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Explicit Calculation Methods for Conjugate Profiles

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Dedicated to Prof. Hellmuth Stachel on the occasion of his 60^{th} birthday

Abstract. The tooth profiles of two meshing spur gears are considered. A given profile may envelop another gear profile. The paper presents an explicit parametrization of the gear profile. It is also shown that the corresponding conjugate rack profile is explicitly determined.
Analogously, the line of action is described. Afterwards, the inverse problem is studied. If the gear centers, the gear ratio, and a line of action are given then conjugate gear profiles are determined by the solution of an ordinary differential equation of first order. An example of application is shown. Finally, this differential equation is geometrically interpreted by introducing the pole distance. At any instant the rate of change of the pole distance relates to the inclination of the rack profile. It is shown that for a specific linear relation, corresponding conjugate profiles are trochoids of 2nd or 3rd order.

Keywords: plane kinematics, conjugate tooth profiles, line of action, rack profile

MSC 2000: 53A17

1. Reference systems

For spur gears in mesh, it is necessary to establish a system of coordinates to parameterize their geometry. For this purpose

- the coordinate system $(O_1; x_1, y_1)$ is attached to a first cylindrical wheel rotating about the point $O_1 = O_0$ with angular velocity ω_1;
- the coordinate system $(O_2; x_2, y_2)$ is attached to a second cylindrical wheel rotating about the point $O_2 = O_4$ with angular velocity ω_2;
- the coordinate systems $(O_0; x_0, y_0)$ and $(O_4; x_4, y_4)$ are fixed to the ground;
- the coordinate system $(O_3; x_3, y_3)$ is attached to a rack which is intended to mesh with the designed spur gears.

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After all, five coordinate systems are needed. Let Σ_k denote the coordinate system $(O_k; x_k, y_k)$ where the subscipt k may be $0, \ldots, 4$. In order to indicate that any point X of the plane is described by the coordinate system Σ_k we also use the subscript k. For instance, X_1 indicates a point which is referred to Σ_1. The subscript 1 indicates that together with the frame Σ_1 the point X_1 is rigidly connected to the first cylindrical wheel. Furthermore, considering the coordinates x_k and y_k of a point X_k to be the real and imaginary part of a complex number, respectively, we obtain

$$X_k = x_k + iy_k = \rho_k\, e^{i\theta_k}, \tag{1}$$

where i is the imaginary unit. Here, the second equation gives the exponential representation of X_k, where ρ_k is the absolute value (modulus) and Θ_k the argument of X_k. This powerful notation for Euclidean kinematics has been systematically developed by Bereis [2]. Some remarks in English language can be found in [4]. Applying this notation we find the following coordinate transformations in dependance on the rotation angle $t := t_1$ of frame Σ_1:

$$X_1 = X_0 e^{it}, \quad X_2 = X_4 e^{i\omega t}$$

$$X_1 = (r_1 - ir_1 t + X_3)e^{it} \tag{2}$$

where $t_2 = \omega t$ is the rotation angle of Σ_2 (see Fig. 1[1]). The constant gear ratio is

$$\omega = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} \tag{3}$$

where r_1 and r_2 are the radii of the centrode circles p_1 and p_2. Note that the center distance a satisfies

$$a = r_1 - r_2. \tag{4}$$

In any case assuming $r_1 > 0$, let r_2 be positive or negative according to the intented spur gear design. The transformation between the ground frames simply reads

$$X_0 = a + X_4. \tag{5}$$

The centrode circles p_1 and p_2 contact each other at the instantaneous rotation pole P which is described by

$$P_1(t) = r_1 e^{it}, \quad P_2(t) = r_2 e^{i\omega t}, \quad P_3(t) = -ir_1 t \tag{6}$$

according to the various coordinate systems Σ_1, Σ_2, and Σ_3. The relative displacement of Σ_2 with respect to Σ_1 is given by

$$X_1 = ae^{it} + X_2 e^{i(1-\omega)t} \tag{7}$$

where t is the parameter of this motion. If X_2 is any fixed point in Σ_2 then the path of X_2 (in the plane of Σ_1) is parametrized by (7) in terms of $X_1 = X_1(t)$. Of course, the path is a trochoid which may be prolate or curtate dependent on the position of X_2.

[1]The figures of the present paper naturally depend on the parameter t. In order to support the understanding of the considered displacements, corresponding movies are presented on the author's home page: http://www.math.tu-dresden.de/~baer. Look at "Selected Papers".

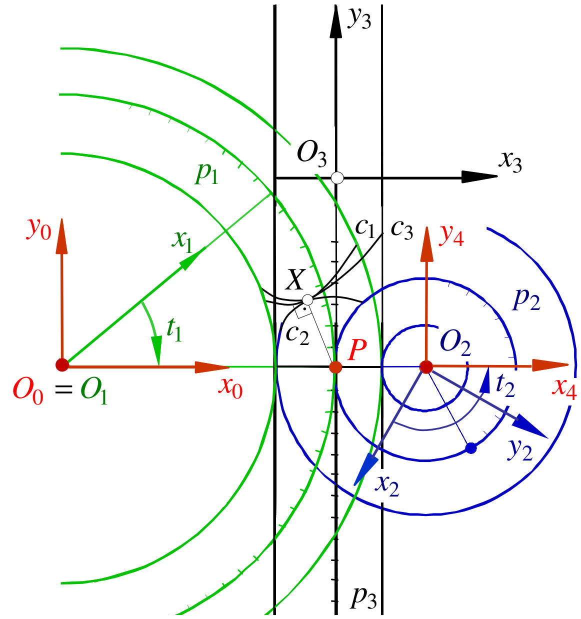

Figure 1: Conjugate profiles c_1, c_2 and c_3 are in contact at X

2. Determination of a conjugate profile

Let a profile (curve) c_2 be given by the parametrization

$$X_2 = x_2(s) + iy_2(s), \quad s \in S \subseteq \mathbb{R} \tag{8}$$

then its various positions in Σ_1 may envelop a profile c_1 in Σ_1. Then, the profiles c_2 and c_1 are called *conjugate*. For any instant t in a certain interval, the two profiles c_1 and c_2 contact each other at a *point of contact*. This contact point is determined by the first law of (planar) gearing that states that the common normal to the contacting profiles has to pass through the instantaneous rotation pole. In equivalent terms: At a contact point the common normal of the profiles is perpendicular to the common tangent line. Hence, any point X_2 of the given profile c_2 becomes a point of contact for an instant t if the direction vector $X_2 - P_2$ of the normal is perpendicular to the tangent vector $X_2' = \frac{d}{ds}X_2(s)$. Working that out with complex numbers we obtain the *contact equation*

$$(X_2 - P_2)\bar{X}_2' + (\bar{X}_2 - \bar{P}_2)X_2' = 0 \tag{9}$$

where the overline denotes the complex conjugate of a complex number.

With the abbreviations

$$\begin{aligned} X_2 &= x_2(s) + iy_2(s) = \rho_2 e^{i\theta_2}, & \theta_2 &:= \mathrm{Arg}(X_2), & \rho_2 &:= \mathrm{Abs}(X_2) \\ X_2' &= x_2'(s) + iy_2'(s) = \lambda e^{i\alpha}, & \alpha &:= \mathrm{Arg}(X_2'), & \lambda &:= \mathrm{Abs}(X_2'), \end{aligned} \tag{10}$$

and substituting P_2 by (6) we obtain

$$(\rho_2 e^{i\theta_2} - r_2 e^{i\omega t})\lambda e^{-i\alpha} + (\rho_2 e^{-i\theta_2} - r_2 e^{-i\omega t})\lambda e^{i\alpha} = 0,$$

which is equivalent to

$$r_2(e^{i(\omega t - \alpha)} + e^{-i(\omega t - \alpha)}) = \rho_2(e^{i(\theta_2 - \alpha)} + e^{-i(\theta_2 - \alpha)}).$$

With the general relation $2\cos\varphi = e^{i\varphi} + e^{-i\varphi}$ we get

$$\begin{aligned} r_2\cos(\omega t - \alpha) &= \rho_2\cos(\theta_2 - \alpha) = \rho_2(\cos\theta_2\cos\alpha + \sin\theta_2\sin\alpha) = \\ &= \rho_2\cos\theta_2\cos\alpha + \rho_2\sin\theta_2\sin\alpha. \end{aligned}$$

Finally, using (10): $X_2 = x_2 + iy_2 = \rho_2\cos\theta_2 + i\rho_2\sin\theta_2$, we obtain the equivalent contact equation

$$r_2\cos(\alpha - \omega t) = x_2\cos\alpha + y_2\sin\alpha.$$

This equation allows the *explicit solution*

$$t = t(s) = \frac{\alpha \pm \beta + 2\pi N}{\omega}, \quad N = 0, \pm 1, \pm 2, \dots, \tag{11}$$

where

$$\beta = \arccos\frac{x_2x_2' + y_2y_2'}{r_2\lambda}$$
$$s \in S_H = \{s \in S : |x_2x_2' + y_2y_2'| \le |r_2\lambda|\}.$$

Consequently, there is always an explicit parametrization of the conjugate profile. For the case $N = 0$, the solution defines the *basic part of the conjugate profile*

$$X_{1B}(s) = (a + X_2\, e^{-i(\alpha\pm\beta)})\, e^{i\,\frac{1}{\omega}(\alpha\pm\beta)}, \quad s \in S_H. \tag{12}$$

The *complete conjugate profile* c_1 is given by

$$X_1(s) = X_{1B}(s)\, e^{i\,\frac{2\pi}{\omega}N}. \tag{13}$$

Geometrically spoken, the complete conjugate profile c_1 is obtained by revolutions of the basic part through any integer multiple of the angle $\frac{2\pi}{\omega}$.

The set of all contact points observed in the ground coordinate system $(O_0; x_0, y_0)$ establish the so-called *line of action*

$$c_0\colon \quad X_0(s) = a + X_2(s)e^{i(\alpha\pm\beta)} = X_1(s)e^{-i(\alpha\pm\beta)/\omega}. \tag{14}$$

Example 1: The given profile c_2 is chosen to be the real axis of the system Σ_2 that is

$$X_2(s) = s, \quad s \in \mathbb{R}.$$

Then by (11) it follows the explicit solution

$$t = \frac{1}{\omega}(\pm\beta + 2\pi N) \quad \text{where} \quad \beta = \arccos\left(\frac{s}{r_2}\right), \quad s \in S_H = \{s: \quad |s| \le |r_2|\}. \tag{15}$$

The solution set S_H shows that the contact points lie within the diameter segment of the centrode circle p_2. By eq. (12), the basic part of the conjugate profile is

$$c_{1B}\colon \quad X_{1B}(s) = (a + se^{-i\beta})e^{\pm i\,\frac{\beta}{\omega}}.$$

With help of the implicit form of the solution

$$s = r_2\cos\beta \tag{16}$$

we find the new parametrization

$$X_{1B}(s) = \left(r_1 - \frac{r_2}{2} + \frac{r_2}{2}\, e^{-i2\omega\tau}\right) e^{i\tau}, \quad 0 \le \tau \le 2\pi.$$

Comparing this parametrization with eq. (7), we see that the conjugate profile c_1 is a cuspidal trochoid. It is traced by the point $X_2 = \frac{r_2}{2}$ attached to a circle of radius $\frac{r_2}{2}$ rolling around the outside of the fixed circle of radius r_1.
Using the substitution (16) in eq. (14) we can conclude that the line of action is the circle

$$c_0\colon \quad X_0(\xi) = r_1 - \frac{r_2}{2} + \frac{r_2}{2}\, e^{i\xi}, \quad 0 \le \xi \le 2\pi.$$

which has radius $\frac{r_2}{2}$ and the mid point $M_0 = r_1 - \frac{r_2}{2}$. Fig. 2 illustrates the results for the case of an enveloped epicycloid with the gear ratio $\omega = -3$.

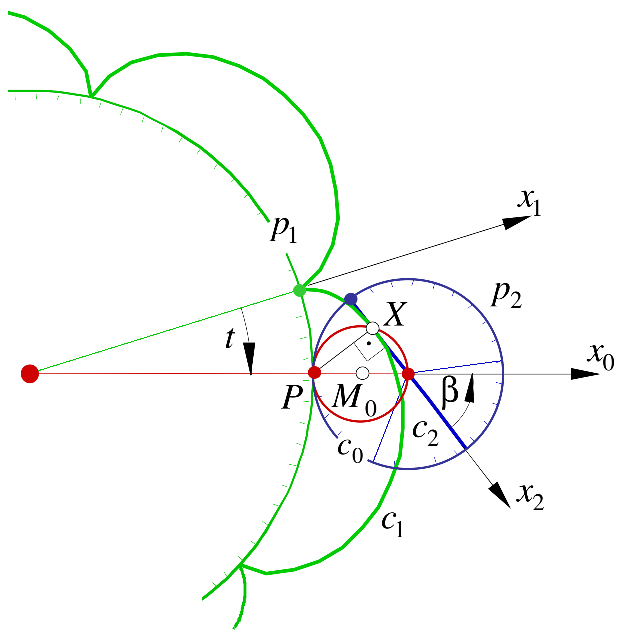

Figure 2: A diameter of the rolling centrode envelops a cuspidal epitrochoid

3. Determination of a rack profile

In this chapter the relative motion of the gear systems and the rack system is again considered as depicted in Fig. 1. The displacement of the gear system Σ_1 with respect to the rack system Σ_3 is given by the rolling of the centrode circle p_1 upon the straight line centrode p_3. So, it is a cyclic motion which is analytically described by (2). The instantaneous rotation centers of all considered displacements coincide at point P. Therefore, if a pair (c_2, c_1) of conjugate profiles contact at a point X then this point can belong to a rack profile c_3 with the same contact normal as the pair at the considered moment. So, this rack profile c_3 is in contact with c_1 and c_2 at the same moment.

Resolving (2) for X_3, we obtain an *explicit parametrization of the rack profile* c_3 for the pair (c_2, c_1) that is

$$X_3(s) = X_1(s)e^{-it(s)} - r_1 + ir_1 t(s). \tag{17}$$

Example 1 (continued)**:** If we substitute the solution (16) in (17) the rack profile c_3 is determined for the pair (c_2, c_1) of the above example. The rack profile c_3 is the cycloid traced out by the pole P attached to the line of action and rolling with this circle along the centrode p_3.

Now let us consider the reverse problem. A rack profile

$$c_3\colon \quad X_3 = x_3(s) + iy_3, \quad s \in S_3 \subseteq \mathbb{R}$$

may be given. We are looking for the conjugate gear profile c_1 in the system Σ_1. According to the first law of gearing applied to the displacement of Σ_3 with respect to Σ_1, the normal vector $X_3 - P_3$ and the tangent vector $X_3' = x_3' + iy_3'$ must be perpendicular, i.e., analytically

$$(X_3 - P_3)\bar{X}_3' + (\bar{X}_3 - \bar{P}_3)X_3' = 0.$$

Here we find the explicit solution

$$t = t_3(s) := \frac{x_3 x_3' + y_3 y_3'}{r_1 y_3'}\,. \tag{18}$$

Therefore, by (2) the profile c_1 is given by

$$X_1 = X_1(s) = (r_1 - ir_1 t_3(s) + X_3(s))e^{it_3(s)}, \quad s \in S_{3H}. \tag{19}$$

Example 2: It is well known that a straight lined rack profile c_3 generates an involute in each gear system. The line of action is then a straight line which is tangent to the basic circles of the gear involutes. It is easy to proof that the presented formulae produce this special case.

4. A differential equation for conjugate profiles

Let

$$X_0(s) = E_0(s) = u(s) + iv(s), \quad s \in S, \tag{20}$$

be a parametrization of an arbitrary line of action with respect to the ground system. With an unknown function of contact, $t = t(s)$, the conjugate profile is

$$X_1(s) = E_0(s)e^{it(s)}. \tag{21}$$

Hence the tangent vector of the conjugate profile c_1 is

$$X_1' = (E_0' + it'E_0)e^{it}. \tag{22}$$

If the profile c_1 contacts a profile c_2 at the point $E_0(s)$ then the normal vector $X_1 - P_1$ and the tangent vector X_1' have to be perpendicular, i.e., analytically

$$(X_1 - P_1)\bar{X}_1' + (\bar{X}_1 - \bar{P}_1)X_1' = 0.$$

Here, we insert (6), (21), and (22), and by a little manipulation we obtain the ordinary differential equation (ODE)

$$\frac{d}{ds}t(s) = \frac{(r_1 - u)u' - vv'}{r_1 v} =: g(s). \tag{23}$$

Assuming the boundary condition $t_0 = t(s_0)$ the ODE has the unique solution

$$t(s) = \int_{s_0}^{s} g(\sigma)d\sigma + t_0. \tag{24}$$

Inserting this solution into equation (21) yields the profile c_1. The conjugate profile c_2 is then given by

$$X_2(s) = (E_0(s) - a)e^{i\omega t(s)}. \tag{25}$$

Example 3: Let the line of action be a polynomial of order n for the real and imaginary part:

$$\begin{aligned} u(s) &= \textstyle\sum_{k=0}^{n} a_k s^k \quad (a_n \neq 0,\ 0 \leq s \leq s_1) \\ v(s) &= \textstyle\sum_{k=0}^{n} b_k s^k \quad (b_n \neq 0,\ 0 \leq s \leq s_1). \end{aligned} \tag{26}$$

Hence the integrand (23) is a rational function with a numerator polynomial of order $n(n-1)$ and a denominator polynomial of order n. Therefore, (24) is integrable by partial fraction decomposition.

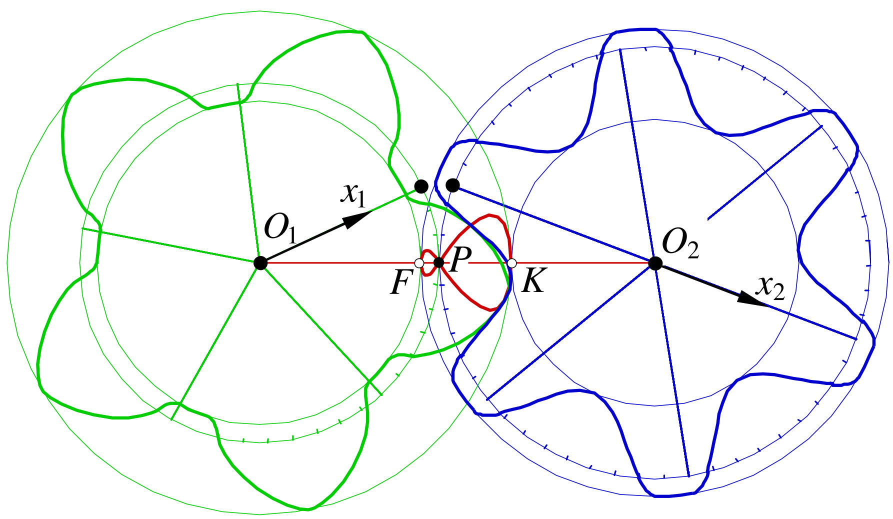

Figure 3: Conjugate profiles by integration

Fig. 3 shows the solution of a numerical example with the gear ratio $\omega = -5/6$. The addendum, pitch, and dedendum radii are $r_{k1} = 70$, $r_1 = 50$, and $r_{f1} = 40$, respectively. The gears are designed without clearance. According to the polynomial approach (26) the line of action c_0 is chosen to start at the foot point F of gear 1 with negative inclination at the moment $t = 0$. Furtheron, it is required that the line of action has to intersect the pole P forming an angle of 60 degree with the centre line. Then, the line of action has to end at the addendum point K. These requirements determine seven coefficients a_k and b_k. So, the order $n = 3$ is sufficient. The second half of the line of action is simply the reflection of the first half. The method facilitates the design of conjugate profiles with a *closed line of action*. For this purpose, the boundary condition for the second integration is $t_0 = t(s_1)$ where s_1 is the endpoint of the definition interval in (26).

5. A differential equation for the pole distance

Let us consider a rack profile

$$c_3: \quad X_3 = X_3(s), \quad s \in S_3.$$

At the instant $\tau = -t(s)$ the point $X_3(s)$ is assumed to be a contact point with a profile c_1. Then the relation to the ground system is given by

$$X_0 = r_1 + ir_1\tau + X_3 = r_1 + X_3(s) - P_3(\tau). \tag{27}$$

The term $X_3 - P_3$ can be expressed by an exponential complex number that is

$$ne^{i\alpha} := X_3 - P_3$$

where $n = n(s)$ is the *pole distance*, i.e., the distance $\text{Abs}(X_3 - P_3)$ between a point of contact $X_3(s)$ and the instantaneous rotation pole P_3. The rack centrode p_3 and the common tangent line of the contacting profiles include the *rack angle* $\alpha = \alpha(s)$.

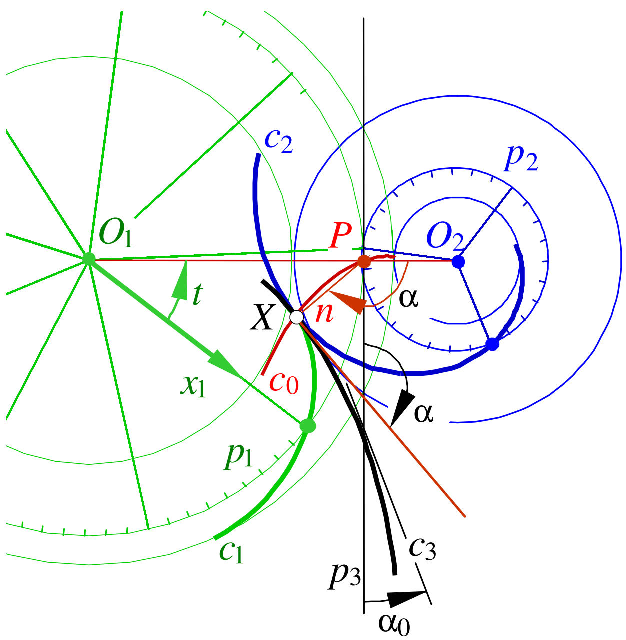

Figure 4: Integrated profiles

In view of the description of the line of action (20) we insert these definitions into (27) and get

$$E_0(s) = r_1 + n(s)e^{i\alpha(s)}. \tag{28}$$

For this special representation of the line of action, we determine (23) and replace the auxiliary parameter $\tau(s)$ by the familiar parameter $-t(s)$. We obtain

$$\frac{dt}{ds} r_1 \sin\alpha = \frac{dn}{ds}. \tag{29}$$

The general curve parameter s can be specified by the arc length of the rack centrode, i.e., $s = y_3$. It follows $s = -r_1 t$, and by differentiation, we have $\frac{ds}{dt} = -r_1$.

Inserting that into (29) we obtain the ODE

$$r_1 \sin\alpha(t) = -\frac{dn}{dt}. \tag{30}$$

Assuming the boundary condition $n_0 = n(s_0)$, (30) has the unique solution

$$n(t) = -r_1 \int_{t_0}^{t} \sin \alpha(x) dx + n_0, \quad n_0 = n(t_0). \tag{31}$$

If the function $\sin(\alpha(x))$ is integrable conjugate profiles c_1 and c_2 are determined by

$$X_1(t) = (r_1 + n(t) e^{i\alpha(t)}) e^{it} \tag{32}$$

and

$$X_2(t) = (r_2 + n(t) e^{i\alpha(t)}) e^{i\omega t}, \tag{33}$$

respectively. The corresponding rack profile c_3 is

$$X_3(t) = P_3(t) + n(t) e^{i\alpha(t)}. \tag{34}$$

In the case

$$\alpha(x) = \alpha_1 x + \alpha_0 \quad (\alpha_k = \text{const.}, \ \alpha_1 \neq 0), \tag{35}$$

in which $\alpha(x)$ is a linear function of x, the function $\sin(\alpha(x))$ is integrable. The solution

$$\int_{t_0}^{t} \sin \alpha(x) dx = \frac{1}{\alpha_1} (\cos \alpha(t_0) - \cos \alpha(t)) \tag{36}$$

is substituted in (31) and (32). We obtain the gear profile c_1:

$$X_1(t) = r_1 (1 - \frac{1}{2\alpha_1}) e^{it} + \left(n_0 - \frac{r_1}{\alpha_1} \cos \alpha(t_0) \right) e^{i((\alpha_1+1)t+\alpha_0)} - \frac{r_1}{2\alpha_1} e^{i((2\alpha_1+1)t+2\alpha_0)}.$$

Therefore, in the subcase $\alpha_1 \neq 1$, the gear profile c_1 is a *trochoid of 3rd order* (in terms of [3, 9]: Radlinie 3-ter Stufe). But in the subcase $\alpha_1 = 1$, the gear profile c_1 is the *trochoid* (of 2nd order)

$$X_1(t) = (n_0 - r_1 \cos(t_0 + \alpha_0)) e^{i(2t+\alpha_0)} - \frac{r_1}{2} e^{i(3t+2\alpha_0)}. \tag{37}$$

In [8], it was shown that the conjugate profile for a trochoid consists of two parts, a trochoid again and also a trochoid of 3rd order. Hence, *in the subcase $\alpha_1 = 1$, the conjugate gear profiles are trochoids of order m* where $m \leq 3$.

If the angle $\alpha(x)$ is a polynom of second order, that is

$$\alpha(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0 \quad (\alpha_k = \text{const.}, \ \alpha_2 \neq 0), \tag{38}$$

the function $\sin \alpha(x)$ is not integrable, but the solution can be written by the help of the Fresnel integrals

$$S(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin u}{\sqrt{u}} du \quad \text{and} \quad C(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos u}{\sqrt{u}} du.$$

The Fresnel integrals are well-known by tabled values. They can practically be handled like explicit functions. With these notations the solution reads

$$\int_{t_0}^{t} \sin \alpha(x) dx = \sqrt{\frac{\pi}{2\alpha_2}} \left[(S(f(t)) - S(f(t_0))) \cos \delta + (C(f(t)) - C(f(t_0))) \sin \delta \right] \tag{39}$$

where the abbreviations

$$\delta = \alpha_0 - \frac{\alpha_1^2}{4\alpha_2} \quad \text{and} \quad f(t) = \frac{\alpha_1 + 2\alpha_2 t}{\sqrt{2\pi\alpha_2}}$$

have been used.

Example 4: A numerical example is given for the linear case (35) with coefficients $\alpha_1 = \frac{1}{2}$ and $\alpha_0 = \frac{\pi}{9}$. So, the gear profile c_1 will be a trochoid of 3rd order. See Fig. 4.
The gear ratio $\omega = -\frac{9}{3}$ and the pitch radius $r_1 = 50$ are chosen. The integrated profiles are meant to contact the rack profile c_3 at P at the instant $t = 0$. Therefore, we assume $t_0 = n_0 = 0$. The resulting profiles are explicitly given by (32) to (34) for $0 \le t \le 1$. The corresponding line of action c_0 is determined by inserting the solution (31) in (28). In Fig. 4 the trochoidal gear profiles are plotted at the instant $t = 0,8$.

6. Conclusion

The presented explicit solutions are not contained in the relevant textbooks [3, 4, 6, 7, 9]. They have been provided to support the design of general spur gear profiles in a concise manner. For given gear centers and a gear ratio, the designer may start with a given profile either of a gear or a rack. Then, the other profiles are explicitly calculable. As a by-product, the line of action is found. On the other hand, the designer may begin with a given line of action. Then he can find conjugate profiles by solving an ordinary differential equation. The latter method shows a profile design starting with a given rack angle function.

All methods allow free design parameters. Therefore, future research may focus on these parameters to meet new requirements of gear design.

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