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Shapes of Space Curves

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Abstract. We study an ordered pair of functions which determines a regular space curve up to a direct similarity. The first function is called a shape curvature and the second one a shape torsion. We prove an extension of the fundamental theorem of space curves relating to the group of direct similarities. We also propose a direct way for recovering a space curve from its shape curvature and shape torsion.

Key Words: space curves, direct similarities, differential-geometric invariants

MSC 2000: 53A55, 53A04

1. Introduction

The direct similarities of the Euclidean space $\mathbb{R}^3$ preserve the orientation and the angles. Geometric objects in $\mathbb{R}^3$ like triangles and tetrahedra have a natural description with respect to the group of direct similarities by measures of two and four angles. The quaternion algebra $\mathbb{H}$ can be used for another description of similar triangles and tetrahedra. There is a well-defined correspondence between the set of all triangles in $\mathbb{R}^3$ and $\mathbb{H}$ so that every two triangles corresponding to the same quaternion are similar. This correspondence associates to any tetrahedron an ordered pair of quaternions so that two tetrahedra corresponding to the same pair are similar (see [3] for details). It is well-known that a regular space curve with non-zero curvature, a so-called *Frenet space curve*, is determined up to a Euclidean motion of $\mathbb{R}^3$ by its curvature and torsion (see [4], ch. 7).

The aim of this paper is to determine a Frenet space curve up to a direct similarity of $\mathbb{R}^3$ by a pair of real functions of class C^1. We first discuss some differential-geometric invariants of curves under the action of the group of direct similarities. Then, for a Frenet space curve we introduce a shape as an ordered pair of two invariants called a shape curvature and a shape torsion. The use of a spherical arc length parameter plays a key role in these considerations. In Section 3 we prove that if for two Frenet space curves, the shape curvature and the shape torsion coincide, then these curves are equivalent modulo a direct similarity. In the next section we study the relation between a Frenet curve and its spherical tangent indicatrix. In particular we show that the shape torsion of the Frenet space curve coincides with the

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geodesic curvature of the spherical indicatrix. Thus we obtain a direct constructive way for determining a Frenet space curve by its shape curvature and shape torsion under some initial condition. Finally we give examples for recovering a space curve with a given shape.

2. Differential geometric invariants of space curves under the group of direct similarities

Any direct similarity of the Euclidean space $\mathbb{R}^3$ is a product of an orientation-preserving homothety and a orthogonal mapping (see [1], p. 220). Denote by $\mathrm{Sim}^+(\mathbb{R}^3)$ the group of the direct similarities of $\mathbb{R}^3$. Then any $f \in \mathrm{Sim}^+(\mathbb{R}^3)$ can be expressed as a product

$$f = g_2 \circ g_1 \circ h,$$

where h is a homothety centered at the origin and with a scaling factor $\lambda > 0$; g_1 is an orthogonal mapping preserving the origin, and g_2 is a translation. Let $\mathbb{H}$ be the quaternion algebra. We identify $\mathbb{R}^3$ with $\mathrm{Im}\,\mathbb{H}$. According to Hamilton's Theorem (see [6], p. 216, or [7], p. 130) there exists a unit quaternion $\boldsymbol{n}$ such that

$$\mathrm{Im}\,\mathbb{H} \ni \boldsymbol{z} \overset{g_1}{\to} \boldsymbol{n}.\boldsymbol{z}.\boldsymbol{n}^{-1} \in \mathrm{Im}\,\mathbb{H}.$$

Thus we obtain

$$f(\boldsymbol{z}) = \lambda \boldsymbol{n}.\boldsymbol{z}.\boldsymbol{n}^{-1} + \boldsymbol{a} \quad (\boldsymbol{z} \in \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3)$$

for some fixed $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\boldsymbol{a} \in \mathrm{Im}\,\mathbb{H}$.

Let

$$\boldsymbol{c}\colon\ (t_1,\, t_2) \ni t \ \to\ \boldsymbol{c}(t) \in \mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H}$$

be a curve of class C^3. We denote the image of $\boldsymbol{c}$ under the direct similarity f by $\boldsymbol{c}_0$, i.e., $\boldsymbol{c}_0 = f \circ \boldsymbol{c}$. Then $\boldsymbol{c}_0$ can be expressed as

$$(t_1,\, t_2) \ni t \overset{\boldsymbol{c}_0}{\to} \lambda \boldsymbol{n}\boldsymbol{c}(t)\boldsymbol{n}^{-1} + \boldsymbol{a} \in \mathrm{Im}\,\mathbb{H}.$$

The arc length functions of $\boldsymbol{c}$ and $\boldsymbol{c}_0$ starting at $t_0 \in (t_1,\, t_2)$ are

$$s(t) = \int_{t_0}^{t} \left\| \frac{d\boldsymbol{c}(u)}{du} \right\| du \ \text{ and } s_0(t) = \int_{t_0}^{t} \left\| \frac{d\boldsymbol{c}_0(u)}{du} \right\| du = \lambda s(t).$$

Both curves have reparametrizations by the arc length parameter $\boldsymbol{c} : (s_1,\, s_2) \to \mathbb{R}^3$ and $\boldsymbol{c}_0 \colon (\lambda s_1,\, \lambda s_2) \to \mathbb{R}^3$. In this section we denote by primes the differentiation with respect to s. It is well-known (see [4], p. 125 and p. 127) that the Frenet curvature κ_1 and the torsion κ_2 of $\boldsymbol{c}$ are given by

$$\kappa_1(s) = \|\boldsymbol{c}''(s)\|, \quad \kappa_2(s) = \frac{\det(\boldsymbol{c}'(s),\, \boldsymbol{c}''(s),\, \boldsymbol{c}'''(s))}{\|\boldsymbol{c}''(s)\|^2}.$$

Since $\frac{ds}{ds_0} = \frac{1}{\lambda}$ (= const.), the curvature $\kappa_{10}(s_0) = \kappa_{10}(\lambda s)$ and the torsion $\kappa_{20}(s_0) = \kappa_{20}(\lambda s)$ of the curve $\boldsymbol{c}_0$ are given by

$$\kappa_{10} = \frac{1}{\lambda}\,\kappa_1(s), \quad \kappa_{20} = \frac{1}{\lambda}\,\kappa_2(s) \tag{1}$$

Thus we obtain $\kappa_1 ds = \kappa_{10} ds_0$ and $\kappa_2 ds = \kappa_{20} ds_0$.

Any curve in the Euclidean space with non-zero curvature, or equivalently a Frenet space curve, admits a reparameterization by an arc length parameter of its spherical image (see [5] and [9]). If we denote by σ and σ_0 the spherical arc length parameters of $\boldsymbol{c}$ and $\boldsymbol{c}_0$, respectively, we have that

$$d\sigma = \kappa_1 ds = \kappa_{10} ds_0 = d\sigma_0. \tag{2}$$

Hence, $d\sigma = \kappa_1 ds$ is invariant under the group of the direct similarities of $\mathbb{R}^3$.

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be a Frenet frame field along the curve $\boldsymbol{c}$ parameterized by the spherical arc length parameter σ. Then the structure equations of $\boldsymbol{c}$ are given by

$$\frac{d\boldsymbol{c}}{d\sigma} = \frac{1}{\kappa_1}\boldsymbol{e}_1\,, \quad \frac{d\boldsymbol{e}_1}{d\sigma} = \boldsymbol{e}_2\,, \quad \frac{d\boldsymbol{e}_2}{d\sigma} = -\boldsymbol{e}_1 + \frac{\kappa_2}{\kappa_1}\boldsymbol{e}_3\,, \quad \frac{d\boldsymbol{e}_3}{d\sigma} = -\frac{\kappa_2}{\kappa_1}\boldsymbol{e}_2\,.$$

Moreover, we have

$$\frac{d^2\boldsymbol{c}}{d\sigma^2} = -\frac{d\kappa_1}{\kappa_1 d\sigma} \cdot \frac{d\boldsymbol{c}}{d\sigma} + \frac{1}{\kappa_1}\boldsymbol{e}_2\,. \tag{3}$$

Similarly, $\dfrac{d^2\boldsymbol{c}_0}{d\sigma_0^2} = -\dfrac{d\kappa_{10}}{\kappa_{10} d\sigma} \cdot \dfrac{d\boldsymbol{c}_0}{d\sigma_0} + \dfrac{1}{\kappa_{10}}\boldsymbol{e}_{20}$, where $\boldsymbol{e}_{10}$, $\boldsymbol{e}_{20}$, $\boldsymbol{e}_{30}$ is a Frenet frame field along the curve $\boldsymbol{c}_0 = f \circ \boldsymbol{c}$. Using (1) and (2) we get

$$-\frac{d\kappa_{10}}{\kappa_{10}\, d\sigma_0} = -\frac{d\frac{1}{\lambda}\kappa_1}{d\sigma} \cdot \frac{1}{\frac{1}{\lambda}\kappa_1} = -\frac{d\kappa_1}{\kappa_1\, d\sigma} \quad \text{and} \quad \frac{\kappa_{20}}{\kappa_{10}} = \frac{\frac{1}{\lambda}\kappa_2}{\frac{1}{\lambda}\kappa_1} = \frac{\kappa_2}{\kappa_1}\,.$$

By above considerations we obtain the following

Lemma 1 *The functions* $\widetilde{\kappa}_1 = -\dfrac{d\kappa_1}{\kappa_1 d\sigma}$ *and* $\widetilde{\kappa}_2 = \dfrac{\kappa_2}{\kappa_1}$ *are invariants under the group of the direct similarities of the Euclidean space.*

The equation (3) and the structure equations of $\boldsymbol{c}$ can be rewritten in the form

$$\frac{d^2\boldsymbol{c}}{d\sigma^2} = \widetilde{\kappa}_1\frac{d\boldsymbol{c}}{d\sigma} + \frac{1}{\kappa_1}\boldsymbol{e}_2\,, \quad \frac{d\boldsymbol{e}_1}{d\sigma} = \boldsymbol{e}_2\,, \quad \frac{d\boldsymbol{e}_2}{d\sigma} = -\boldsymbol{e}_1 + \widetilde{\kappa}_2\boldsymbol{e}_3\,, \quad \frac{d\boldsymbol{e}_3}{d\sigma} = -\widetilde{\kappa}_2\boldsymbol{e}_2$$

and then the invariants $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ can be expressed as

$$\widetilde{\kappa}_1(\sigma) = \frac{\left\langle \frac{d^2\boldsymbol{c}}{d\sigma^2}, \frac{d\boldsymbol{c}}{d\sigma} \right\rangle}{\left\langle \frac{d\boldsymbol{c}}{d\sigma}, \frac{d\boldsymbol{c}}{d\sigma} \right\rangle}, \tag{4}$$

$$\widetilde{\kappa}_2(\sigma) = \det\left(\frac{d\boldsymbol{c}}{d\sigma}, \frac{d^2\boldsymbol{c}}{d\sigma^2}, \frac{d^3\boldsymbol{c}}{d\sigma^3}\right) \cdot \left[\frac{\left(\frac{d\boldsymbol{c}}{d\sigma}\right)^2}{\left(\frac{d\boldsymbol{c}}{d\sigma}\right)^2\left(\frac{d^2\boldsymbol{c}}{d\sigma^2}\right)^2 - \left\langle \frac{d\boldsymbol{c}}{d\sigma}, \frac{d^2\boldsymbol{c}}{d\sigma^2} \right\rangle^2}\right]^{3/2}. \tag{5}$$

Definition 1 *Let* $\boldsymbol{c}\colon I \to \mathbb{R}^3$ *be a Frenet space curve of the class* C^3 *parameterized by a spherical arc length parameter* σ. *Let* $\kappa_1(\sigma)$ *and* $\kappa_2(\sigma)$ *be the curvature and the torsion of* $\boldsymbol{c}$*, respectively. The functions*

$$\widetilde{\kappa}_1 = -\frac{d\kappa_1}{\kappa_1 d\sigma} \quad \text{and} \quad \widetilde{\kappa}_2 = \frac{\kappa_2}{\kappa_1}$$

are called shape curvature and shape torsion of $\boldsymbol{c}$. *The ordered pair* $(\widetilde{\kappa}_1, \widetilde{\kappa}_2)$ *is called a (local) shape of the curve* $\boldsymbol{c}$.

For an arc length parameter s we have

$$\widetilde{\kappa}_1 = -\frac{d\kappa_1}{ds}\cdot\frac{ds}{d\sigma}\cdot\frac{1}{\kappa_1} = -\frac{\kappa_1'}{\kappa_1^2} = \left(\frac{1}{\kappa_1}\right)' = \frac{(\log\|\boldsymbol{c}''(s)\|^{-1})'}{\|\boldsymbol{c}''(s)\|},$$

$$\widetilde{\kappa}_2 = \frac{\kappa_2}{\kappa_1} = \frac{\det(\boldsymbol{c}'(s),\, \boldsymbol{c}''(s),\, \boldsymbol{c}'''(s))}{\|\boldsymbol{c}''\|^3},$$

and more generally for an arbitrary parameter t

$$\widetilde{\kappa}_1 = -\frac{d\kappa_1}{dt}\cdot\frac{dt}{d\sigma}\cdot\frac{1}{\kappa_1} = \frac{3\left\|\frac{d\boldsymbol{c}}{dt}\times\frac{d^2\boldsymbol{c}}{dt^2}\right\|^2\left\langle\frac{d\boldsymbol{c}}{dt},\,\frac{d^2\boldsymbol{c}}{dt^2}\right\rangle - \left\|\frac{d\boldsymbol{c}}{dt}\right\|^2\left\langle\frac{d\boldsymbol{c}}{dt}\times\frac{d^2\boldsymbol{c}}{dt^2},\,\frac{d\boldsymbol{c}}{dt}\times\frac{d^3\boldsymbol{c}}{dt^3}\right\rangle}{\left\|\frac{d\boldsymbol{c}}{dt}\times\frac{d^2\boldsymbol{c}}{dt^2}\right\|^3},$$

$$\widetilde{\kappa}_2 = \frac{\kappa_2}{\kappa_1} = \frac{\left\|\frac{d\boldsymbol{c}}{dt}\right\|^3 .\ \det\left(\frac{d\boldsymbol{c}}{dt},\,\frac{d^2\boldsymbol{c}}{dt^2},\,\frac{d^3\boldsymbol{c}}{dt^3}\right)}{\left\|\frac{d\boldsymbol{c}}{dt}\times\frac{d^2\boldsymbol{c}}{dt^2}\right\|^3}$$

In case of plane curves other formulas for $\widetilde{\kappa}_1$ are given in ([8], p. 53) and ([10], p. 105). Note that the invariants $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ were introduced by É. Cartan in [2].

It is easy to see that both $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ are not affine invariants. For example, a circle and an ellipse are always affine equivalent but they have different shape curvature functions. Similarly, two circular helices $\boldsymbol{c}_1(t) = (a\cos t,\, a\sin t,\, bt)$ and $\boldsymbol{c}_2(t) = (b\cos t,\, b\sin t,\, at)$, where $a > b > 0$, are affine equivalent but their shape torsion functions b/a and a/b are different.

3. Uniqueness theorem

The definition of the "shape curvature" in terms of a spherical arc length parameter suggests to recognize the curve from its "shape data". Two Frenet curves with the same torsion and the same always positive curvature are equivalent modulo a Euclidean motion. This statement can be extended for the Frenet curves with the same shape curvature and shape torsion.

Theorem 1 (Uniqueness Theorem) *Let $I \subset \mathbb{R}$ be an open interval and let $\boldsymbol{c}_i : I \to \mathbb{R}^3$, $i = 1, 2$, be two Frenet curves of class C^3 parameterized by the same spherical arc length parameter σ. Assume that $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ have the same shape curvature $\widetilde{\kappa}_1$ and the same shape torsion $\widetilde{\kappa}_2$ for any $\sigma \in I$. Then there exists a direct similarity f of $\mathbb{R}^3$ such that $\boldsymbol{c}_2 = f \circ \boldsymbol{c}_1$.*

Proof: Let $\kappa_{i1} \neq 0$ and κ_{i2} be the curvature and the torsion of the curve $\boldsymbol{c}_i$, $i = 1, 2$. Denote by $\widetilde{\kappa}_{i1}$ and $\widetilde{\kappa}_{i2}$ the shape curvature and the shape torsion of $\boldsymbol{c}_i$. Using $\widetilde{\kappa}_{11} = \widetilde{\kappa}_{21}$ we obtain $d\kappa_{11}/\kappa_{11} = d\kappa_{21}/\kappa_{21}$ or equivalently $\log\kappa_{11} = \log\kappa_{21} + \log\lambda$, where λ is a positive real constant. Then $\kappa_{11} = \lambda\kappa_{21}$ for any $\sigma \in I$. Applying $\widetilde{\kappa}_{12} = \widetilde{\kappa}_{22}$ we also get $\kappa_{12} = \lambda\kappa_{22}$ for any $\sigma \in I$. Let $\boldsymbol{e}_{ij}$, $j = 1, 2, 3$, be a Frenet frame field on $\boldsymbol{c}_i$, $i = 1, 2$, and let $\sigma_0 \in I$. Since $\|\boldsymbol{e}_{ij}\| = 1$ there exists an orientation-preserving Euclidean motion h of $\mathbb{R}^3$ such that $h(\boldsymbol{c}_1(\sigma_0)) = \boldsymbol{c}_2(\sigma_0)$ and $h(\boldsymbol{e}_{1j}(\sigma_0)) = \boldsymbol{e}_{2j}(\sigma_0)$ for $j = 1, 2, 3$. Consider the function $\phi\colon I \to \mathbb{R}$ given by

$$\phi(\sigma) = \|h(\boldsymbol{e}_{11}(\sigma)) - \boldsymbol{e}_{21}(\sigma)\|^2 + \|h(\boldsymbol{e}_{12}(\sigma)) - \boldsymbol{e}_{22}(\sigma)\|^2 + \|h(\boldsymbol{e}_{13}(\sigma)) - \boldsymbol{e}_{23}(\sigma)\|^2$$

for $\sigma \in I$. Then

$$\begin{aligned}
\frac{d\phi}{d\sigma} &= 2\left\langle \frac{d}{d\sigma}h(\boldsymbol{e}_{11}(\sigma)) - \frac{d}{d\sigma}\boldsymbol{e}_{21}(\sigma),\ h(\boldsymbol{e}_{11}(\sigma)) - \boldsymbol{e}_{21}(\sigma) \right\rangle + \\
&+ 2\left\langle \frac{d}{d\sigma}h(\boldsymbol{e}_{12}(\sigma)) - \frac{d}{d\sigma}\boldsymbol{e}_{22}(\sigma),\ h(\boldsymbol{e}_{12}(\sigma)) - \boldsymbol{e}_{22}(\sigma) \right\rangle + \\
&+ 2\left\langle \frac{d}{d\sigma}h(\boldsymbol{e}_{13}(\sigma)) - \frac{d}{d\sigma}\boldsymbol{e}_{23}(\sigma),\ h(\boldsymbol{e}_{13}(\sigma)) - \boldsymbol{e}_{23}(\sigma) \right\rangle .
\end{aligned}$$

Since $\|h(\boldsymbol{e}_{1j})\|^2 = \|\boldsymbol{e}_{1j}\|^2 = 1$ and $\|\boldsymbol{e}_{2j}\|^2 = 1$ we have

$$\left\langle \frac{d}{d\sigma}h(\boldsymbol{e}_{1j}(\sigma)),\ h(\boldsymbol{e}_{1j}(\sigma)) \right\rangle = 0 \ \text{ and } \ \left\langle \frac{d\boldsymbol{e}_{2j}}{d\sigma}(\sigma),\, \boldsymbol{e}_{2j} \right\rangle = 0, \quad j = 1,2,3 .$$

Hence,

$$\begin{aligned}
\frac{d\phi}{d\sigma} = & -2\left\langle h\Big(\frac{d\boldsymbol{e}_{11}}{d\sigma}(\sigma)\Big),\, \boldsymbol{e}_{21}(\sigma) \right\rangle - 2\left\langle \frac{d\boldsymbol{e}_{21}}{d\sigma}(\sigma),\, h(\boldsymbol{e}_{11}(\sigma)) \right\rangle - 2\left\langle h\Big(\frac{d\boldsymbol{e}_{12}}{d\sigma}(\sigma)\Big),\, \boldsymbol{e}_{22}(\sigma) \right\rangle - \\
& -2\left\langle \frac{d\boldsymbol{e}_{22}}{d\sigma}(\sigma),\ h(\boldsymbol{e}_{12}(\sigma)) \right\rangle - 2\left\langle h\Big(\frac{d\boldsymbol{e}_{13}}{d\sigma}(\sigma)\Big),\ \boldsymbol{e}_{23}(\sigma) \right\rangle - 2\left\langle \frac{d\boldsymbol{e}_{23}}{d\sigma}(\sigma),\ h(\boldsymbol{e}_{13}(\sigma)) \right\rangle .
\end{aligned}$$

Using the structure equations

$$\frac{d\boldsymbol{e}_{i1}}{d\sigma} = \boldsymbol{e}_{i2}\,, \quad \frac{d\boldsymbol{e}_{i2}}{d\sigma} = -\boldsymbol{e}_{i1} + \widetilde{\kappa}_{i2}\boldsymbol{e}_{i3}\,, \quad \frac{d\boldsymbol{e}_{i3}}{d\sigma} = -\widetilde{\kappa}_{i2}\boldsymbol{e}_{i2}\,, \quad i = 1,2\,,$$

we obtain

$$\begin{aligned}
\frac{d\phi}{d\sigma} &= -2\widetilde{\kappa}_{12}\left\langle h(\boldsymbol{e}_{13}(\sigma)),\ \boldsymbol{e}_{22}(\sigma)\right\rangle - 2\widetilde{\kappa}_{22}\left\langle \boldsymbol{e}_{23}(\sigma),\ h(\boldsymbol{e}_{12}(\sigma))\right\rangle + \\
&\quad +2\widetilde{\kappa}_{12}\left\langle h(\boldsymbol{e}_{12}(\sigma)),\ \boldsymbol{e}_{23}(\sigma)\right\rangle + 2\widetilde{\kappa}_{22}\left\langle \boldsymbol{e}_{22}(\sigma),\ h(\boldsymbol{e}_{13}(\sigma))\right\rangle .
\end{aligned}$$

But $\widetilde{\kappa}_{12} = \widetilde{\kappa}_{22}$ and then $d\phi/d\sigma = 0$ for any $\sigma \in I$. On the other hand $\phi(\sigma_0) = 0$ and hence $\phi(\sigma) = 0$ for any $\sigma \in I$. This implies that $h(\boldsymbol{e}_{1j}(\sigma)) = \boldsymbol{e}_{2j}(\sigma)$ for any $\sigma \in I$, $j = 1,2,3$. The map $g = \lambda h\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a direct similarity of $\mathbb{R}^3$ because $\lambda = \kappa_{11}/\kappa_{21}$ is a positive real constant.

We also consider the function $\psi\colon I \to \mathbb{R}$, defined by

$$\psi(\sigma) = \left\| \frac{d}{d\sigma}g(\boldsymbol{c}_1(\sigma)) - \frac{d}{d\sigma}\boldsymbol{c}_2(\sigma) \right\|^2 \ \text{ for any } \ \sigma \in I.$$

Then

$$\begin{aligned}
\frac{d\psi}{d\sigma} &= 2\left\langle \frac{d^2}{d\sigma^2}g(\boldsymbol{c}_1(\sigma)) - \frac{d^2}{d\sigma^2}\boldsymbol{c}_2(\sigma),\ \frac{d}{d\sigma}g(\boldsymbol{c}_1(\sigma)) - \frac{d}{d\sigma}\boldsymbol{c}_2(\sigma) \right\rangle = \\
&= 2\left\langle g\Big(\frac{d^2\boldsymbol{c}_1}{d\sigma^2}(\sigma)\Big),\ g\Big(\frac{d\boldsymbol{c}_1}{d\sigma}(\sigma)\Big) \right\rangle - 2\left\langle g\Big(\frac{d^2\boldsymbol{c}_1}{d\sigma^2}(\sigma)\Big),\ \frac{d\boldsymbol{c}_2}{d\sigma}(\sigma) \right\rangle - \\
&\quad -2\left\langle \frac{d^2\boldsymbol{c}_2}{d\sigma^2}(\sigma),\ g\Big(\frac{d\boldsymbol{c}_1}{d\sigma}(\sigma)\Big) \right\rangle + 2\left\langle \frac{d^2\boldsymbol{c}_2}{d\sigma^2}(\sigma),\ \frac{d\boldsymbol{c}_2}{d\sigma}(\sigma) \right\rangle .
\end{aligned}$$

Using the properties of h we obtain

$$\frac{d\psi}{d\sigma} = -2\lambda^2 \frac{\widetilde{\kappa}_{11}}{\kappa_{11}^2} - 2\frac{\widetilde{\kappa}_{21}}{\kappa_{21}^2} + 2\lambda \frac{\widetilde{\kappa}_{11}}{\kappa_{11}\kappa_{21}} + 2\lambda \frac{\widetilde{\kappa}_{21}}{\kappa_{11}\kappa_{21}}\,.$$

From $\widetilde{\kappa}_{11} = \widetilde{\kappa}_{21}$ and $\kappa_{11} = \lambda\kappa_{21}$ we conclude that

$$\frac{d\psi}{d\sigma} = -4\frac{\widetilde{\kappa}_{21}}{\kappa_{21}^2} + 4\frac{\widetilde{\kappa}_{21}}{\kappa_{21}^2} = 0\,.$$

Applying the structure equations of the curves $g \circ \boldsymbol{c}_1$ and $\boldsymbol{c}_2$ we calculate

$$\frac{d}{d\sigma} g(\boldsymbol{c}_1(\sigma_0)) = \frac{\lambda}{\kappa_{11}}\boldsymbol{e}_{21}(\sigma_0), \quad \frac{d}{d\sigma}(\boldsymbol{c}_2)(\sigma_0) = \frac{1}{\kappa_{21}}\boldsymbol{e}_{21}(\sigma_0)$$

and then $\psi(\sigma_0) = 0$. Hence, $\psi(\sigma) = 0$ for any $\sigma \in I$, or equivalently $\frac{d}{d\sigma}\boldsymbol{c}_2(\sigma) \equiv \frac{d}{d\sigma} g(\boldsymbol{c}_1(\sigma))$. This means that there exists a constant vector $\boldsymbol{q} \in \mathbb{R}^3$ such that $\boldsymbol{c}_2(\sigma) = g(\boldsymbol{c}_1(\sigma)) + \boldsymbol{q}$. Let $g_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the translation determined by the vector $\boldsymbol{q}$. Then the image of $\boldsymbol{c}_1$ under the direct similarity $f = g_1 \circ g$ is the curve $\boldsymbol{c}_2$. □

4. Representations of space curves by curves on the unit sphere

Izumiya and Takeuchi [5] show that any Bertrand curve can be constructed from the spherical curve whose spherical evolute coincides with the spherical Darboux image of the Bertrand curve. Using a shape curvature and a shape torsion we obtain a similar representation for any Frenet curve.

Let $\boldsymbol{\gamma} : I \to S^2$ be a unit speed spherical curve with σ as arc length parameter of $\boldsymbol{\gamma}$. Then $\boldsymbol{t}(\sigma) = d\boldsymbol{\gamma}(\sigma)/d\sigma$ is the unit tangent vector of $\boldsymbol{\gamma}$ at σ. Consider the unit vector field $\boldsymbol{p}(\sigma) = \boldsymbol{\gamma}(\sigma) \times \boldsymbol{t}(\sigma)$. The orthogonal frame $\{\boldsymbol{\gamma}(\sigma), \boldsymbol{t}(\sigma), \boldsymbol{p}(\sigma)\}$ along $\boldsymbol{\gamma}$ is called the Sabban frame of $\boldsymbol{\gamma}$ (see [5]). Then the following Frenet-Serret formulas hold

$$\frac{d\boldsymbol{\gamma}(\sigma)}{d\sigma} = \boldsymbol{t}(\sigma), \quad \frac{d\boldsymbol{t}(\sigma)}{d\sigma} = -\boldsymbol{\gamma}(\sigma) + k_g(\sigma)\boldsymbol{p}(\sigma), \quad \frac{d\boldsymbol{p}(\sigma)}{d\sigma} = -k_g(\sigma)\boldsymbol{t}(\sigma), \tag{6}$$

where $k_g(\sigma) = \det\left(\boldsymbol{\gamma}(\sigma),\ \boldsymbol{t}(\sigma),\ d\boldsymbol{t}(\sigma)/d\sigma\right)$ is the geodesic curvature of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(\sigma)$.

Let $k \colon I \to \mathbb{R}$ be a function of class C^1. Then we may define a space curve $\boldsymbol{c} \colon I \to \mathbb{R}^3$ given by

$$\boldsymbol{c}(\sigma) = b\int e^{\int k(\sigma)d\sigma}\boldsymbol{\gamma}(\sigma)d\sigma + \boldsymbol{a}\,, \tag{7}$$

where $\boldsymbol{a}$ is a constant vector and b is a real constant. Obviously, σ is a spherical arc length parameter of $\boldsymbol{c}$, because $\frac{d\boldsymbol{c}}{d\sigma} : \left\|\frac{d\boldsymbol{c}}{d\sigma}\right\| = \boldsymbol{\gamma}(\sigma)$. Under the above assumptions we can give a description of all Frenet curves in $\mathbb{R}^3$.

Proposition 1 *The curve $\boldsymbol{c}$ defined by (7) is a Frenet curve with shape curvature $\widetilde{\kappa}_1(\sigma) = k(\sigma)$ and shape torsion $\widetilde{\kappa}_2(\sigma) = k_g(\sigma)$. Moreover, all Frenet curves can be obtained in this way.*

Proof. First, we have that

$$\frac{d\boldsymbol{c}}{d\sigma} = b\,e^{\int k(\sigma)d\sigma}\boldsymbol{\gamma}(\sigma), \quad \frac{d^2\boldsymbol{c}}{d\sigma^2} = b\,e^{\int k(\sigma)d\sigma}\left\{k(\sigma)\boldsymbol{\gamma}(\sigma) + \frac{d\boldsymbol{\gamma}}{d\sigma}\right\}$$

and

$$\frac{d^3\boldsymbol{c}}{d\sigma^3} = b\, e^{\int k(\sigma)d\sigma}\Big\{\Big(k^2(\sigma) + \frac{dk}{d\sigma}\Big)\boldsymbol{\gamma}(\sigma) + 2k(\sigma)\,\frac{d\boldsymbol{\gamma}}{d\sigma} + \frac{d^2\boldsymbol{\gamma}}{d\sigma^2}\Big\}.$$

Since $\|\boldsymbol{\gamma}(\sigma)\| = 1$ and $\|d\boldsymbol{\gamma}/d\sigma\| = 1$ then $\big\langle\boldsymbol{\gamma}(\sigma),\ \frac{d\boldsymbol{\gamma}}{d\sigma}\big\rangle = 0$ and $\boldsymbol{\gamma}(\sigma) \times \frac{d\boldsymbol{\gamma}}{d\sigma} \neq 0$ for any $\sigma \in I$. Hence,

$$\frac{d\boldsymbol{c}}{d\sigma} \times \frac{d^2\boldsymbol{c}}{d\sigma^2} = b^2\, e^{2\int k(\sigma)d\sigma}\Big(\boldsymbol{\gamma} \times \frac{d\boldsymbol{\gamma}}{d\sigma}\Big) \neq 0\,.$$

This means that $\boldsymbol{c}$ is a Frenet curve. Applying the equations (4) and (5) from Section 2 we find that

$$\widetilde{\kappa}_1(\sigma) = k(\sigma) \ \text{ and } \ \widetilde{\kappa}_2(\sigma) = \det\left(\boldsymbol{\gamma},\ \frac{d\boldsymbol{\gamma}}{d\sigma},\ \frac{d^2\boldsymbol{\gamma}}{d\sigma^2}\right) = k_g(\sigma).$$

Conversely, let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a regular curve parameterized by a spherical arc length parameter σ. Denote by $\kappa_1(\sigma)$ and $\kappa_2(\sigma)$ the curvature and the torsion of $\boldsymbol{c}$, respectively. Then

$$\widetilde{\kappa}_1(\sigma) = -\frac{d\kappa_1(\sigma)}{\kappa_1 d\sigma} \ \text{ and } \ \widetilde{\kappa}_2(\sigma) = \frac{\kappa_2(\sigma)}{\kappa_1(\sigma)}$$

are the shape curvature and the shape torsion of $\boldsymbol{c}$. Consider the spherical indicatrix $\boldsymbol{\gamma}$ of $\boldsymbol{c}$, i.e., the curve $\boldsymbol{\gamma}\colon I \to S^2$ is given by

$$\boldsymbol{\gamma}(\sigma) = \boldsymbol{e}_1 = \frac{d\boldsymbol{c}}{d\sigma} : \left\|\frac{d\boldsymbol{c}}{d\sigma}\right\| = \kappa_1(\sigma)\frac{d\boldsymbol{c}}{d\sigma}\,.$$

Obviously, σ is an arc length parameter of $\boldsymbol{\gamma}$ and $k_g = \det\Big(\boldsymbol{\gamma}(\sigma),\ \boldsymbol{t}(\sigma),\ \frac{d\boldsymbol{t}(\sigma)}{d\sigma}\Big) = \widetilde{\kappa}_2(\sigma)$ is the geodesic curvature of $\boldsymbol{\gamma}$. If $k(\sigma) = \widetilde{\kappa}_1(\sigma)$, then

$$\begin{aligned}\int e^{\int k(\sigma)d\sigma}\boldsymbol{\gamma}(\sigma)d\sigma &= \int e^{\int \widetilde{\kappa}_1(\sigma)d\sigma}\boldsymbol{e}_1(\sigma)d\sigma = \int e^{-\int \frac{d\kappa_1}{\kappa_1 d\sigma}d\sigma}\boldsymbol{e}_1(\sigma)d\sigma = \\ &= \int e^{-\int \frac{d\kappa_1}{\kappa_1}}\boldsymbol{e}_1(\sigma)d\sigma = e^{b_0}\int \frac{1}{\kappa_1}\,\boldsymbol{e}_1(\sigma)d\sigma = e^{b_0}\int \frac{d\boldsymbol{c}}{d\sigma}\,d\sigma = e^{b_0}\boldsymbol{c}(\sigma) + \boldsymbol{c}_0\,,\end{aligned}$$

where $\boldsymbol{c}_0$ is a constant vector and b_0 is a real constant. Thus,

$$\boldsymbol{c}(\sigma) = b\int e^{\int k(\sigma)d\sigma}\boldsymbol{\gamma}(\sigma)d\sigma + \boldsymbol{a}. \qquad \square$$

Now we have a simple description of all cylindrical helices.

Corollary 1 *The spherical curve $\boldsymbol{\gamma}$ is a circle if and only if the corresponding space curves defined by (7) are cylindrical helices.*

Proof: $\boldsymbol{\gamma}$ is a circle if and only if $k_g(\sigma) = \widetilde{\kappa}_2(\sigma) = \dfrac{\kappa_2(\sigma)}{\kappa_1(\sigma)} = \text{const.}$ $\square$

In order to give another relation between the spherical curve $\boldsymbol{\gamma}$ and the corresponding space curve defined by (7) we recall the definitions of a spherical evolute and a Darboux indicatrix (see [5] for more details). The spherical evolute of $\boldsymbol{\gamma}$ is given by

$$\boldsymbol{\epsilon}_{\boldsymbol{\gamma}(\sigma)} = \frac{1}{\sqrt{k_g^2(\sigma) + 1}}\,\left(k_g(\sigma)\boldsymbol{\gamma}(\sigma) + \boldsymbol{p}(\sigma)\right). \tag{8}$$

Let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a curve parameterized by a spherical arc length parameter σ. If $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the Frenet frame field of $\boldsymbol{c}$, then the Darboux vector of $\boldsymbol{c}$ is

$$\boldsymbol{D}(\sigma) = \kappa_2(\sigma)\boldsymbol{e}_1 + \kappa_1(\sigma)\boldsymbol{e}_3.$$

The normalization of the Darboux vector

$$\boldsymbol{D}^0(\sigma) = \frac{\boldsymbol{D}(\sigma)}{\|\boldsymbol{D}(\sigma)\|} = \frac{1}{\sqrt{\widetilde{\kappa}_2^2+1}}\left(\widetilde{\kappa}_2\boldsymbol{e}_1 + \boldsymbol{e}_3\right) \tag{9}$$

is called the spherical Darboux image or the Darboux indicatrix of $\boldsymbol{c}$.

Proposition 2 *Let $\boldsymbol{\gamma}\colon I \to S^2$ be a spherical curve and let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a corresponding space curve defined by (7). Then, the spherical Darboux image of $\boldsymbol{c}$ coincides with the spherical evolute of $\boldsymbol{\gamma}$.*

Proof. We observe that $\boldsymbol{e}_1(\sigma) = \boldsymbol{\gamma}(\sigma)$ and

$$\boldsymbol{e}_3(\sigma) = \frac{\frac{d\boldsymbol{c}}{d\sigma} \times \frac{d^2\boldsymbol{c}}{d\sigma^2}}{\left\|\frac{d\boldsymbol{c}}{d\sigma} \times \frac{d^2\boldsymbol{c}}{d\sigma^2}\right\|} = \boldsymbol{\gamma}(\sigma) \times \frac{d\boldsymbol{\gamma}}{d\sigma}(\sigma) = \boldsymbol{p}(\sigma).$$

Hence, using (8) and (9) we get

$$\boldsymbol{D}^0(\sigma) = \frac{1}{\sqrt{k_g^2+1}}\left(k_g\boldsymbol{\gamma}(\sigma) + \boldsymbol{p}(\sigma)\right) = \boldsymbol{\epsilon}_{\boldsymbol{\gamma}}(\sigma). \qquad \square$$

Now we can show that the shape curvature and the shape torsion determine a Frenet curve under some initial conditions.

Theorem 2 (Existence Theorem) *Let $f_i : I \to \mathbb{R}$, $i = 1, 2$, be two functions of class C^1. Let $\boldsymbol{e}_1^0$, $\boldsymbol{e}_2^0$, $\boldsymbol{e}_3^0$ be an right-handed orthonormal triad of vectors at a point $\boldsymbol{c}_0$ in the Euclidean space $\mathbb{R}^3$. Up to an orientation-preserving homothety with center $\boldsymbol{c}_0$ there exists a unique Frenet curve $\boldsymbol{c}\colon I \to \mathbb{R}^3$, which satisfies the conditions:*

(i) There is $\sigma_0 \in I$ such that $\boldsymbol{c}(\sigma_0) = \boldsymbol{c}_0$ and the Frenet frame of $\boldsymbol{c}$ at $\boldsymbol{c}_0$ is $\{\boldsymbol{e}_1^0, \boldsymbol{e}_2^0, \boldsymbol{e}_3^0\}$.

(ii) For any $\sigma \in I$, $\widetilde{\kappa}_1(\sigma) = f_1(\sigma)$ and $\widetilde{\kappa}_2(\sigma) = f_2(\sigma)$.

Proof: We consider the system of differential equations

$$\frac{d\boldsymbol{\gamma}}{d\sigma} = \boldsymbol{t}(\sigma), \quad \frac{d\boldsymbol{t}}{d\sigma} = -\boldsymbol{\gamma}(\sigma) + f_2(\sigma)\boldsymbol{p}(\sigma), \quad \frac{d\boldsymbol{p}}{d\sigma} = -f_2(\sigma)\boldsymbol{t}(\sigma) \tag{10}$$

with respect to the vectorial functions $\boldsymbol{\gamma}(\sigma)$, $\boldsymbol{t}(\sigma)$ and $\boldsymbol{p}(\sigma)$. If we rewrite the coordinate functions of these vectorial functions as rows of the matrix $X(\sigma)$ then the system (10) may be represented in the form

$$\frac{dX}{d\sigma}(\sigma) = M(\sigma)X(\sigma), \quad \text{where} \quad M(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & f_2(\sigma) \\ 0 & -f_2(\sigma) & 0 \end{pmatrix}. \tag{11}$$

It is well-known that the system (11) has an unique solution

$$X(\sigma) = (\boldsymbol{\gamma}(\sigma), \boldsymbol{t}(\sigma), \boldsymbol{p}(\sigma)) \tag{12}$$

satisfying the initial conditions $X(\sigma_0) = (\boldsymbol{e}_1^0,\, \boldsymbol{e}_2^0,\, \boldsymbol{e}_3^0)$ for $\sigma_0 \in I$. If ${}^tX(\sigma)$ denotes the transposed matrix of $X(\sigma)$ and E is the unit matrix then $\frac{d}{d\sigma}({}^tX(\sigma)) = {}^tX(\sigma)\,{}^tM(\sigma)$ and

$$\frac{d}{d\sigma}\left({}^tX(\sigma)X(\sigma)\right) = {}^tX(\sigma)\left({}^tM(\sigma) + M(\sigma)\right)X(\sigma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

because the matrix $M(\sigma)$ is skew-symmetric. But ${}^tX(\sigma_0)X(\sigma_0) = E$ since $\boldsymbol{e}_1^0,\, \boldsymbol{e}_2^0,\, \boldsymbol{e}_3^0$ are orthonormal. Consequently, ${}^tX(\sigma)X(\sigma) = E$ for any $\sigma \in I$. This implies that the vectorial fields $\boldsymbol{\gamma}(\sigma),\ \boldsymbol{t}(\sigma),\ \boldsymbol{p}(\sigma)$ form a right-handed orthonormal frame field.

Let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be the space curve given by

$$\boldsymbol{c}(\sigma) = \boldsymbol{c}_0 + b\int_{\sigma_0}^{\sigma} e^{\int f_1(\sigma)d\sigma}\boldsymbol{\gamma}(\sigma)\,d\sigma, \quad \sigma \in I,\ b > 0.$$

From Proposition 1 we have that the Frenet frame field of $\boldsymbol{c}$ is $\{\boldsymbol{e}_1 = \boldsymbol{\gamma}(\sigma),\ \boldsymbol{e}_2 = \boldsymbol{t}(\sigma),\ \boldsymbol{e}_3 = \boldsymbol{p}(\sigma)\}$ and the Frenet frame at $\boldsymbol{c}_0 = \boldsymbol{c}(\sigma_0)$ is $\{\boldsymbol{e}_1^0 = \boldsymbol{\gamma}(\sigma_0), \boldsymbol{e}_2^0 = \boldsymbol{t}(\sigma_0),\ \boldsymbol{e}_3^0 = \boldsymbol{p}(\sigma_0)\}$. Moreover, the shape curvature and the shape torsion of $\boldsymbol{c}$ are the functions f_1 and f_2, respectively. $\square$

Combining Theorems 1 and 2 we obtain an analogue of the fundamental theorem of space curves:

Theorem 3 *Let $f_i\colon\ I \to \mathbb{R}$, $i = 1, 2$, be two functions of class C^1. Modulo a direct similarity of $\mathbb{R}^3$ there exists a unique Frenet curve with the shape curvature f_1 and the shape torsion f_2.*

5. Recovering a space curve from its shape

As in previous sections, let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a Frenet curve of class C^3 defined in an open interval $I \subset \mathbb{R}$ and parameterized by a spherical arc length parameter σ. Then, the shape of $\boldsymbol{c}$ is the pair $(\widetilde{k}_1(\sigma), \widetilde{k}_2(\sigma))$, where $\widetilde{k}_i : I \to \mathbb{R}$ $(i = 1, 2)$ are functions of class C^1 defined by (4) and (5). From Theorem 3 we have that the curve $\boldsymbol{c}$ is determined uniquely by its shape up to a direct similarity of the Euclidean space. In this section we shall construct space curves with given shape using the procedure from the proof of Theorem 2. First we fix a right-handed orthonormal triad of vectors $\boldsymbol{e}_1^0,\, \boldsymbol{e}_2^0,\, \boldsymbol{e}_3^0$. The unique solution of the system of differential equations

$$\frac{d\boldsymbol{\gamma}}{d\sigma} = \boldsymbol{t}(\sigma), \quad \frac{d\boldsymbol{t}}{d\sigma} = -\boldsymbol{\gamma}(\sigma) + \widetilde{k}_2(\sigma)\boldsymbol{p}(\sigma), \quad \frac{d\boldsymbol{p}}{d\sigma} = -\widetilde{k}_2(\sigma)\boldsymbol{t}(\sigma) \tag{13}$$

with initial conditions $\boldsymbol{e}_1^0,\, \boldsymbol{e}_2^0,\, \boldsymbol{e}_3^0$, determine a spherical curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\sigma)$ such that $\boldsymbol{\gamma}(\sigma_0) = \boldsymbol{e}_1^0$ for some $\sigma_0 \in I$. Let $\mu(\sigma) = \int_{\sigma_1}^{\sigma} \widetilde{k}_1(\sigma)\,d\sigma$ for fixed $\sigma_1 \in I$. Then applying the equation (7) and Proposition 1 we find that the curve

$$\boldsymbol{c}(\sigma) = \boldsymbol{c}_0 + \int_{\sigma_0}^{\sigma} e^{\mu(\sigma)}\boldsymbol{\gamma}(\sigma)\,d\sigma \tag{14}$$

has a shape $(\widetilde{k}_1(\sigma), \widetilde{k}_2(\sigma))$ and passes through a point $\boldsymbol{c}_0 = \boldsymbol{c}(\sigma_0)$. In simple cases the system (13) and the equation (14) can be solved explicitly, but in the general case only a numerical solution is possible. We now consider few examples of space curves constructed by the above method.

Example 1 Let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a space curve with shape $(0, \mathrm{a})$, where a is a non-zero real constant. Then $\boldsymbol{c}$ is a circular helix.
We have $\mu(\sigma) = 0$ for any $\sigma \in I$. Choose initial conditions

$$\boldsymbol{e}_1^0 = \left(0,\ -\frac{1}{\sqrt{1+a^2}},\ \frac{a}{\sqrt{1+a^2}}\right), \quad \boldsymbol{e}_2^0 = (1,\, 0,\, 0), \quad \boldsymbol{e}_3^0 = \left(0,\ \frac{a}{\sqrt{1+a^2}},\ \frac{1}{\sqrt{1+a^2}}\right).$$

Then the system (13) defines a spherical curve

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}(\sigma) = \left(\frac{1}{\sqrt{1+a^2}} \sin\left(\sqrt{1+a^2}\,\sigma\right),\ -\frac{1}{\sqrt{1+a^2}} \cos\left(\sqrt{1+a^2}\,\sigma\right),\ \frac{a}{\sqrt{1+a^2}}\right)$$

with $\boldsymbol{\gamma}(0) = \boldsymbol{e}_1^0$. Solving the equation (14) we get

$$\boldsymbol{c}(\sigma) = \left(-\frac{1}{1+a^2}\cos q,\ -\frac{1}{1+a^2}\sin q,\ \frac{a}{1+a^2}\, q\right), \quad \text{where} \quad q = \sigma\sqrt{1+a^2}, \quad \sigma \in I.$$

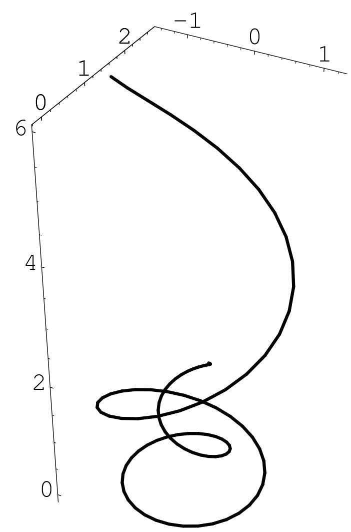

Fig. 1.

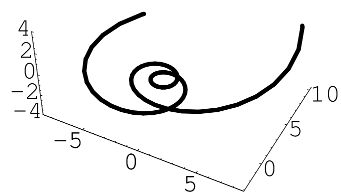

Fig. 2.

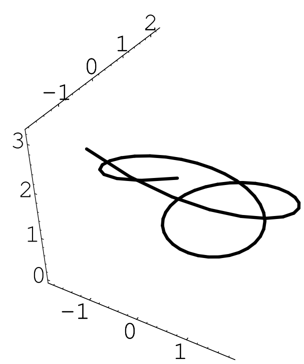

Fig. 3.

Example 2 Let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a space curve with shape $(b,\, a)$, where $a \neq 0$ and $b \neq 0$ are real constants. Then $\boldsymbol{c}$ is a conic spiral in $\mathbb{R}^3$.
We have $\mu(\sigma) = \int_0^\sigma b\, d\sigma = b\sigma$ for $\sigma \in I$. Choosing the same initial conditions as in Example 1 we get the same spherical curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\sigma)$ which is a circle with a radius $1/\sqrt{1+a^2}$. Solving the equation (14) and setting $q = \sigma\sqrt{1+a^2}$ we obtain

$$\boldsymbol{c}(\sigma) = \left(\frac{e^{mq}\, \sin(q-n)}{(1+a^2)\sqrt{1+m^2}},\ -\frac{e^{mq}\, \cos(q-n)}{(1+a^2)\sqrt{1+m^2}},\ \frac{a e^{mq}}{m(1+a^2)}\right),$$

where $m = b/\sqrt{1+a^2}$ and $n = \arccos(b/\sqrt{1+a^2+b^2})$.

The examined examples show that the only space curves with a constant shape are circular helices and conic spirals. In the next example we consider a space curve with a non-constant shape curvature.

Example 3 Let $I \subset \mathbb{R}$ be an open interval, $0 \notin I$, and let $\boldsymbol{c}\colon I \to \mathbb{R}^3$ be a space curve with a shape $(\widetilde{\kappa}_1 = 1/\sigma,\ \widetilde{\kappa}_2 = a)$, where a is a non-zero real constant. Then $\boldsymbol{c}$ is a cylindrical helix given by

$$\boldsymbol{c}(\sigma) = \left(\frac{\sin q - q\cos q}{(1+a^2)^{3/2}},\ -\frac{\cos q + q\sin q}{(1+a^2)^{3/2}},\ \frac{aq^2}{2(1+a^2)^{3/2}} \right), \quad \text{where} \quad q = \sigma\sqrt{1+a^2}. \tag{15}$$

As in the previous examples we take the same spherical curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\sigma)$. Since $\mu(\sigma) = \log\sigma$ then from (14) we get (15).

A lot of computer programs can be used effectively to determine numerically a space curve with a given shape and then to construct it. The above figures illustrate space curves with shapes $(b,\, a\sigma)$ (see Fig. 1), $(b\sigma,\, a)$ (see Fig. 2) and $(b\sigma,\, a\sigma)$ (see Fig. 3), where a, b are non-zero real constants.

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