Journal for Geometry and Graphics
Volume 7 (2003), No. 2, 191–199.

Desargues' Configuration in a Special Layout

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Abstract. In the paper a special case of DESARGUES' configuration will be discussed, where one triangle is inscribed into a conic and the other is circumscribed about the same conic. These two triangles are in a correspondence called a DESARGUES collineation K_D. Three theorems have been formulated and proved. One of them characterizes the central collineation K_D. In a DESARGUES collineation the base conic will be transformed into another conic. Different cases are discussed. In the case of a base circle the center of the collineation K_D coincides with the GERGONNE point.

Key words: DESARGUES' configuration, collineation

MSC 2000: 51M35, 51M05

1. Introduction

If two triangles are in DESARGUES' configuration then the straight lines connecting pairs of corresponding vertices intersect at a single point while the three points of intersection between corresponding sides are collinear (Fig. 1). This planar figure consists of ten points and ten straight lines. This figure creates a configuration $[10_3]$ as each point of this configuration coincides with three different straight lines and simultaneously on each line there are three specific points creating a DESARGUES' configuration [1].

In this work a special case of DESARGUES' configuration will be discussed where one of the two triangles is inscribed into the other.

2. Description of a special case of DESARGUES' configuration

Let line p be given together with three points A, B, C coinciding with this line. We then specify three lines a_1, b_1, c_1 such that they make sides of a triangle with vertices $A_1 = b_1c_1$, $B_1 = a_1c_1$, $C_1 = a_1b_1$. Let us now draw lines $a_2 = AA_1$, $b_2 = BB_1$, $c_2 = CC_1$. Lines a_2, b_2 and c_2 make a triangle with vertices $A_2 = b_2c_2$, $B_2 = a_2c_2$, $C_2 = a_2b_2$ (Fig. 2).

ISSN 1433-8157/\$ 2.50

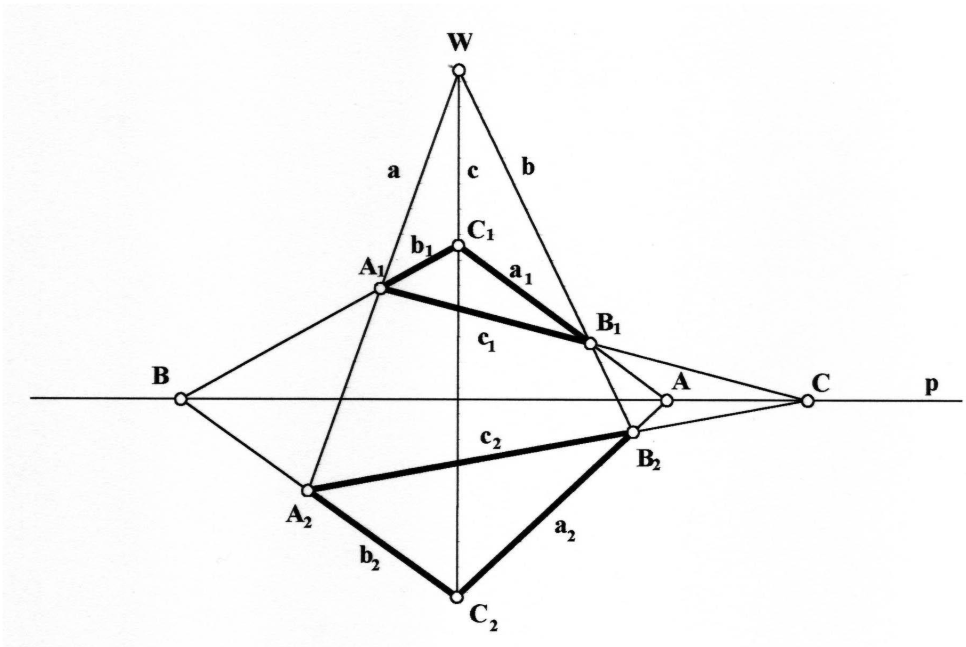

Figure 1: DESARGUES' configuration

The pairs of corresponding sides of the given triangles intersect on the line p at points A, B, C, respectively. Therefore according to DESARGUES' Theorem the straight lines connecting opposite vertices, namely $c = C_1C_2$, $b = B_1B_2$ and $a = A_1A_2$, meet at one point W.

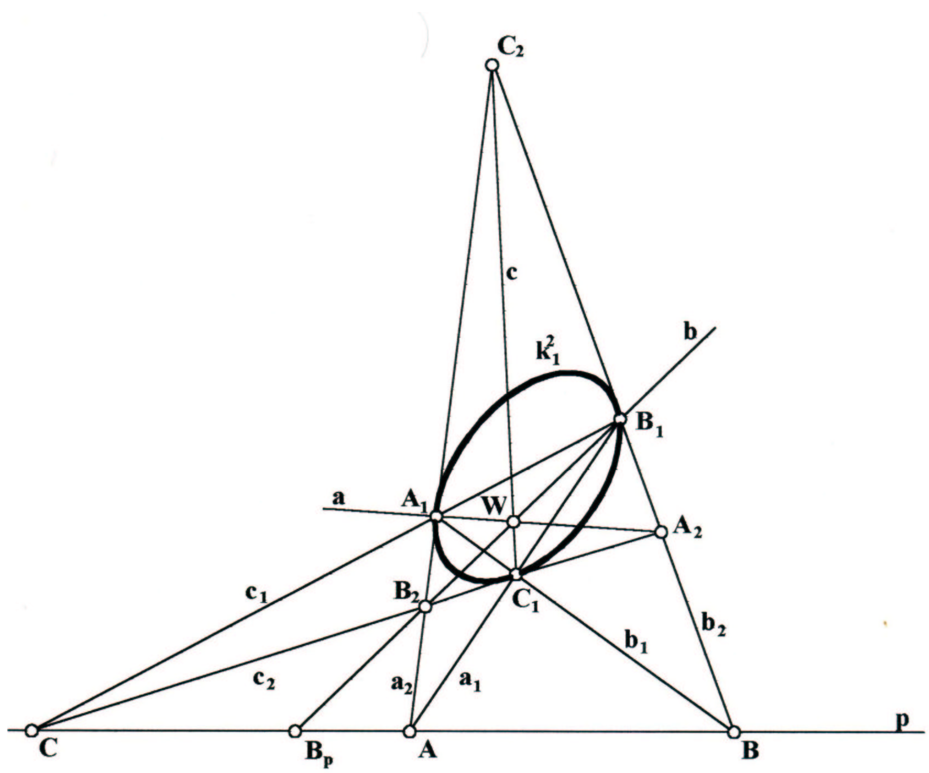

Figure 2: Special case of DESARGUES' configuration

3. On some properties of Desargues' configuration

Theorem 1 *There is one and only one conic k_1^2 that is circumscribed about the triangle $A_1B_1C_1$ and inscribed into the triangle $A_2B_2C_2$.*

Proof: The tangent lines a_2, b_2, c_2 and two respective tangency points $A_1 \in a_2$, $B_1 \in b_2$ determine a conic s_1^2, which contacts line c at some point. We will prove that this point coincides with C_1 and thus the conics k_1^2 and s_1^2 coincide.

Let us consider the straight lines a_2, b_2 and c_2 to be the sides of a degenerated hexagon circumscribed about conic s_1^2. According to Brianchon's Theorem, opposite sides of this hexagon determine three lines, which meet at one point. The two joins of pairs of opposite vertices of the hexagon, namely $a = A_1A_2$ and $b = B_1B_2$, intersect at Brianchon's point W. Thus, if we join point W with point C_2, a line c is created. Line c defines on the opposite line c_2 the opposite vertex C_1. At this point the conic s_1^2 touches line c_2. This completes our proof to Theorem 1 concluding that the conics k_1^2 and s_1^2 coincide as stated. □

Theorem 2 *Point W is the pole of line p with respect to the conic k_1^2.*

Proof: Point B_1 is the pole of line b_2, while B_2 is the pole of b_1. The lines b_1 and b_2 meet at point B, and thus point B is the pole of $b = B_1B_2$. In analogy point A is the pole of line $a = A_1A_2$. Hence point W as the meeting point of the polars a, b is the pole of line p joining A, B. □

It is easy to notice that the Desargues' point W is the center and the Desargues' line p is the axis of a collineation K_D, which transforms one triangle into the other. Let us call K_D the Desargues' *central collineation.*

Theorem 3 *The characteristic cross ratio of the central collineation K_D, which transforms the triangle $A_1B_1C_1$ into $A_2B_2C_2$, equals $-1/2$.*

Proof: Let us denote with B_p the point of intersection of any ray b of the collineation with the axis p (Fig. 2). It is sufficient to prove that the cross ratio $(B_2B_1B_pW)$ of these four points equals $-1/2$.

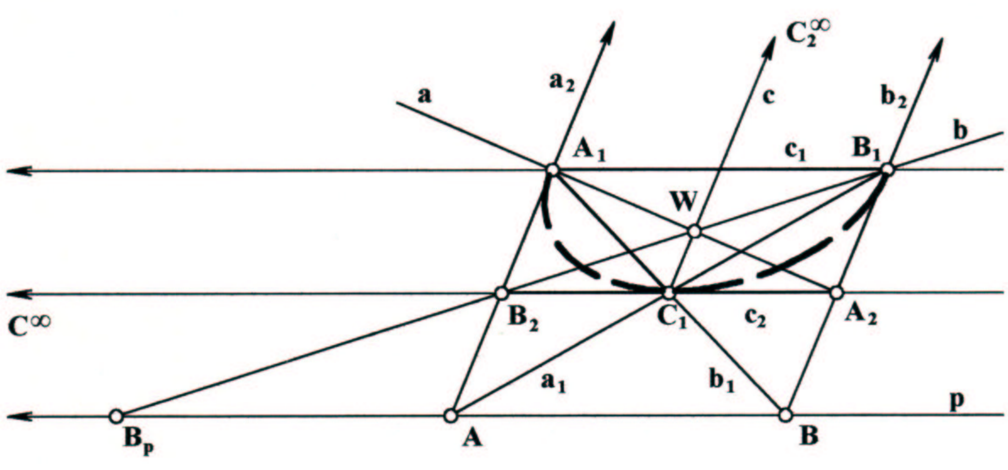

Figure 3: Proving the characteristic cross ratio of K_D in a special position

We first assume that A_1B_1 is a diameter of an ellipse and C_1 is located on the conjugate diameter. Let then the triangle $A_2B_2C_2^{\infty}$ be circumscribed to the triangle $A_1B_1C_1$ (see Fig. 3)

We determine the center W and the axis p of the central collineation K_D and the point $B_p = bp$. Then the cross ratio under consideration is

$$(B_2B_1B_pW) = \frac{\overleftrightarrow{B_2B_p}}{\overleftarrow{B_1B_p}} : \frac{\overrightarrow{B_2W}}{\overleftarrow{B_1W}} = \frac{1}{2} : -1 = -\frac{1}{2}\,.$$

If we transform the configuration presented in Fig. 2 in a central collineation so that the vanishing line of this collineation passes through points C and C_2, then we obtain the configuration presented in Fig. 3. Since cross ratios are invariant under projective transformations, the cross ratio of the four points $(B_2B_1B_pW)$ in Fig. 2 and in Fig. 3 are equal to each other and equal to $-1/2$ as stated. □

The discussed transformation has been presented in Fig. 4, in which the vanishing line $g = CC_2$, the center S and the axis t have been specified. Since ranges b and b' of points are mutually perspective, the cross ratio for any four corresponding points on these lines are equal, i.e.,

$$b(B_2B_1B_pW) = b'(B_2'B_1'B_p'W') = -\frac{1}{2}\,.$$

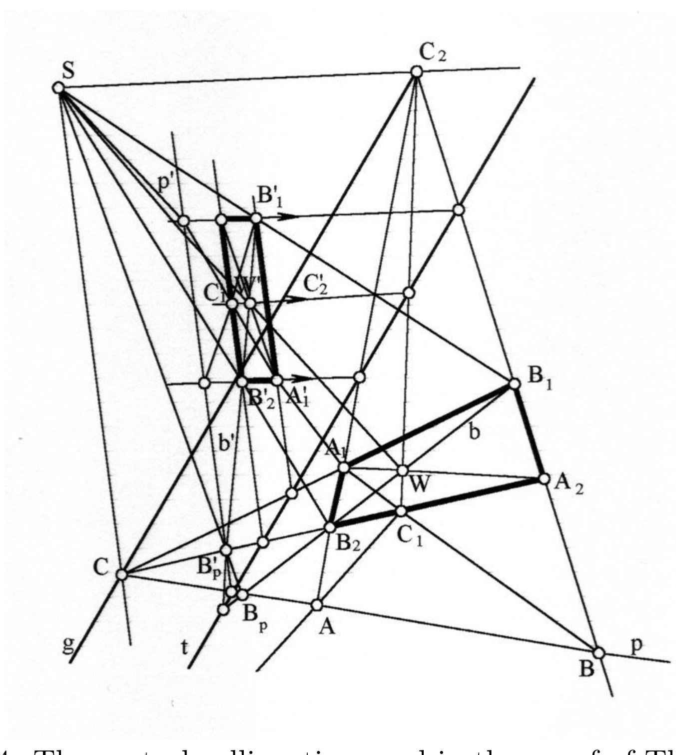

Figure 4: The central collineation used in the proof of Theorem 3

If the characteristic cross ratio of a central collineation between two planar sets (ω_1) and (ω_2) is equal to $-1/2$, then the distance between the vanishing line g_1 of the set (ω_1) and the center W of the collineation is equal to one third of the distance between the axis p and center W (Fig. 5), as

$$(WPG_1G_2^{\infty}) = \frac{\overrightarrow{WG_1}}{\overleftarrow{PG_1}} : \frac{\overrightarrow{WG_2^{\infty}}}{\overleftrightarrow{PG_2^{\infty}}} = -\frac{1}{2} : 1 = -\frac{1}{2}\,.$$

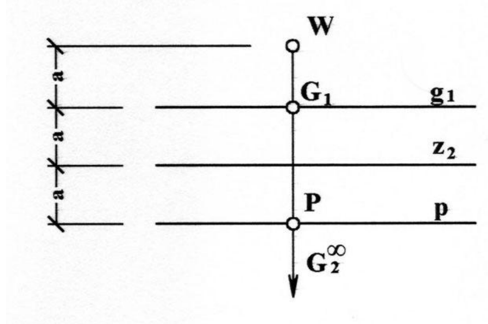

Figure 5: The DESARGUES collineation has the characteristic cross ratio $-\frac{1}{2}$

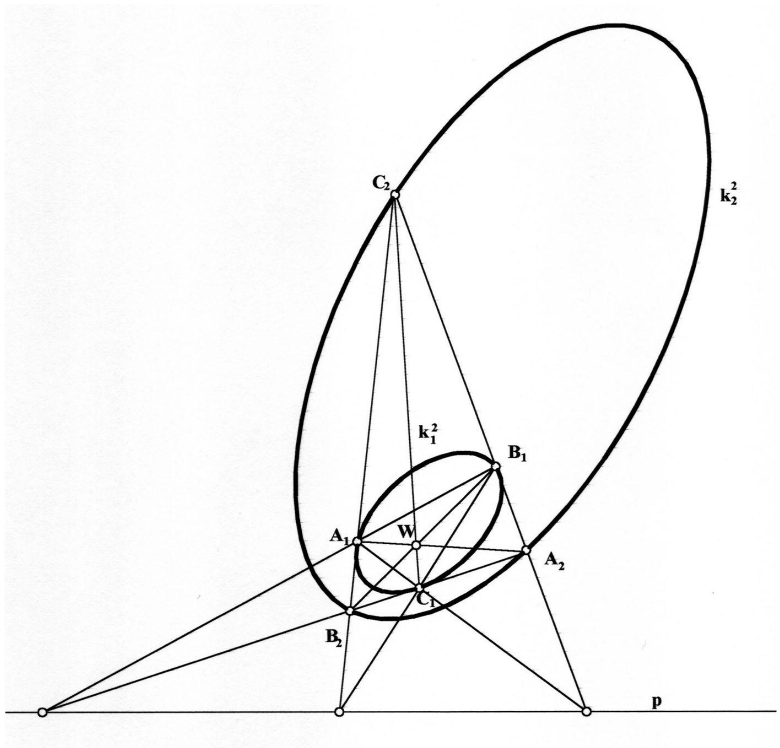

Figure 6: k_1^2 and k_2^2 are two conics in closure position according to PONCELET

The central collineation K_D transforms the conic k_1^2, which is circumscribed to the triangle $A_1B_1C_1$, into a conic k_2^2 circumscribed to $A_2B_2C_2$ (Fig. 6). According to PONCELET's 'Closure Theorem' there is an infinite set of triangles $A_2B_2C_2$ inscribed to k_2^2 and circumscribed to k_1^2 with contact points $A_1B_1C_1$ (note the triangles $A_2B_2C_2$ and $L_2M_2N_2$ in Figs. 10 and 12).

The DESARGUES center W is the pole of the DESARGUES axis p with respect to the conic k_1^2. The conics k_1^2 and k_2^2 have W and p as a common pair of pole and polar line. Continuing the procedure of constructing a triangle $A_3B_3C_3$ from $A_2B_2C_2$ and so on will lead to a special set of conics within the pencil of conics spanned by k_1^2 and k_2^2.

4. GERGONNE point in DESARGUES' configuration

One of the special points in the configuration of a triangle inscribed into a circle is the so called GERGONNE point. The three segments joining the vertices of a triangle with the points of tangency with the incircle intersect at this GERGONNE point [3] (Fig. 7).

We now consider the following construction (Fig. 8): Let a circle k_1^2 and an internal point

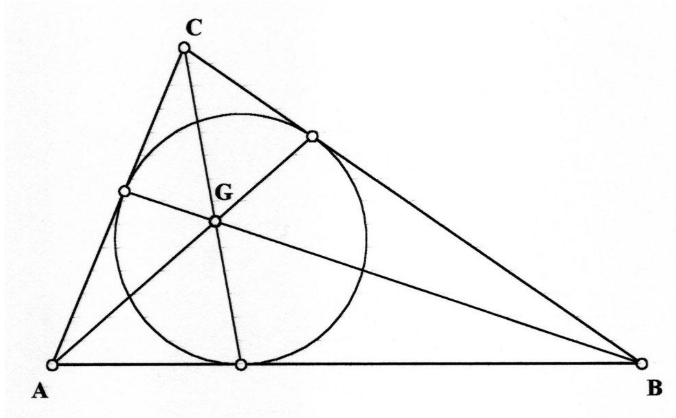

Figure 7: GERGONNE point G of the triangle ABC

G_r be given. Determine a triangle $A_2B_2C_2$ circumscribed about this circle so that the point G_r is the GERGONNE point of this triangle. Then point $G_r = W$ is the center of the collineation K_D of two planar sets (ω_1) and (ω_2) while its polar p is the axis.

Let us determine the vanishing line z_2 of the set (ω_2) for which the distance from line p is one third of the distance between point W and line p. Through an optional point $A_1 \in k_1^2$ we draw the tangent line a_2 to the circle. The line a_1 corresponding to a_2 in the collineation K_D intersects the circle at points B_1 and C_1. Conversely, we obtain the corresponding points B_2 and C_2 on line a_2. The lines $b_2 = B_1C_2$ and $c_2 = C_1B_2$ are the two other sides of the triangle to be constructed. They intersect at point A_2, which lies on a ray of collineation passing through point A_1.

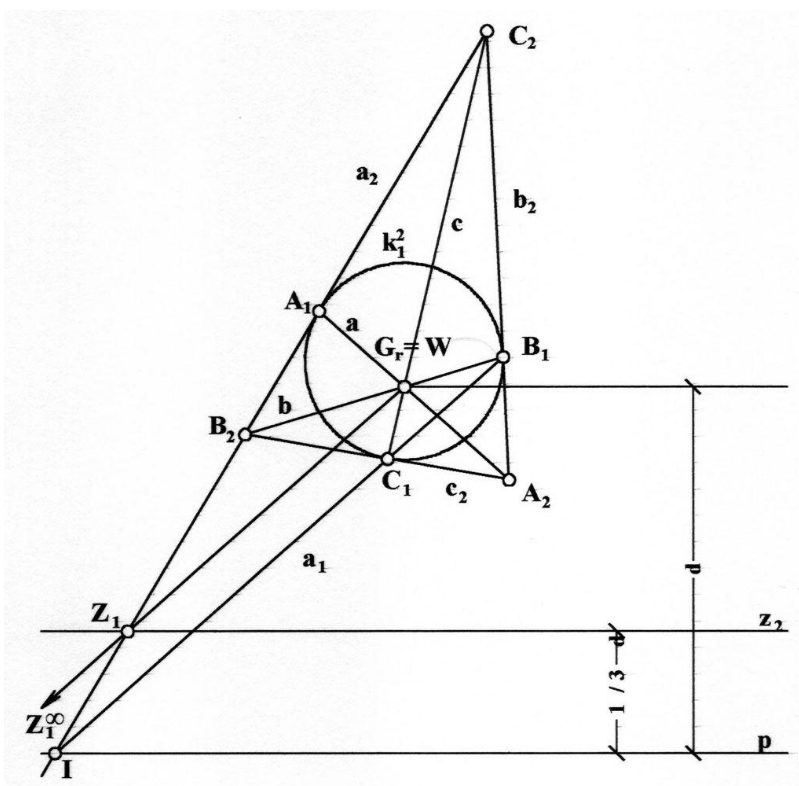

Figure 8: For given circle k_1^2 and GERGONNE point G_r find a triangle $A_2B_2C_2$

5. Particular cases of Desargues' configuration

The conic k_2^2 can be a parabola, a hyperbola or an ellipse depending on the position of point W in relation to the circle k_1^2: Let two circles k_1^2 and k^2 with the common center O be given. Let the radius R of k_1^2 double the radius r of k^2 (see Fig. 9).

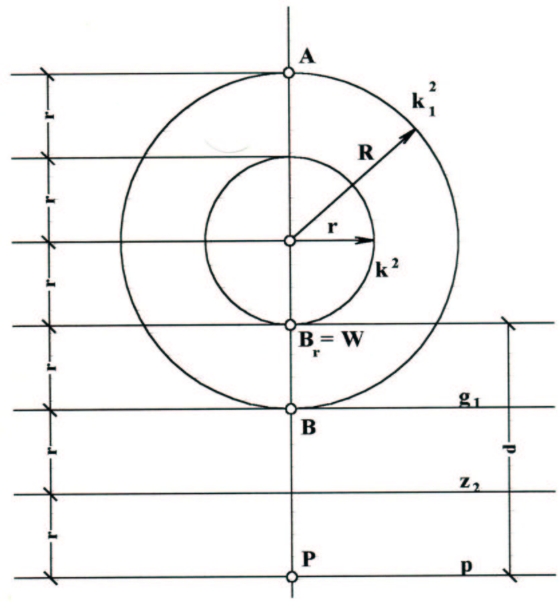

Figure 9: Discussing special cases

Case I. If point $W = G_r$ lies on the circle k^2, then the distance d between the axis p and the center W of the collineation equals $3r$. This means that the vanishing line g_1 (passing at distance d from point W) is tangent to the circle k_1^2. Hence the conic k_2^2 is a parabola (Fig. 10). However, if line p passing at the distance $d = 3r$ from point W is an axis of the collineation K_D, it should be proved that it is the polar of point W: The line $a = OW$ intersects the conic k_1^2 at points A and B, and line p at point P. For the cross ratio of these four points we obtain

$$(ABWP) = \frac{\overrightarrow{AW}}{\overleftarrow{BW}} : \frac{\overrightarrow{AP}}{\overrightarrow{BP}} = \frac{\overrightarrow{3r}}{\overleftarrow{r}} : \frac{\overrightarrow{6r}}{\overrightarrow{2r}} = \frac{-3}{3} = -1\,.$$

We conclude that p is the polar of W (Fig. 9).

Case II. If point $W = G_r$ lies between the circles k_1^2 and k^2, then line p is nearer to point W and thus we have the distance $d < 3r$. We may conclude that the vanishing line g_1 (parallel to p at distance d from W) intersects the circle k_1^2 and thus the conic k_2^2 is a hyperbola (Fig. 11).

Case III. If point $W = G_r$ lies inside the circle k^2, then the distance d between the line p and the center W obeys $d > 3r$. The vanishing line g_1 will be external in relation to the circle k_1^2, and thus the conic k_2^2 is an ellipse (Fig. 12).

Case IIIa. If point $W = G_r$ coincides with the center O of the circles k_1^2 and k^2, then the central collineation K_D is a central homothety with the characteristic ratio $-1/2$. The triangle circumscribed about k_1^2 becomes equilateral and the conic k_2^2 is a circle.

All the cases discussed above respectively illustrated in Figs. 10–13.

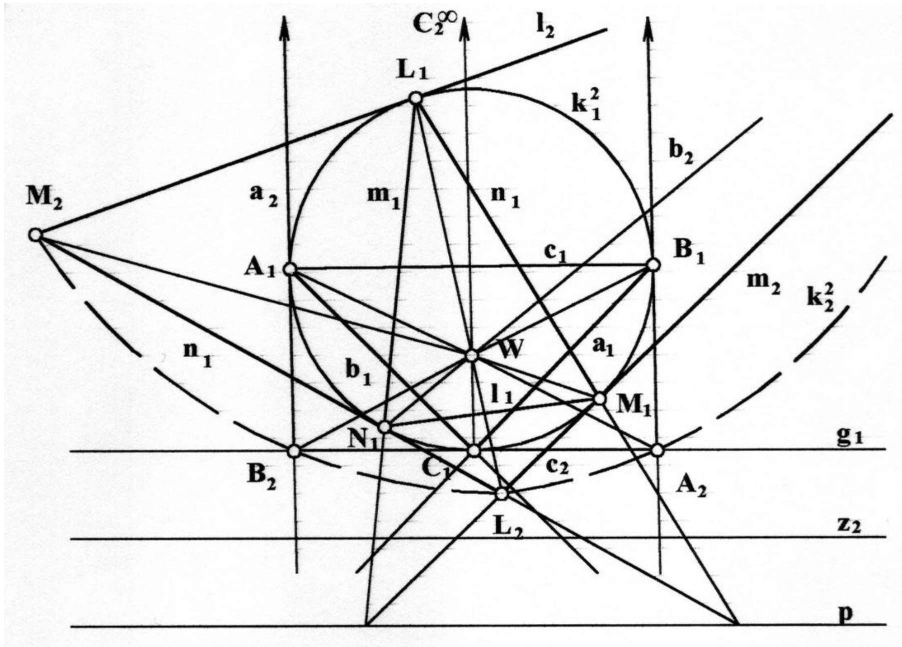

Figure 10: Case I: k_2^2 is a parabola. Both triangles $A_2B_2C_2$ and $L_2M_2N_2$ are circumscribed k_1^2 and inscribed k_2^2

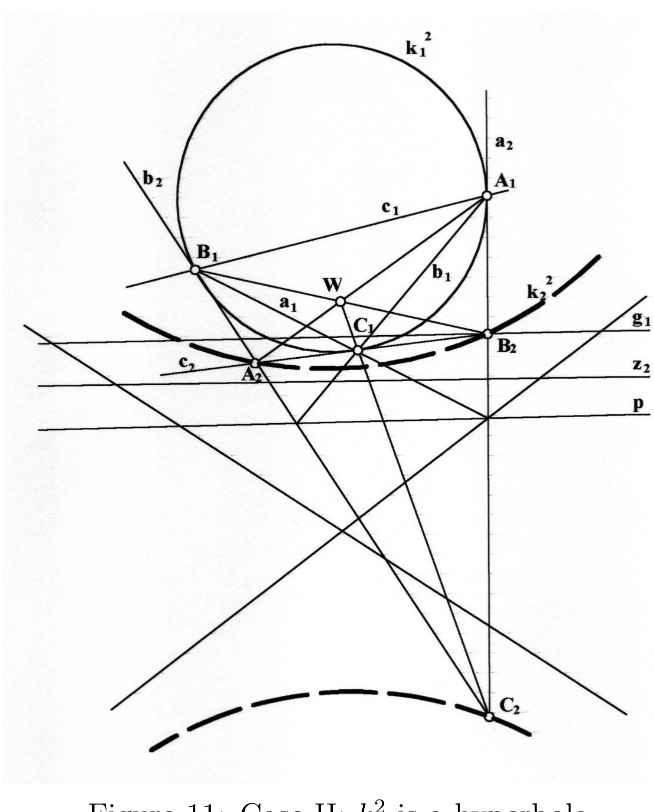

Figure 11: Case II: k_2^2 is a hyperbola

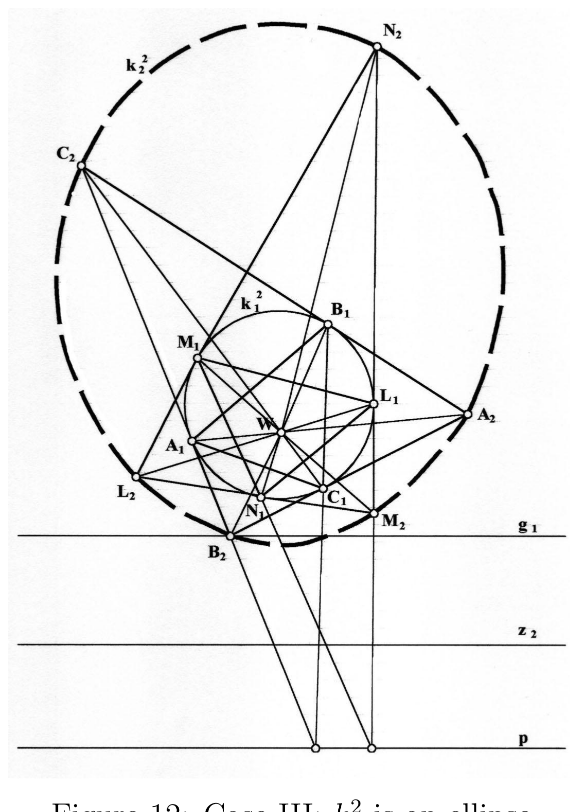

Figure 12: Case III: k_2^2 is an ellipse

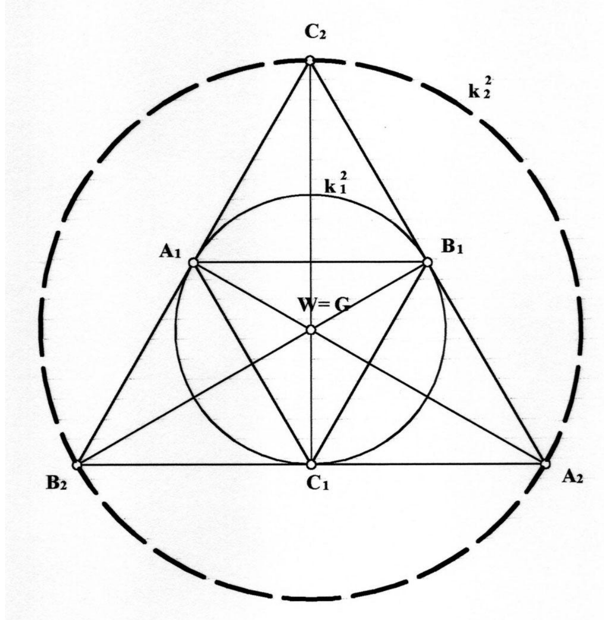

Figure 13: Case IIIa: k_2^2 is a circle

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Received September 17, 2002; final form July 3, 2003