(6)} x_0 x_1 + 2a_{02}^{(6)} x_0 x_2 + 2a_{12}^{(6)} x_1 x_2 = 0 \tag{2}$$

describes the conic c_6 defined by the first five points P_i, whereby the coefficients $a_{jk}^{(6)}$ of (2) result from the system of linear equations

$$\begin{array}{c|cccccc|c}
x & x_0^2 & x_1^2 & x_2^2 & x_0x_1 & x_0x_2 & x_1x_2 & \\
\hline
P_1 & (x_0^{(1)})^2 & \ldots & & x_0^{(1)}x_0^{(1)} & \ldots & x_1^{(1)}x_2^{(1)} & 0 \\
P_2 & (x_0^{(2)})^2 & & & & & x_1^{(2)}x_2^{(2)} & 0 \\
\vdots & \vdots & & & & & \vdots & \vdots \\
P_5 & (x_0^{(5)})^2 & & & & & x_1^{(5)}x_2^{(5)} & 0 \\
\hline
 & a_{00}^{(6)} & a_{11}^{(6)} & a_{22}^{(6)} & a_{01}^{(6)} & a_{02}^{(6)} & a_{12}^{(6)} &
\end{array} \tag{3}$$

The coefficients $a_{jk}^{(6)}$ are the six main minors of the 6×5-matrix (3).

Finally, the polar line l_6 to point P_6, with respect to the conic c_6 has (dual) projective coordinates

$$(y_0^{(6)}, y_1^{(6)}, y_2^{(6)}) = \begin{pmatrix} a_{00}^{(6)} & a_{01}^{(6)} & a_{02}^{(6)} \\ a_{01}^{(6)} & a_{11}^{(6)} & a_{12}^{(6)} \\ a_{02}^{(6)} & a_{12}^{(6)} & a_{22}^{(6)} \end{pmatrix} \cdot \begin{pmatrix} x_0^{(6)} \\ x_1^{(6)} \\ x_2^{(6)} \end{pmatrix}. \tag{4}$$

Thus, the first step ends with line-coordinates $y_k^{(i)}$ which are forms of degree 11 in the coordinates of the given points $P_1, \ldots, P_6$. Therefore, by the analogous dual step, the coordinates of a point P_i' are forms of degree 121 in the coordinates of the initial points P_j. Consequently, there hardly seems to be any chance to describe the iteration process and its limit figure $\mathcal{P}^\infty$ by means of computer aided algebraic manipulations.

1.1. Some experiments

Numerical treatment shows that for any (general) initial hexagon $\mathcal{P}$ there exists a "limit" figure $\mathcal{P}^\infty$ (see Fig. 1). We will show that all the limiting figures of different initial figures are projectively equivalent (see Section 2).

Obviously, a hexagon $\mathcal{P}$ consisting of six points of one conic, — we call such a hexagon Pascal-hexagon, as the six points obey Pascal's condition (c.f. [5]) — should be fixed under the iterated polarizing process IPP described above. Pascal-hexagons turn out to be *repulsive* initial figures (Fig. 2).

An initial hexagon $\mathcal{P}$ with (Euclidean) symmetries will lead to derived hexagons $\mathcal{P}'$, $\mathcal{P}''$, … with the same symmetries. Especially a regular pentagon together with its center is *fixed* under IPP. This can be shown by elementary geometric arguments (Fig. 3).

Another special Euclidean representative $\mathcal{P}_{\text{tri}}$ of an *attractive fixed hexagon* with a three-fold symmetry consists of two concentric equilateral triangles with a side length ratio of

$$s_1 : s_2 = -1 : \varphi^4, \quad (\varphi = 0{,}618...). \tag{5}$$

Fig. 4 shows remarkable incidences of sides of $\mathcal{P}_{\text{tri}}$:

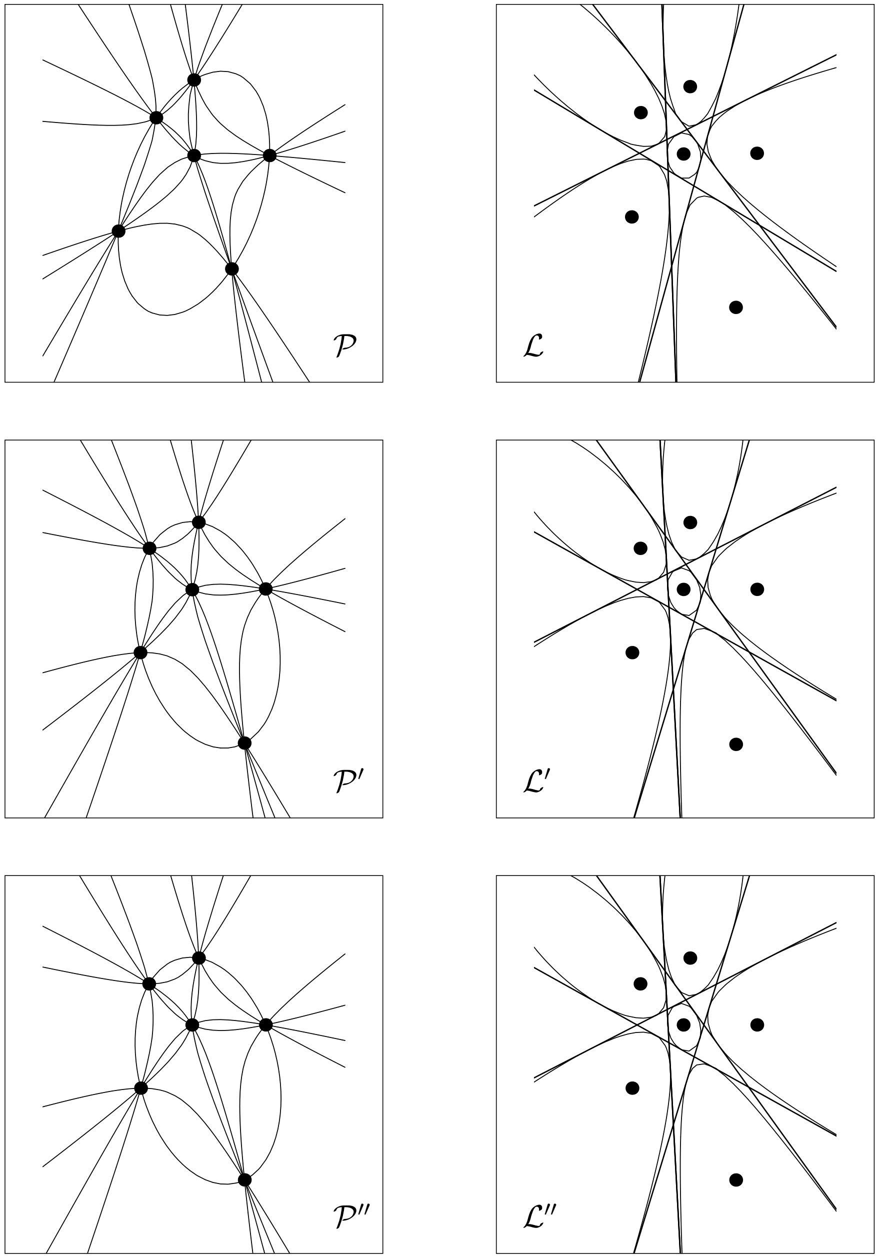

Figure 1: First six steps of an iterated polarizing process IPP applied to the initial hexagon $\mathcal{P}$

- The point A of intersection between any side of the "inner" triangle and a side of the "outer" one divides the outer (and the inner) side according to the golden section.

- Connecting A with any point of $\mathcal{P}_{\text{tri}}$ gives a line containing two points of $\mathcal{P}_{\text{tri}}$.

So, in spite of the fact that collinear transformations in π do not preserve ratios (only cross-ratios), a perspective collineation transforming $\mathcal{P}_{\text{tri}}$ into $\mathcal{P}_{\text{pent}}$ (see Fig. 5) seemingly preserves the *"golden proportions"*. So it seems worthwhile to take a closer look to golden proportions from the projective point of view (see Section 3). Because of their golden proportions we will call $\mathcal{P}_{\text{pent}}$, $\mathcal{P}_{\text{tri}}$ and all their projectively equivalent hexagons $\mathcal{P}^{\infty}$ "golden hexagons" $\mathcal{P}_{\text{gold}}$.

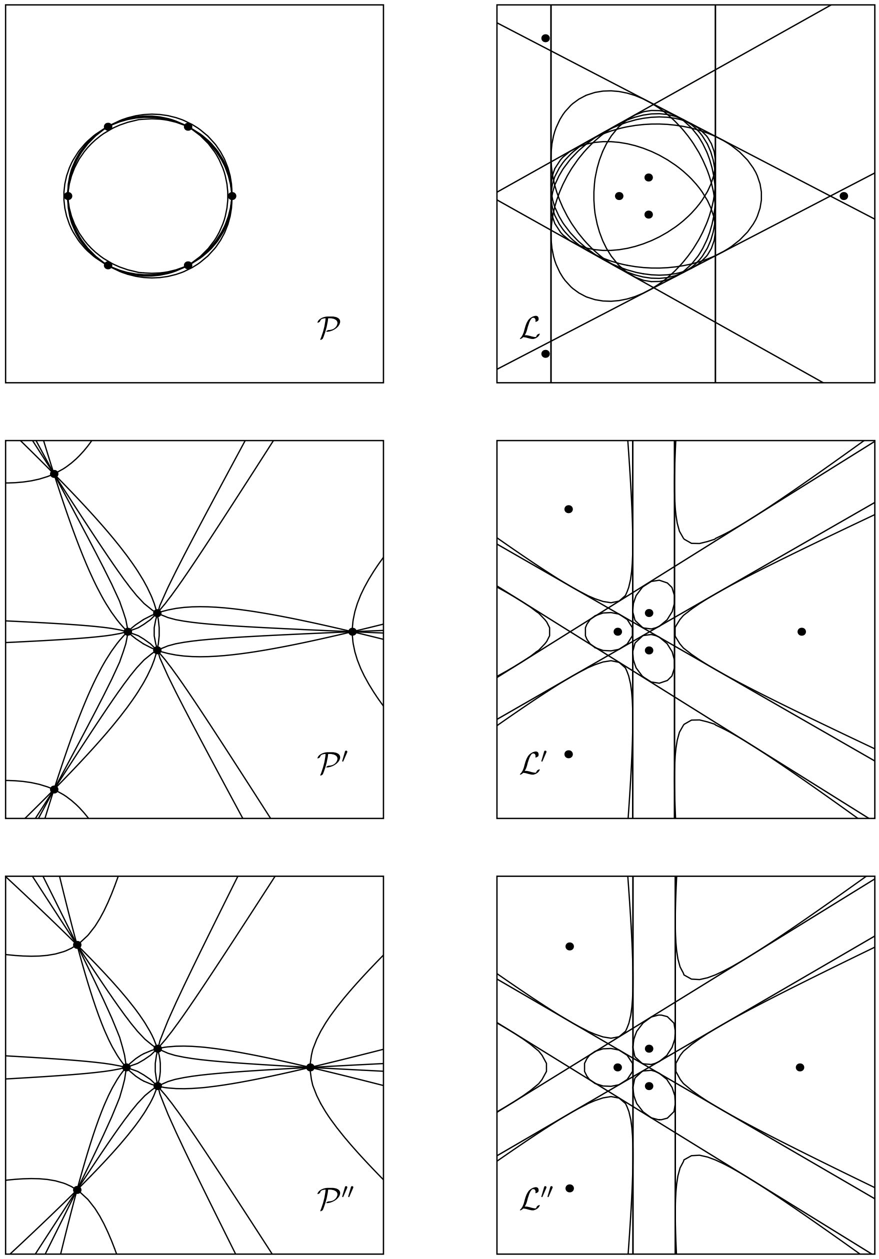

Figure 2: The iterated polarizing process IPP applied
to an initial hexagon $\mathcal{P}$, which is almost a Pascal-hexagon

2. Projective geometric properties of golden hexagons

One might "standardize" the iterative polarizing process IPP by adding a collinear transformation κ to each pair of dual steps, which maps points $P_1', \ldots, P_4'$ onto $P_1, \ldots, P_4$, respectively. This means that the "extended IPP" keeps the first four points of $\mathcal{P}$ fixed. In the following these points will serve as the projective coordinate frame in π. So the question arises: To which limit points P_5^∞, P_6^∞ will an arbitrarily chosen pair of points P_5, P_6 tend?

Fig. 6 shows that, according to the initial position of the pair (P_5, P_6), there are six possibilities (P_5^∞, P_6^∞) completing $\{P_1, \ldots, P_4\}$ to a limit hexagon $\mathcal{P}^\infty$, and we learn from the coordinates of (P_5^∞, P_6^∞) that $\mathcal{P}^\infty$ is always a golden hexagon. Experiments show that ac-

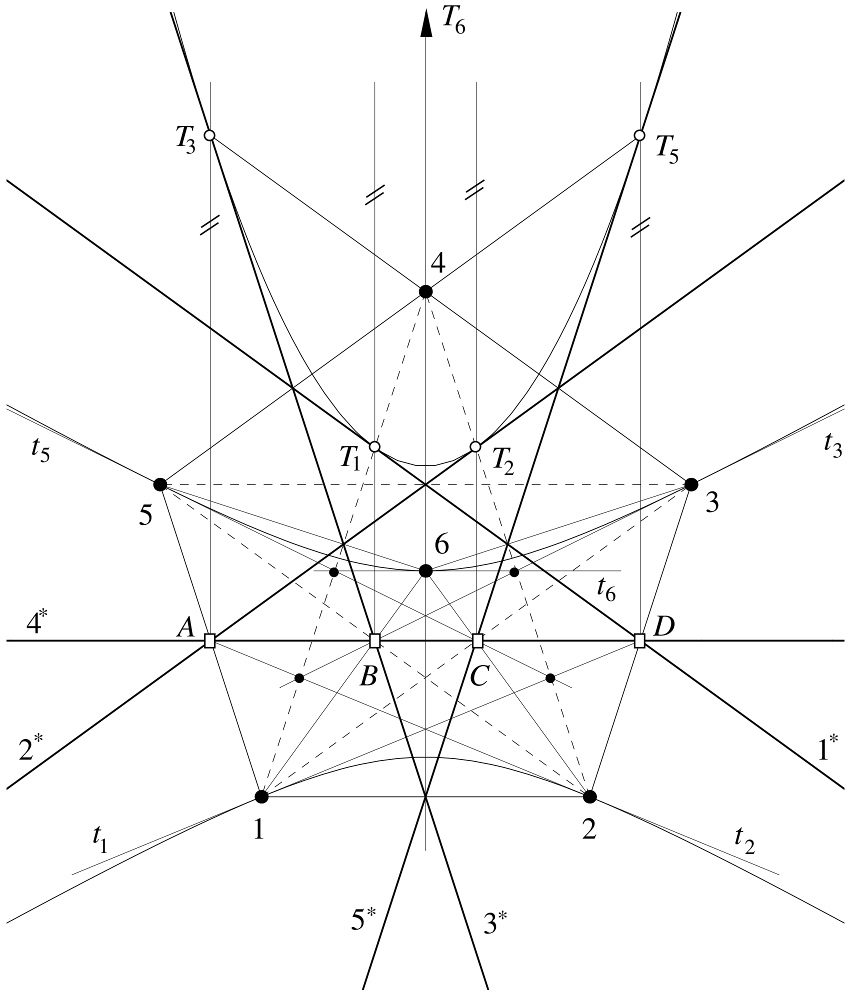

Figure 3: A regular pentagon and its center as an *attractive fixed hexagon* $\mathcal{P}_{\text{pent}}$ under the iterative polarizing process IPP, together with the conic c_4 and its polar parabola c_4^*

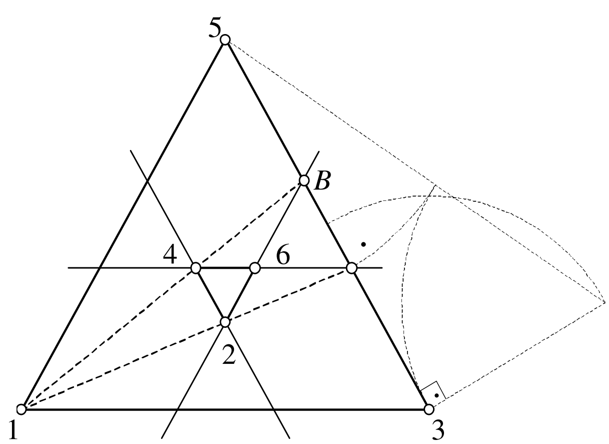

Figure 4: The attractive fixed hexagon $\mathcal{P}_{\text{tri}}$ and its golden proportions

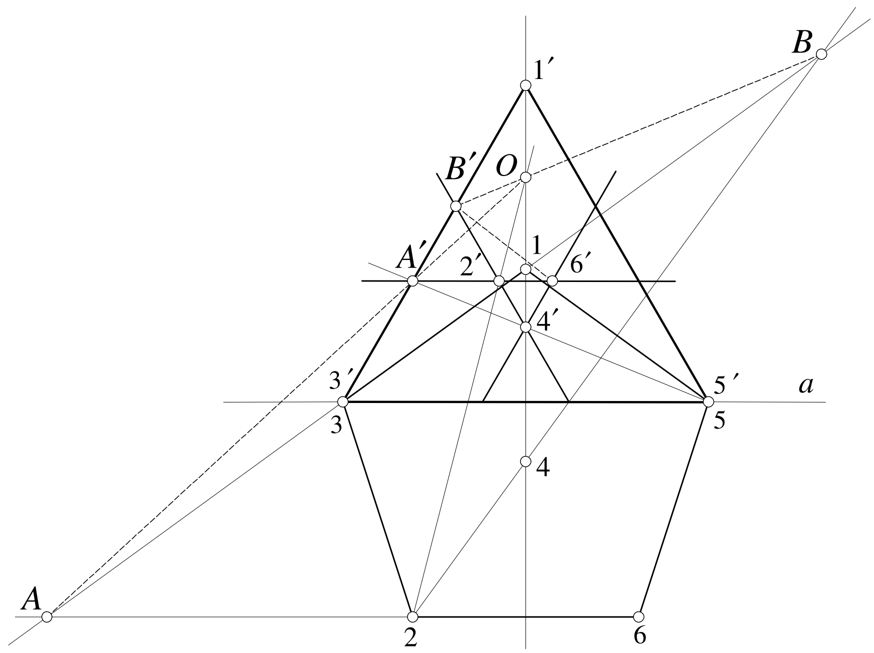

Figure 5: Perspective collinear transformation of $\mathcal{P}_{\text{pent}}$ into $\mathcal{P}_{\text{tri}}$ (center O and axis a)

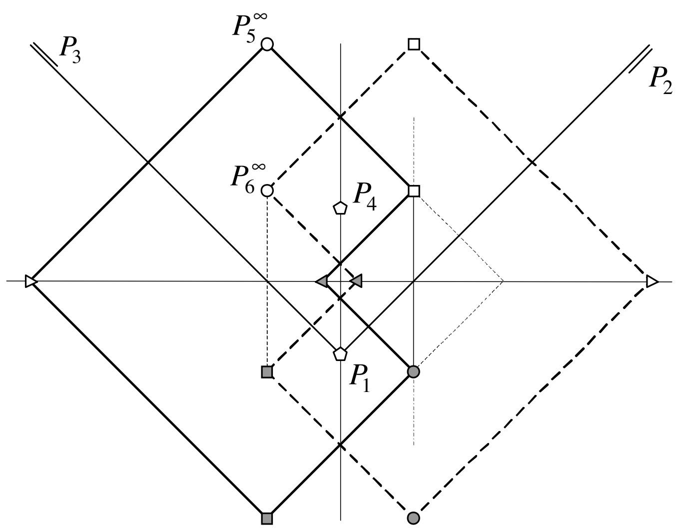

Figure 6: Limit figures $\mathcal{P}_{\text{gold}}$ of an extended iterative polarizing process IPP applied to a hexagon $\mathcal{P}$ with four points of $\mathcal{P}$ remaining fixed at all steps of IPP

cording to the initial position of the randomly chosen pair (P_5, P_6) there occur six possibilities for the pair of limit points (P_5^∞, P_6^∞), each point having "golden" coordinates. Combinatorial arguments for choosing start positions support the

Conjecture *that there exist exactly these six pairs of limit points and that* $P_1, \ldots, P_4, P_5^\infty, P_6^\infty$ *in any case form a golden hexagon.*

In Fig. 6 the pairs of limit points are marked by the same symbol and one read off their coordinates that the set of all possibilities P_5^∞ resp. P_6^∞ each forms a golden hexagon, too. These new golden hexagons can be based on a square as follows:

Let three vertices of a square be the first three points of a golden hexagon $\mathcal{P}_{\text{gold}}$. Divide the sides having the fourth vertex in common in accordance to the golden ratio, the "minor" closer to that fourth vertex, and receive so the 4th and 5th point of $\mathcal{P}_{\text{gold}}$. Connect these points by a line and reflect the fourth vertex of the square at this line receiving finally the 6th point of $\mathcal{P}_{\text{gold}}$. Note that this special Euclidean representative of $\mathcal{P}_{\text{gold}}$ again shows golden ratios in spite of its projective origin!

As a main result we state:

Theorem 1 *Each iterative polarizing process IPP has a golden hexagon as attractor $\mathcal{P}^\infty$, which is projectively equivalent to $\mathcal{P}_{pent}$.*

Golden hexagons have another remarkable property, which is characteristic with respect to the group of projective transformations:

Theorem 2 *Each partition of a golden hexagon into two triangles allows a homology from one triangle to the other, i.e., all pairs of triangles are in "Desarguesian position". Conversely, all hexagons with this property are golden.*

This statement is obviously true for $\mathcal{P}_{\text{pent}}$ (and $\mathcal{P}_{\text{tri}}$). Consequently it is true for every projectively equivalent hexagon. Golden hexagons are therefore the six-points-configurations in the plane π, with the maximum number of pairs of triangles in Desarguesian position. Each of the 20 pairs defines two homologies, but all the homologies together give rise to only 10 centers.

3. Golden section, golden ratio, golden cross ratio, golden objects

The "golden section" is a well known concept since the classical hellenistic period. It dates back to ancient Egypt and is used in Architecture and Fine Arts. But also biologists point out that some laws of branching and of composite flowers and cones obey Fibonacci-series and the golden section. The related literature is immense, so we will restrict ourselves to cite [2], [7] and [6] as references.

In the following, our goal is to give a projective-geometric characterization of what is usually called the golden section φ:

Historically, φ is a geometric figure of three collinear points A, X, B, such that

(G1) *the length ratio of the minor segment to the major segment equals the ratio of the major part to the total segment $[A, B]$.*

This ratio takes the well known numerical value $\varphi = 0.618\ldots$, and, of course, there are two symmetric partitions of $[A, B]$ with this property (Fig. 7):

As long as we deal with unoriented segments, the points C and D (Fig. 7) obey the condition (G1). But as soon as we use ordered segments, by defining

$$\varphi := \text{dist}[A, X] : \text{dist}[A, B] = \text{dist}[X, B] : \text{dist}[A, X] \tag{6}$$

the ratio φ fulfils the quadratic equation (see e.g. [2])

$$\varphi^2 + \varphi - 1 = 0\,. \tag{7}$$

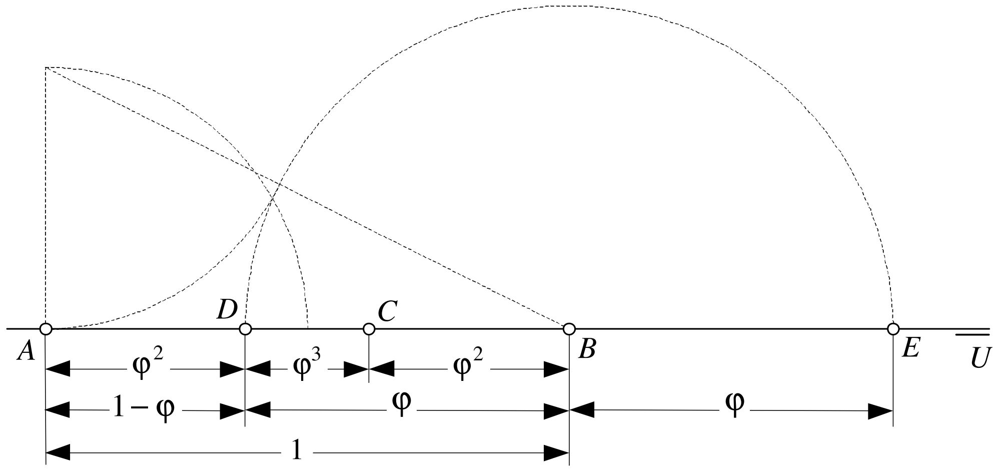

Figure 7: Golden section, golden ratio, golden cross ratio

The two signed solutions are $\varphi = -1.618...$ and $+0.618...$, and they lead to an inner (C) and an outer (E) subdividing point of $[A, B]$. So, from a mathematical point of view, it seems natural to apply the (affinely invariant) concept of "ratio" $\mathrm{r}(A, B, C)$ to the triple of collinear points A, B, C. Note that $\mathrm{r}(X, A, B) = x$ results in the coordinate representation of X on the line AB based on a coordinate frame with origin A and unit point B (see e.g. [4(1)]).

Thus, a reformulation of (G1) based on the concept of the ratio, should read as follows:

(G2) *The ratio of three collinear points A, B, C is called a golden ratio, iff $r(A, B, C) = -r(B, C, A)$.*

In the projective extension of the (real) affine line AB the ratio of three points A, B, C is represented by the cross ratio $\mathrm{cr}(A, B, C, U)$ of these three points together with the ideal point U of AB. (G2) stimulates the question to what extent a quadruple of collinear points defining a cross ratio of say 1.618... is special? E.g., in Fig. 7 the points A, B, C, D define this cross ratio $\mathrm{cr}(A, D, C, B) = 1.618...$.

Remark. "Cross ratio" is a (projectivly invariant) concept of *ordered* collinear quadruples of points. The 24 permutations of $\{A, B, C, D\}$ lead to only six (in general different) cross ratio values, namely

$$x, \quad \frac{1}{x}, \quad 1 - x, \quad \frac{1}{1-x}, \quad \frac{x-1}{x}, \quad \frac{x}{x-1}, \quad (x \in \mathbb{R} \setminus \{0, 1\}), \tag{8}$$

while e.g.

$$\mathrm{cr}(A, B, C, D) = \mathrm{cr}(B, A, D, C) = \mathrm{cr}(D, C, B, A) = \mathrm{cr}(C, D, A, B) = x, \tag{9}$$

(c.f. [4(2)]).

It seems natural to ask for quadruples $\{A, B, C, D\}$ which define less than the six cr-values (8). More generally we ask for quadruples $\{A, B, C, D\}$ having at least two equal or absolutely equal cr-values (8).

Case 1: *Equal (nontrivial and real)* cr-*values occur exactly for harmonic quadruples. There are only three* cr-*values, namely* $x = -1,\ 2,\ \frac{1}{2}$.

Case 2: *Negatively equal (nontrivial and real)* cr-*values occur exactly for golden quadruples. The* cr-*values are* $x = \varphi = 0.618...,\ \frac{1}{\varphi},\ \varphi^2,\ \frac{1}{\varphi^2},\ -\varphi,\ -\frac{1}{\varphi},\ (\varphi^2 = 1 - \varphi)$.

Other cases with $x \in \mathbb{R} \setminus \{0,1\}$ do not exist!

Theorem 3 *Harmonic and golden quadruples of (unordered) collinear points are projectively characterized by the fact that for their possibly six cross ratios (8) (at least) two of them have the same absolute value.*

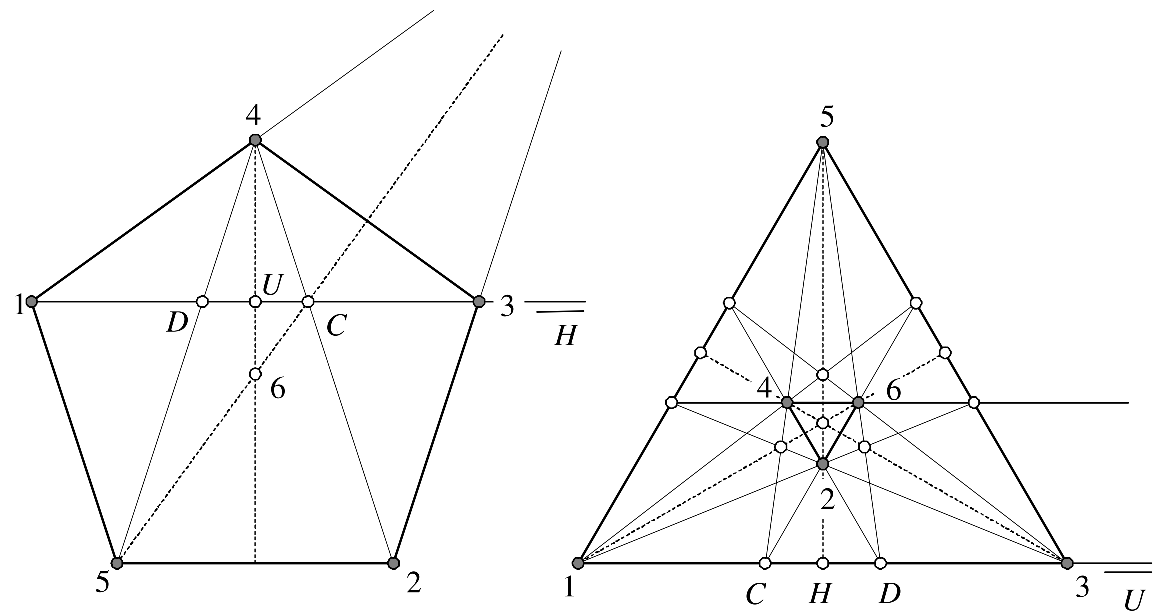

Figure 8: Golden quadruples with golden cross ratio occurring in golden hexagons $\mathcal{P}_{\text{pent}}$ and $\mathcal{P}_{\text{tri}}$

From Fig. 8 we deduce that a golden hexagon consists of an immense set of golden quadruples and therefore it is justified to see $\mathcal{P}_{\text{gold}}$ as a two-dimensional extension of the classical golden section.

Remark. The "invariance" of golden sections under the collinear transformation from $\mathcal{P}_{\text{pent}}$ to $\mathcal{P}_{\text{tri}}$ (Fig. 5) together with the important meaning of the golden section for branching and flower forms, makes the coexistence of flowers with fivefold and threefold symmetry geometrically "understandable".

4. Golden hexagons as images of regular polyhedra

Let $\mathcal{O}$ be a *regular octahedron* positioned like an antiprism on a (horizontal) facet plane π. Then there exist two central projections ξ_1, ξ_2 onto π with centers Z_1, Z_2, resp., such that the vertices of $\mathcal{O}$ are mapped onto the pointset of a golden hexagon $\mathcal{P}_{\text{tri}}$ (see Fig. 9a). These two centers Z_i are located on the "axis" of the antiprism $\mathcal{O}$ ($=z$-axis) symmetrically to the center M of $\mathcal{O}$ with

$$\operatorname{dist}(M, Z_i) = \frac{1}{\sqrt{6}} \cdot \frac{1+\varphi^4}{1-\varphi^4} = \ \ldots \ = \frac{\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}}{10}. \tag{10}$$

(For the final calculation we use $\varphi^2 = 1-\varphi$ and $2\varphi = \sqrt{5}-1$ and a unit edge length of $\mathcal{O}$.)

As $\mathcal{O}$ is an antiprism in four different ways, we can state the

Theorem 4 *For any regular octahedron $\mathcal{O}$ there exist eight projection centers Z such that the projection ξ_i of $\mathcal{O}$ onto a facet plane is a golden hexagon $\mathcal{P}_{tri}$. These eight centers Z_i are the vertices of a cube polar to $\mathcal{O}$ with respect to a concentric sphere with radius $\sqrt{1/(2\sqrt{5})}$.*

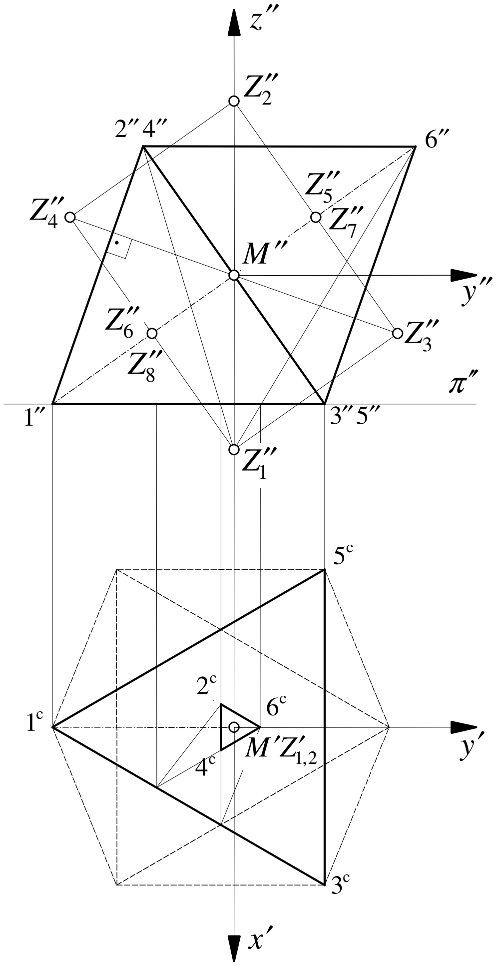

Figure 9a: Central projection mapping a regular octahedron $\mathcal{O}$ onto a golden hexagon $\mathcal{P}_{\text{tri}}$ (top and front view). The points Z_i, $i = 1, \ldots, 8$, are the possible centers of such projections.

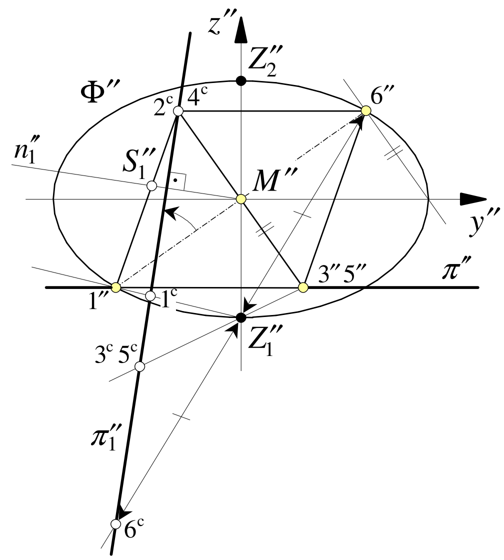

Figure 9b: The central projection with center Z_1 and image plane π_1 maps the regular octahedron $\mathcal{O}$ onto a golden hexagon $\mathcal{P}_{\text{pent}}$ (i.e., a *regular pentagon* plus its center) (front view with π_1 as projecting plane). The obtuse ellipsoid Φ with poles Z_1, Z_2 is circumscribed $\mathcal{O}$.

Remark 1. The perspective collineation between $\mathcal{P}_{\text{tri}}$ and $\mathcal{P}_{\text{pent}}$ (see Fig. 5) implies that the "projection pyramid" $Z_1 \vee \mathcal{O}$ can be intersected by planes such that there occur regular pentagons. Fig. 9b shows such a plane π_1 corresponding to the center Z_1 and passing through an edge of $\mathcal{O}$. It turns out that π_1 makes 45° with the diagonal 16 of $\mathcal{O}$.

As any plane parallel to π_1 (besides the one through Z_1) suits as image plane such that $\xi_1(\mathcal{O}) = \mathcal{O}^c$ is a $\mathcal{P}_{\text{pent}}$, we represent π_1 by its normal vector $\mathbf{n}_1$. Because of the threefold symmetry of $\mathcal{O}$ with respect to the axis MZ_1 there are three such direction vectors corresponding to Z_1 and they are the same for Z_2, too. Therefore there are altogether 12 such normal vectors $\mathbf{n}_j$, and they intersect the facets of $\mathcal{O}$ three by three at the vertices of equilateral triangles. A dilation with factor $\delta = \frac{3}{2}\sqrt{2} - 2 \sim 0,1\ldots$ maps these triangles to the facets of $\mathcal{O}$. The convex hull of these small triangles is a "trimmed cube" consisting of 8 squares (with side length $\sqrt{2}-1$), together with the 8 equilateral triangles in the facets of $\mathcal{O}$ (with side length $3/\sqrt{2}-2$), and 12 rectangles.

Remark 2. We replace the sphere circumscribed to $\mathcal{O}$ by a quadric of revolution having Z_1 and Z_2 as North and South pole. This quadric is an obtuse ellipsoid Φ with $\sqrt{5}:\sqrt{2}$ as ratio of semiaxes. This means that the equator of Φ has the radius $\sqrt{3}/2$ (see Fig. 9b).

In analogy to the ordinary stereographic projection of a sphere the planar sections of Φ will be mapped under ξ_1 onto circles of π. Thus, especially the intersections of Φ with the facet planes of $\mathcal{O}$ are mapped onto circumcircles of any three points of $\mathcal{P}_{\text{tri}}$. In the following Chapter 5 we will investigate $\mathcal{P}_{\text{tri}}$ from a Moebius-geometric point of view.

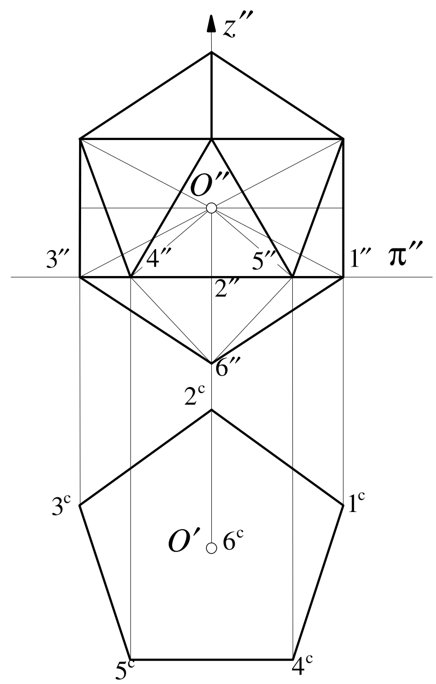

Figure 10: Central projection of a regular icosahedron producing a golden hexagon $\mathcal{P}_{\text{pent}}$ (top and front view)

Another candidate for a polyhedron to be connected with golden hexagons is the *regular icosahedron* $\mathcal{I}$. In order to project its twelve vertices onto the six points of a golden hexagon, the center of the projection must coincide with the center of $\mathcal{I}$ (see Fig. 10).

By using e.g. a facet plane of $\mathcal{I}$ as the image plane π (instead of a plane orthogonal to a diameter like in Fig. 10) we would receive a $\mathcal{P}_{\text{tri}}$ as the image of $\mathcal{I}$. For an arbitrary projection plane π we get a hexagon projectively equivalent to the golden hexagons in π. Because of this property and because of the extraordinary frequency of golden ratios and golden cross ratios connected with $\mathcal{I}$ it would be justified to see in $\mathcal{I}$ a *three-dimensional golden object.*

Remark 3. From the view point of Projective Geometry $\mathcal{I}$ is dual to the (regular) dodecahedron $\mathcal{D}$, which also shows golden ratios and golden cross ratios. By dualizing the concept "projection" to linear mappings of planes (as elements) to their trace lines within a fixed "trace plane", one can also connect $\mathcal{D}$ with golden hexagons: The traces of the facet planes of $\mathcal{D}$ in the ideal plane of the projectively extended space form the six lines of the dual of a golden hexagon. Similarly, the traces of a cube's facets in a (special) plane orthogonal to a diagonal of the cube form the dual of a $\mathcal{P}_{\mathrm{tri}}$.

5. The golden hexagon $\mathcal{P}_{\mathrm{tri}}$ in the Moebius plane

By intersecting one side of a golden hexagon $\mathcal{P}_{\mathrm{gold}}$ with the other 14 sides one obtaines "derived points" of $\mathcal{P}_{\mathrm{gold}}$. Let us consider side 13 of $\mathcal{P}_{\mathrm{tri}}$ in Fig. 8: The remaining points 2, 4, 5, 6 form a quadrangle such that 13 is one of its diagonals with the diagonal points C, D. The third pair of quadrangle sides intersect 13 in H and U. Thus we can distinguish three types of derived points:

(1) at each of the vertices 1, 3 five sides are meeting,

(2) each of the points C and D is the intersection of three sides, and

(3) H and U are intersections of only two sides.

When lines connecting derived points are intersected with each other, we obtain points *"derived of higher order"*.

In the following we associate circles to $\mathcal{P}_{\mathrm{tri}}$ passing through original and/or derived points of $\mathcal{P}_{\mathrm{tri}}$. The golden hexagon $\mathcal{P}_{\mathrm{pent}}$ seems to be already well discussed from this point of view (see e.g. [3]), while $\mathcal{P}_{\mathrm{tri}}$ may offer new and unexpected circle geometric configurations.

Let us start with the system of three circles, each passing trough four points of $\mathcal{P}_{\mathrm{tri}}$: After denoting the points of $\mathcal{P}_{\mathrm{tri}}$ with the numbers $1, \dots, 6$ and the first order derived points with the letters $A, \dots, F, \dots$ (according to Fig. 11), we state the following:

The *four-point circles* c_X, c_Y, c_Z are centered at points $X := BC \cap DE$, $Y := DE \cap AF$, and $Z := AF \cap BC$, respectively (see Fig. 11). Note that e.g. X is the mirror image of 6 reflected in 35, and that the triangle $(X, 5, 4)$ is equilateral. The circle c_X intersects 13 and 15 at points X'' and X'. Setting $\mathrm{dist}[1,5] = 1$ implies that

$$\mathrm{dist}[5, E] = \varphi^2 \quad \text{and} \quad \mathrm{dist}[5, X'] = \varphi^4.$$

The circle c_E centered at E and passing through X' contains also the points $F, 4$ and Y. It passes through the intersection $DE \cap 56$. This intersection point divides the segments $[5, 6]$, $[E, D]$ and $[E, X]$ according to the golden ratio; so again we get many golden quadruples of derived points. Similar coincidences occur when we look at the six circles with radius φ^2, which are centered at the derived points $A, \dots, F$.

Another interesting configuration of circles is formed by the system of eight circles, each passing through three points of $\mathcal{P}_{\mathrm{tri}}$ (see Fig. 12):

There are two triples of congruent circles and two concentric circles, each touching the circles of one triple. The remaining six intersections of the triples form a regular hexagon. It is remarkable that the sides of this regular hexagon pass through the derived points $A, \dots, F$ and that they are orthogonal to the sides 13, 35, 51. E.g., the side through D intersects the circumcircles of $(3, 5, 6)$ and $(2, 3, 6)$ at the points $U \in 15$, $V \in 24$ (Fig. 12) and D is the center of the segment $[U, V]$. This configuration of 8 circles represents the image of the facet planes of an octahedron $\mathcal{O}$ under the stereographic projection as mentioned in Section 4, Remark 2.

Among all circles passing through two or one original point of $\mathcal{P}_{\mathrm{tri}}$ and through one or two derived points of a certain type we consider just the nine which are shown in Fig. 13. (Otherwise this figure would be too confusing.) Each of those circles passing through one

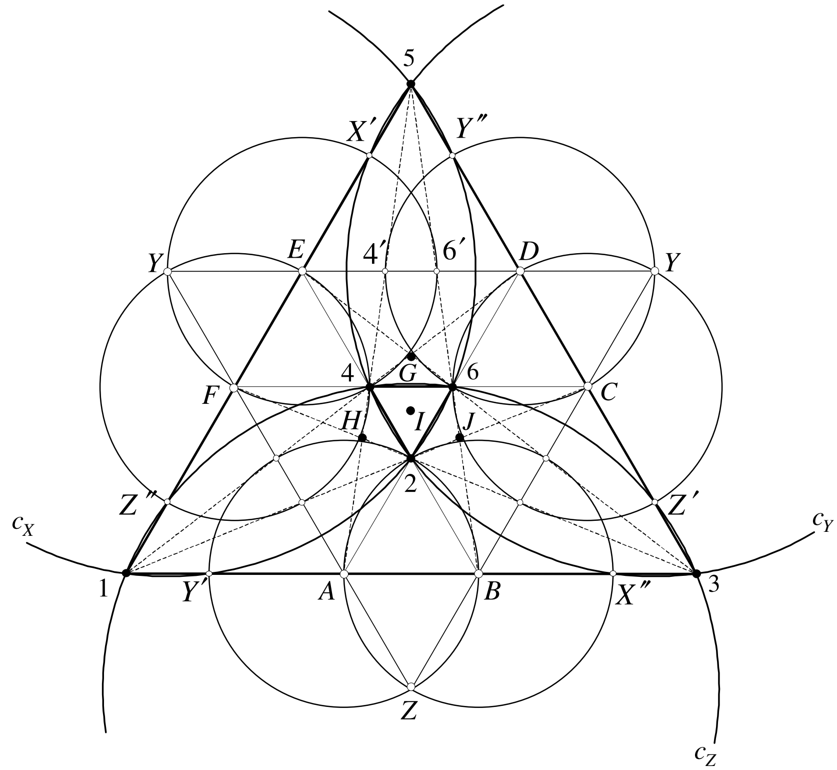

Figure 11: System of the three circles passing through 4 points of $\mathcal{P}_{\text{tri}}$ together with a configuration of six congruent circles centered at the derived points of the first order

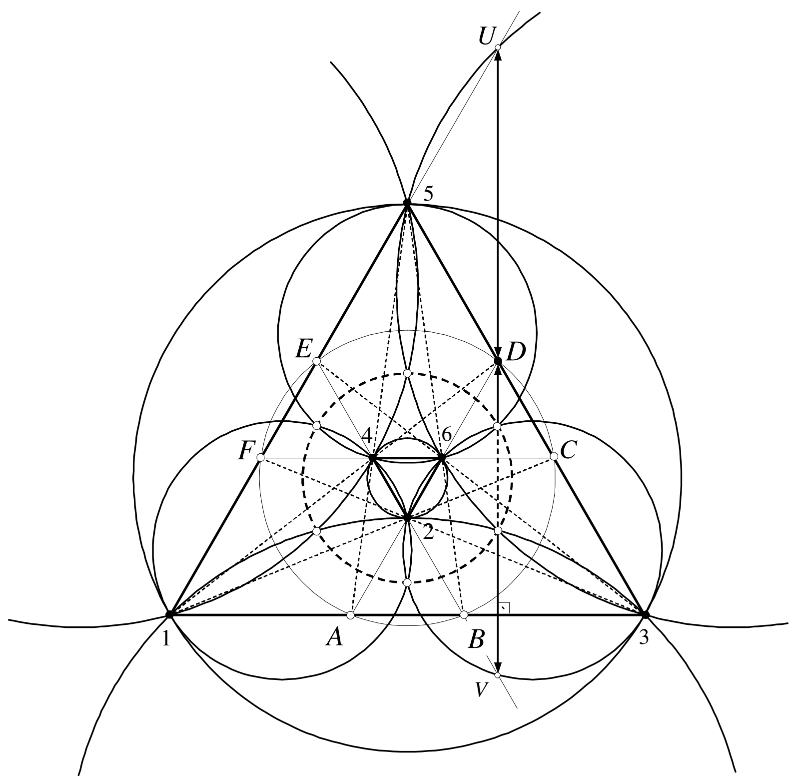

Figure 12: The configuration of the eight circles passing through three points of $\mathcal{P}_{\text{tri}}$

original point contains a pair of derived points of type (2) and a pair of type (3). Those passing through two original points contain one point of type (2), a pair of type (3) points, and (at least) one derived point of higher order. The nine circles can be combined by five to three "rings" twinned into each other. Each ring has a pair of touching circles, the outer ones belonging to the complete system of circles that we are dealing with; the inner circle is not in this system (see Fig. 14).

Additionally there exists a circle i orthogonal to each ring; the 10 intersection points of the circles of a ring are located on two circles of the complete system. Each ring defines thus five additional circles, which belong to one pencil of circles with the"opposite" side of $\mathcal{P}_{\text{tri}}$ as the common cord. (In Fig. 3 this cord is the side 13.) As the ring remains fixed under inversion at the orthogonal circle i, the other four circles are coupled in inverse pairs.

Remark 1. It is worth mentioning that there exists a projective mapping from the five "derived" circles of the ring and their common cord onto the points $1, 3, C, D, H, U$ of line 13 in Fig. 8. Hence, they form a Moebius-geometric analogon to the (complete) pointset of a golden cross ratio.

Remark 2. Each pair of circles of the ring shown in Fig. 14 has its (outer) similarity center on the common cord of the derived circles. These centers coincide exactly with the original and derived points of this cord. This leads us to another circle-geometric interpretation of this ring, based on the so called *cyclographic projection* (see e.g. [1]). Here a spatial point $P = (x, y, z)$ is mapped onto an oriented circle P^z (or onto the point P itself) in the xy-plane with center $P' = (x, y)$ and radius $|z|$. The sign of z is represented by the orientation of P^z. If the five circles of the ring are oriented in the same sense, the cyclographic preimage points form a pentagon in an inclined plane, having side 13 as its trace in the xy-plane. This pentagon is regular in the *pseudo-Euclidean* sense. (Concerning the pseudo-Euclidean structure, which is canonically induced in the space by the cyclographic projection, see also [1]). However, it would go beyond the scope of this paper to discuss in detail the polyhedron formed by the cyclographic preimages of the complete set of the considered circles. Let us just point out that within this set of points there occur topological equivalents to regular icosahedra.

6. Summary and Conclusion

We showed that the golden section has a projective-geometric meaning as a special cross ratio. Four points of a line defining a golden cross ratio represent the one-dimensional fundamental design, which is generalized to objects in two and three dimensions, to golden hexagons $\mathcal{P}_{\text{gold}}$ and the (regular) icosahedron $\mathcal{I}$, respectively. Golden hexagons have a projective characterization as sets of six points, where any partition into pairs of triangles leads to a Desargues configuration. Golden hexagons are limit figures of an iteration process, which is remarkable for its own. They are projection images of regular octahedra, regular icosahedra, and of Kepler's star polyhedra and dually — replacing projections by a mapping of planes onto their traces — also of regular dodecahedra.

We can associate circles to the Euclidean golden hexagon $\mathcal{P}_{\text{tri}}$ and obtain sets of planes, when projecting these circles onto a quadric (via stereographic projection), or points, when using the cyclographic projection. The resulting configurations allow interpretations as (three-dimensional) images of polytopes from multidimensional spaces. It is a matter of taste, whether these polytopes are seen as higher-dimensional analogues of the golden cross ratio.

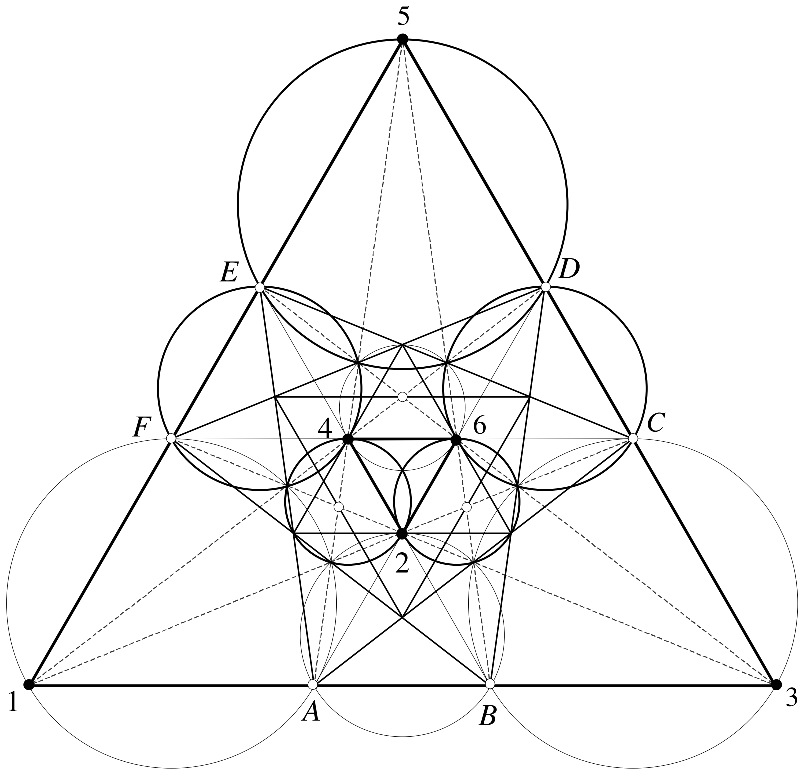

Figure 13: System of 12 circles connected with $\mathcal{P}_{\text{tri}}$. They form three rings each containing five circles

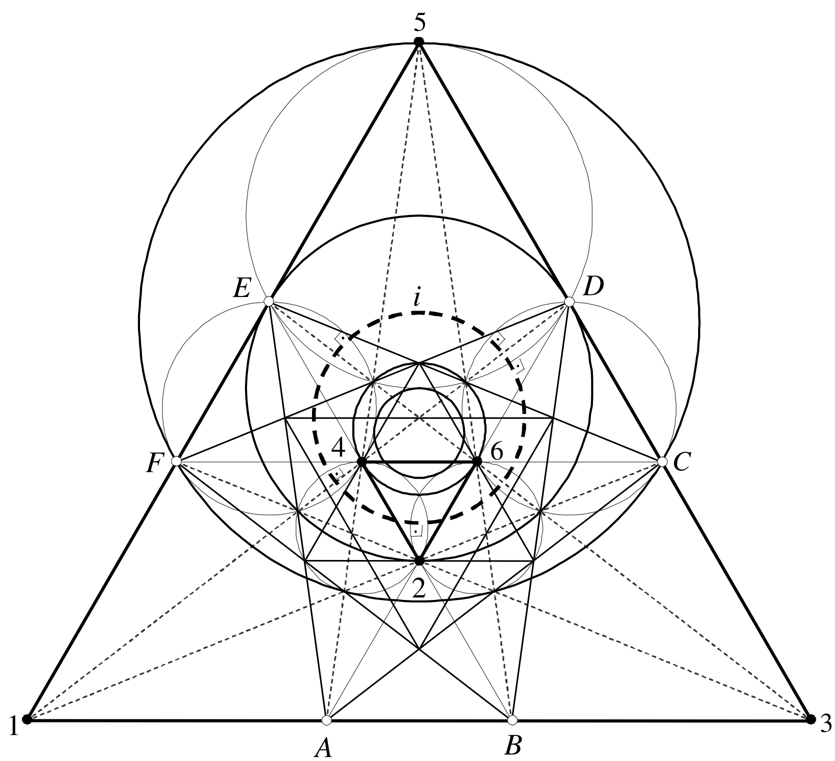

Figure 14: Ring of five circles of Fig. 13 together with the pair of touching circles and a circle orthogonal to each of the five circles

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