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An Algebraic Equation for the Central Projection

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Abstract. A necessary and sufficient condition for a central-axonometric reference system to be the image of a positively oriented orthonormal frame under a central projection is given by a single complex equation. The presented equation is an analogon of the Gauss equation that characterizes the reference system of an orthogonal axonometry.

Key Words: central projection, central axonometry, Gauss equation

MSC 2000: 51N05

1. Introduction

Let X be an oriented Euclidean space of dimension 3 and let Y be an affine plane in X. We choose a projective completion $\hat{X}$ of X and denote the projective closure of Y in $\hat{X}$ by $\overline{Y}$.

Given an arbitrary point z of $\hat{X}$ that does not lie in $\overline{Y}$, the *projection* from $\hat{X}$ onto $\overline{Y}$ with center z is the projective map

$$\pi\colon\ \hat{X} \setminus \{z\} \to \overline{Y}\,, \quad p \mapsto (p \vee z) \cap \overline{Y}\,,$$

where $p \vee z$ denotes the line joining p and z. The map π is called a *central projection* if z is finite or a *parallel projection* if z is infinite.

A positively oriented orthonormal frame in X is a quadruplet (q, r_1, r_2, r_3) of points of X such that the vectors $r_1 - q,\ r_2 - q,\ r_3 - q$ form a positively oriented orthonormal basis of the vector space $\vec{X}$ of X. For $j = 1, 2, 3$ let s_j denote the infinite point of the line joining q and r_j. In this paper we study the images of the points

$$q, r_1, r_2, r_3, s_1, s_2, s_3$$

under a central projection from $\hat{X}$ onto $\overline{Y}$ (see Fig. 1). The problem of characterizing these images has been discussed by several authors [4], [7], [8], [3]. In Section 2 we present an algebraic equation for this problem, and in Section 3 and 4 we show how the geometric conditions of [8] and [7] follow from our algebraic equation. Central projections of orthonormal frames in higher-dimensional spaces are treated in [2] and [6].

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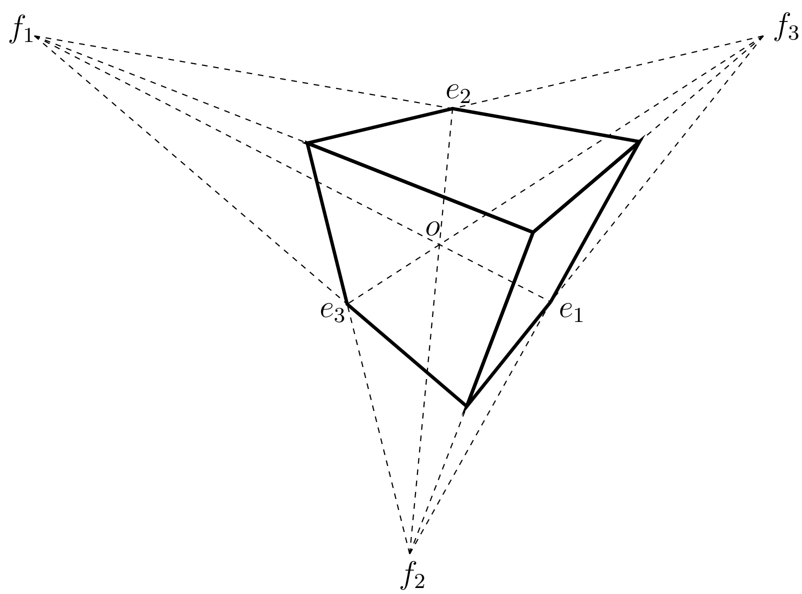

Figure 1: A central-axonometric reference system

2. A complex equation for the image

A *central-axonometric reference system* in $\overline{Y}$ is a septuplet $(o, e_1, e_2, e_3, f_1, f_2, f_3)$ of non-collinear points in $\overline{Y}$ such that o, e_1, e_2, e_3 are finite and each triple (o, e_j, f_j), $j = 1, 2, 3$, consists of three pairwise distinct collinear points. In [1] it is shown how central axonometry can be used for geometric constructions. For $j = 1, 2, 3$ let u_j denote the infinite point of the line joining o and e_j, and let

$$\rho_j := \text{cross-ratio}(o, e_j, f_j, u_j) = \frac{f_j - o}{f_j - e_j} \in \mathbb{R} \setminus \{0\}. \tag{1}$$

Then $\rho_j = 1$ if and only if $f_j = u_j$, i.e., f_j is infinite. For convenience, we use the notation

$$\rho'_j := 1 - \rho_j \quad \text{and} \quad w_j := e_j - o \quad \text{for } j = 1, 2, 3.$$

Theorem 1 *The central-axonometric reference system $(o, e_1, e_2, e_3, f_1, f_2, f_3)$ is the image of a positively oriented orthonormal frame under a central projection if and only if*

$$(\rho'_2\rho_1 w_1 - \rho'_1\rho_2 w_2)^2 + (\rho'_3\rho_1 w_1 - \rho'_1\rho_3 w_3)^2 + (\rho'_3\rho_2 w_2 - \rho'_2\rho_3 w_3)^2 = 0 \tag{2}$$

and ρ_1, ρ_2, ρ_3 are not all equal to 1. Here we use w_1, w_2, w_3 as complex numbers by identifying the Euclidean vector space $\vec{Y}$ with the complex plane $\mathbb{C}$. (Since the equation is invariant with respect to complex multiplication and conjugation, all identifications are equivalent).

Furthermore, for a reference system $(o, e_1, e_2, e_3, f_1, f_2, f_3)$ where at least one of the points f_1, f_2, f_3 is finite and which satisfies equation (2), there are exactly two solutions for the frame and the center, and one solution can be transformed into the other by first reflecting the center of the projection and the frame in the image plane and then reflecting the frame in the centre.

Due to Gauss the complex equation $w_1^2 + w_2^2 + w_3^2 = 0$ characterizes the reference system of an orthogonal axonometry.[1] Therefore, the above equation can be seen as a central-axonometric analogon of the Gauss equation.

For parallel axonometry, a calculation similar to that in the proof of Theorem 1 yields the complex equation

$$w_1^2 + w_2^2 + w_3^2 = \frac{1}{2}\left(|w_1|^2 + |w_2|^2 + |w_3|^2 - |{w_1}^2 + {w_2}^2 + {w_3}^2|\right) v^2$$

which gives a necessary and sufficient condition that the parallel-axonometric reference system (o, e_1, e_2, e_3) is, up to a uniform scaling in the image plane, the image of a positively oriented orthonormal frame under the parallel projection in the direction $n + v$ where n is a unit normal vector of the image plane.

Proof of Theorem 1. Following [5] we choose $\hat{X}$ as the projective space of the vector space $\mathbb{R} \times \vec{X}$ with the embedding

$$X \to \hat{X}\,, \quad p \mapsto o[(1, p-o)] = \mathbb{R}(1, p-o).$$

Writing $z = o + u$ with $u \in \vec{X}$, the central projection from $\hat{X}$ onto $\overline{Y}$ with center z is given analytically by

$$\pi\colon\ \hat{X} \setminus \{z\} \to \overline{Y}, \quad [(\alpha, v)] \mapsto [\vec{\pi}(\alpha, v)],$$

where

$$\vec{\pi}\colon\ \mathbb{R} \times \vec{X} \to \mathbb{R} \times \vec{Y}$$

denotes the linear projection with respect to the direct sum

$$\mathbb{R} \times \vec{X} = \mathbb{R}(1, u) \oplus \mathbb{R} \times \vec{Y}$$

with the kernel $\mathbb{R}(1, u)$, i.e., for $y \in \vec{Y}\colon\ \vec{\pi}(\lambda(1, u) + (\mu, y)) = (\mu, y)$. Then $\pi(q) = o$ if and only if $\vec{\pi}(1, q-o) = \lambda(1, 0)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Equivalently, $(1, q-o) - \lambda(1, 0) \in \mathbb{R}(1, u)$, i.e.,

$$q = o + (1-\lambda)u\,. \tag{3}$$

For $j = 1, 2, 3$, let $r_j = q + v_j$ with $v_j \in \vec{X}$. Then

$$\pi(r_j) = [\vec{\pi}(1, q + v_j - o)] = [\vec{\pi}(1, q-o) + \vec{\pi}(0, v_j)] = [\lambda(1, 0) + \vec{\pi}(0, v_j)].$$

Thus $\pi(r_j) = e_j$ iff

$$\vec{\pi}(0, v_j) = (\lambda_j - \lambda,\ \lambda_j w_j)$$

for some $\lambda_j \in \mathbb{R} \setminus \{0\}$. Equivalently, $(\lambda - \lambda_j,\ v_j - \lambda_j w_j) \in \mathbb{R}(1, u)$, i.e.,

$$v_j = (\lambda - \lambda_j)u + \lambda_j w_j\,. \tag{4}$$

Finally, the cross-ratio of the points

$$\begin{aligned} o &= [1(1, 0) + 0(0, w_j)] \\ e_j &= [1(1, 0) + 1(0, w_j)] \\ f_j &= [(\lambda_j - \lambda)(1, 0) + \lambda_j(0, w_j)] \\ u_j &= [0(1, 0) + 1(0, w_j)] \end{aligned}$$

[1] The author wishes to thank the referee for this information.

is

$$\rho_j = \frac{\begin{vmatrix} 1 & \lambda_j - \lambda \\ 0 & \lambda_j \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & \lambda_j - \lambda \\ 1 & \lambda_j \end{vmatrix}} = \frac{\lambda_j}{\lambda}. \tag{5}$$

From the equations (3)–(5) we conclude that the reference system $(o, e_1, e_2, e_3, f_1, f_2, f_3)$ is the image of a frame $(q,\ q+v_1,\ q+v_2,\ q+v_3)$ under the central projection with center $z = o+u$ if and only if

$$q = o + (1-\lambda)u \tag{6}$$

$$v_j = \lambda((1-\rho_j)u + \rho_j w_j) \quad \text{for } j = 1, 2, 3 \tag{7}$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. To prove the theorem, we examine the existence of some $\lambda \in \mathbb{R} \setminus \{0\}$ and some vector $u \in \vec{X}$ such that the vectors

$$\lambda((1-\rho_j)u + \rho_j w_j)$$

are orthonormal and positively oriented. Obviously not all of ρ_1, ρ_2, ρ_3 can be equal to 1. Therefore,

$$\rho := {\rho_1'}^2 + {\rho_2'}^2 + {\rho_3'}^2 \neq 0.$$

Choosing a suitable positively oriented orthonormal basis of the vector space $\vec{X}$ we can assume that

$$w_1 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} c \\ d \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} e \\ f \\ 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$$

with $a, b, c, d, e, f, r, s, t \in \mathbb{R}$. Then it suffices to check that the rows of the matrix

$$\lambda \begin{pmatrix} \rho_1' r + \rho_1 a & \rho_2' r + \rho_2 c & \rho_3' r + \rho_3 e \\ \rho_1' s + \rho_1 b & \rho_2' s + \rho_2 d & \rho_3' s + \rho_3 f \\ \rho_1' t & \rho_2' t & \rho_3' t \end{pmatrix}$$

are orthonormal and positively oriented. The third row has length 1 iff

$$t = \pm \frac{1}{\lambda\sqrt{\rho}}. \tag{8}$$

The sign of t can be chosen such the determinant of the matrix is positive. The second and the third row are orthogonal iff

$$s = -\frac{\rho_1'\rho_1 b + \rho_2'\rho_2 d + \rho_3'\rho_3 f}{\rho}. \tag{9}$$

The first and the third row are orthogonal iff

$$r = -\frac{\rho_1'\rho_1 a + \rho_2'\rho_2 c + \rho_3'\rho_3 e}{\rho}. \tag{10}$$

In the sequel we use these formulas to eliminate r, s and t. The first row has length 1 iff

$$(\rho_1'\rho_1 a + \rho_2'\rho_2 c + \rho_3'\rho_3 e)^2 = (\rho_1^2 a^2 + \rho_2^2 c^2 + \rho_3^2 e^2 - \lambda^{-2})\rho. \tag{11}$$

The second row has length 1 iff

$$(\rho_1'\rho_1 b + \rho_2'\rho_2 d + \rho_3'\rho_3 f)^2 = (\rho_1^2 b^2 + \rho_2^2 d^2 + \rho_3^2 f^2 - \lambda^{-2})\rho\,. \tag{12}$$

Finally, the first and the second row are orthogonal iff

$$(\rho_1'\rho_1 a + \rho_2'\rho_2 c + \rho_3'\rho_3 e)(\rho_1'\rho_1 b + \rho_2'\rho_2 d + \rho_3'\rho_3 f) = (\rho_1^2 ab + \rho_2^2 cd + \rho_3^2 ef)\rho\,. \tag{13}$$

Using the real bilinear form

$$\langle x, y\rangle := x_1y_1 + x_2y_2 + x_3y_3$$

and the vectors

$$A := (\rho_1 a, \rho_2 c, \rho_3 e), \quad B := (\rho_1 b, \rho_2 d, \rho_3 f) \quad \text{and} \quad N := (\rho_1', \rho_2', \rho_3')$$

we can rewrite the equations (11)–(13) as

$$\begin{aligned} \langle N, A\rangle^2 &= (\langle A, A\rangle - \lambda^{-2})\langle N, N\rangle \\ \langle N, B\rangle^2 &= (\langle B, B\rangle - \lambda^{-2})\langle N, N\rangle \\ \langle N, A\rangle\langle N, B\rangle &= \langle A, B\rangle\langle N, N\rangle \end{aligned}$$

or, equivalently,

$$\begin{aligned} \langle N\times A,\ N\times A\rangle\lambda^2 &= \langle N, N\rangle & (14)\\ \langle N, B\rangle^2 - \langle N, A\rangle^2 &= (\langle B, B\rangle - \langle A, A\rangle)\langle N, N\rangle & (15)\\ \langle N, A\rangle\langle N, B\rangle &= \langle A, B\rangle\langle N, N\rangle & (16) \end{aligned}$$

where $\times$ denotes the cross product and we have used that

$$\langle N\times A,\ N\times A\rangle = \langle N, N\rangle\langle A, A\rangle - \langle N, A\rangle^2.$$

If $\langle N\times A,\ N\times A\rangle = 0$, then by (15) also $\langle N\times B,\ N\times B\rangle = 0$ and the vectors A, B are linearly dependent which contradicts the assumption that $(o, e_1, e_2, e_3, f_1, f_2, f_3)$ is a reference system. Therefore,

$$\lambda = \pm\sqrt{\frac{\langle N, N\rangle}{\langle N\times A,\ N\times A\rangle}} \tag{17}$$

where the denominator cannot vanish. By equation (3), the $\pm$ in this formula corresponds to the symmetry of the problem.

To simplify the remaining equations (15) and (16), we identify the Euclidean vector space $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, i.e.,

$$w_1 = a + ib\,, \quad w_2 = c + id\,, \quad w_3 = e + if \quad \text{and} \quad A + iB = (\rho_1 w_1,\ \rho_2 w_2,\ \rho_3 w_3).$$

Then the equations (15)–(16) can be combined in the single equation

$$\langle N,\ A + iB\rangle^2 = \langle A + iB,\ A + iB\rangle\langle N, N\rangle$$

where $\langle\rangle$ now is the corresponding complex bilinear form. Equivalently,

$$\langle N\times(A + iB),\ N\times(A + iB)\rangle = 0 \tag{18}$$

where $\times$ denotes the cross-product on $\mathbb{C}^3$. Applying the determinant formula for the cross product now yields equation (2) and finishes the proof.

3. The conditions of Szabó-Stachel-Vogel

In [8] all the points f_1, f_2, f_3 are supposed to be finite. Then the conditions of [8] can be derived from equation (2) as follows. From equation (1) we get, for $j = 1, 2, 3$,

$$f_j - o = \frac{\rho_j}{\rho_j - 1}\,(e_j - o)$$

and

$$\frac{\mathrm{d}(o, e_j)}{\mathrm{d}(e_j, f_j)} = \frac{|e_j - o|}{|f_j - e_j|} = \frac{|e_j - o|}{|(f_j - o) - (e_j - o)|} = |\rho'_j| \tag{19}$$

where d denotes the Euclidean distance in Y. Thus equation (2) can be rewritten as

$$\left(\frac{f_3 - f_2}{\rho'_1}\right)^2 + \left(\frac{f_3 - f_1}{\rho'_2}\right)^2 + \left(\frac{f_2 - f_1}{\rho'_3}\right)^2 = 0\,. \tag{20}$$

Denoting the angles in the triangle of vanishing points by $\alpha_1, \alpha_2, \alpha_3$, we have

$$\frac{f_3 - f_1}{f_2 - f_1} = \frac{|f_3 - f_1|}{|f_2 - f_1|}\exp(-i\alpha_1)\,, \qquad \frac{f_3 - f_2}{f_1 - f_2} = \frac{|f_3 - f_2|}{|f_1 - f_2|}\exp(i\alpha_2)$$

and

$$\rho_1'^{\,-2}\left(\frac{|f_3 - f_2|}{|f_2 - f_1|}\right)^2 \exp(2i\alpha_2) + \rho_2'^{\,-2}\left(\frac{|f_3 - f_1|}{|f_2 - f_1|}\right)^2 \exp(-2i\alpha_1) + \rho_3'^{\,-2} = 0\,.$$

Extracting the imaginary parts gives

$$\left(\frac{\rho'_1}{\rho'_2}\right)^2 = \left(\frac{|f_3 - f_2|}{|f_3 - f_1|}\right)^2 \frac{\sin(2\alpha_2)}{\sin(2\alpha_1)} = \left(\frac{\sin(\alpha_1)}{\sin(\alpha_2)}\right)^2 \frac{2\sin(\alpha_2)\cos(\alpha_2)}{2\sin(\alpha_1)\cos(\alpha_1)} = \frac{\tan(\alpha_1)}{\tan(\alpha_2)}$$

and, by the symmetry of equation (20) with respect to permutations,

$$\left(\frac{\rho'_j}{\rho'_k}\right)^2 = \frac{\tan(\alpha_j)}{\tan(\alpha_k)}\,. \tag{21}$$

Combining the equations (19) and (21) now yields the conditions of [8]:

$$\left(\frac{\mathrm{d}(o, e_j)}{\mathrm{d}(e_j, f_j)}\right)^2 : \left(\frac{\mathrm{d}(o, e_k)}{\mathrm{d}(e_k, f_k)}\right)^2 = \tan(\alpha_j) : \tan(\alpha_k)\,.$$

4. The condition of Stiefel

In [7] it is supposed that the points f_1 and f_2 are finite, the point f_3 is infinite and that $f_2 - f_1$ is perpendicular to w_3. Then the condition of [7] can be derived from equation (2) as follows. Using

$$f_1 - o = \frac{\rho_1}{\rho_1 - 1}\,(e_1 - o)\,, \quad f_2 - o = \frac{\rho_2}{\rho_2 - 1}\,(e_2 - o) \quad \text{and} \quad \rho_3 = 1\,,$$

equation (2) can be rewritten as

$$(f_2 - f_1)^2 + (\rho_1'^{\,-2} + \rho_2'^{\,-2})w_3^2 = 0\,. \tag{22}$$

Since $f_2 - f_1$ is perpendicular to w_3, it follows that

$$\rho_1'^{-2} + \rho_2'^{-2} = \left(\frac{|f_2 - f_1|}{|w_3|}\right)^2.$$

Substituting

$$\frac{\mathrm{d}(o, e_j)}{\mathrm{d}(e_j, f_j)} = |\rho_j'| \quad \text{for } j = 1, 2$$

gives the formula of [7]:

$$\left(\frac{\mathrm{d}(e_1, f_1)}{\mathrm{d}(o, e_1)}\right)^2 + \left(\frac{\mathrm{d}(e_2, f_2)}{\mathrm{d}(o, e_2)}\right)^2 = \left(\frac{\mathrm{d}(f_1, f_2)}{\mathrm{d}(o, e_3)}\right)^2.$$

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