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Curves related to triangles: The Balaton-Curves

Hans Dirnböck, Johannes Schoißengeier

Institute for Mathematics, University of Klagenfurt
Universitätsstraße 65-67, A 9020 Klagenfurt, Austria
emails: hans.dirnboeck@uni-klu.ac.at, johannes.schoissengeier@uni-klu.ac.at

Abstract. The remarkable points orthocentre H, circumcentre U, in-centre I, Torricelli's point T_1 and the first isodynamic point D_1 (see [3, 4]) of a given triangle Δ in the Euclidean plane lie on a naturally defined curve f which we call the *Balaton-curve* of Δ. We determine all triangles for which this curve is algebraic and investigate it when it is algebraic, and when it is transcendental as well. In the algebraic case we determine its irreducible equation in the projective plane over $\mathbb{C}$.

Key words: triangle, Balaton-curve

MSC 2000: 51M04, 51N35

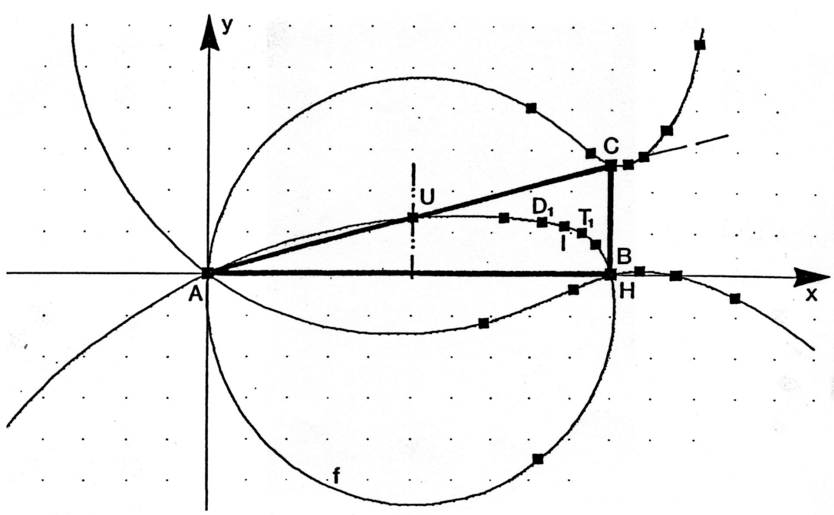

Figure 1: The Balaton-curve f of a triangle ABC

Fig. 1 shows the Balaton-curve of a triangle; the right angle at B does not affect generality. Other points of the curve (indicated by small black squares) may also be considered as being "remarkable".

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1. The angle coordinates of a triangle

Let a triangle Δ with angles α, β, γ and side-lengths a, b, c be given. For many purposes of triangle geometry it is favourable to introduce a coordinate system (depending on the given triangle) by relating to every point $P(x, y)$ of the triangle those three angles α^*, β^* and γ^* under which the sides a, b and c can be seen from the given point P. As $\alpha^* + \beta^* + \gamma^* = 2\pi$, it suffices to assign only two of the three angles $\alpha^*, \beta^*, \gamma^*$. We have made the arbitrary choice (α^*, β^*) (see also Fig. 2).

In the following let us work out the coordinate transformation between the Cartesian coordinate system (x, y) and the *angle coordinates* (α^*, β^*). In the sense of geodesy we use back-cuts. In order to measure angles we shall use the radian measure; furthermore we restrict ourselves to triangles with the vertices $A = (0, 0)$, $B = (2, 0)$, i.e., $c = 2$, and whose third vertex lies in the upper half plane.

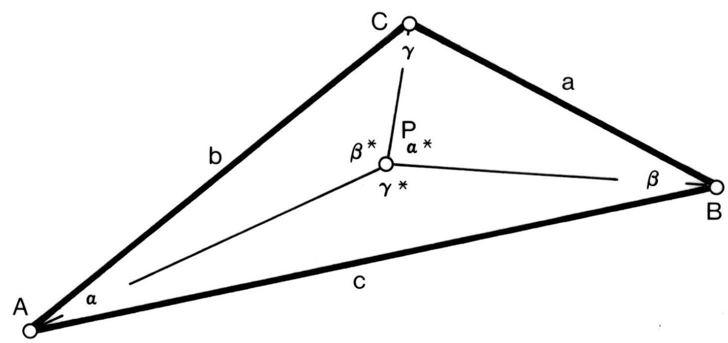

Figure 2: The angle coordinates of the point P

Proposition 1.1 *We have*
$$\begin{cases} \tan(\alpha^* + \beta^*) &= -\dfrac{2y}{x^2 + y^2 - 2x}, \\ \tan\beta^* &= b\,\dfrac{x\sin\alpha - y\cos\alpha}{x^2 + y^2 - b(x\cos\alpha + y\sin\alpha)}. \end{cases}$$

Proof: We follow the notation of Fig. 3 and make repeatedly use of

$$\tan(v + w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$

and notice that $\tan\alpha_1 = y/x$, $\tan(\frac{\pi}{2} - v) = \cot v$, and $\tan(\alpha^* + \beta^* - \pi - \alpha_1) = y/(2 - x)$. Therefore (see Fig. 3)

$$\begin{aligned} -\tan(\alpha^* + \beta^*) &= \tan(2\pi - \alpha^* - \beta^*) = \tan\left(\tfrac{\pi}{2} - \alpha_1 + \tfrac{\pi}{2} - (\alpha^* + \beta^* - \pi - \alpha_1)\right) = \\ &= \frac{\tan(\frac{\pi}{2} - \alpha_1) + \tan(\frac{\pi}{2} - (\alpha^* + \beta^* - \pi - \alpha_1))}{1 - \tan(\frac{\pi}{2} - \alpha_1)\tan(\frac{\pi}{2} - (\alpha^* + \beta^* - \pi - \alpha_1))} = \\ &= \frac{\cot\alpha_1 + \cot(\alpha^* + \beta^* - \pi - \alpha_1)}{1 - \cot\alpha_1 \cot(\alpha^* + \beta^* - \pi - \alpha_1)} = \frac{\frac{x}{y} + \frac{2-x}{y}}{1 - \frac{x(2-x)}{y^2}} = \frac{2y}{x^2 + y^2 - 2x}. \end{aligned}$$

This is the first formula. Furthermore, we have $\cos\alpha_1 = x/\sqrt{x^2 + y^2}$, $\sin\alpha_1 = y/\sqrt{x^2 + y^2}$ and therefore

$$h = \sqrt{x^2 + y^2}\sin(\alpha - \alpha_1) = \sqrt{x^2 + y^2}(\sin\alpha\cos\alpha_1 - \cos\alpha\sin\alpha_1) = x\sin\alpha - y\cos\alpha.$$

Furthermore

$$u = \sqrt{x^2+y^2}\cos(\alpha-\alpha_1) = \sqrt{x^2+y^2}(\cos\alpha\cos\alpha_1+\sin\alpha\sin\alpha_1) = x\cos\alpha + y\sin\alpha.$$

From

$$\tan(\alpha-\alpha_1) = \frac{h}{u} \quad \text{and} \quad \tan(\pi-\beta^*-\alpha+\alpha_1) = \frac{h}{b-u}$$

we get

$$\begin{aligned}
\tan\beta^* &= \tan\left(\tfrac{\pi}{2}-(\alpha-\alpha_1)+\tfrac{\pi}{2}-(\pi-\beta^*-\alpha+\alpha_1)\right) = \\
&= \frac{\tan\left(\frac{\pi}{2}-(\alpha-\alpha_1)\right)+\tan\left(\frac{\pi}{2}-(\pi-\beta^*-\alpha+\alpha_1)\right)}{1-\tan\left(\frac{\pi}{2}-(\alpha-\alpha_1)\right)\tan\left(\frac{\pi}{2}-(\pi-\beta^*-\alpha+\alpha_1)\right)} = \\
&= \frac{\cot(\alpha-\alpha_1)+\cot(\pi-\beta^*-\alpha+\alpha_1)}{1-\cot(\alpha-\alpha_1)\cot(\pi-\beta^*-\alpha+\alpha_1)} = \\
&= \frac{\frac{u}{h}+\frac{b-u}{h}}{1-\frac{u(b-u)}{h^2}} = b\,\frac{h}{h^2+u^2-ub} = b\,\frac{x\sin\alpha-y\cos\alpha}{x^2+y^2-b(x\cos\alpha+y\sin\alpha)}\,.
\end{aligned}$$

□

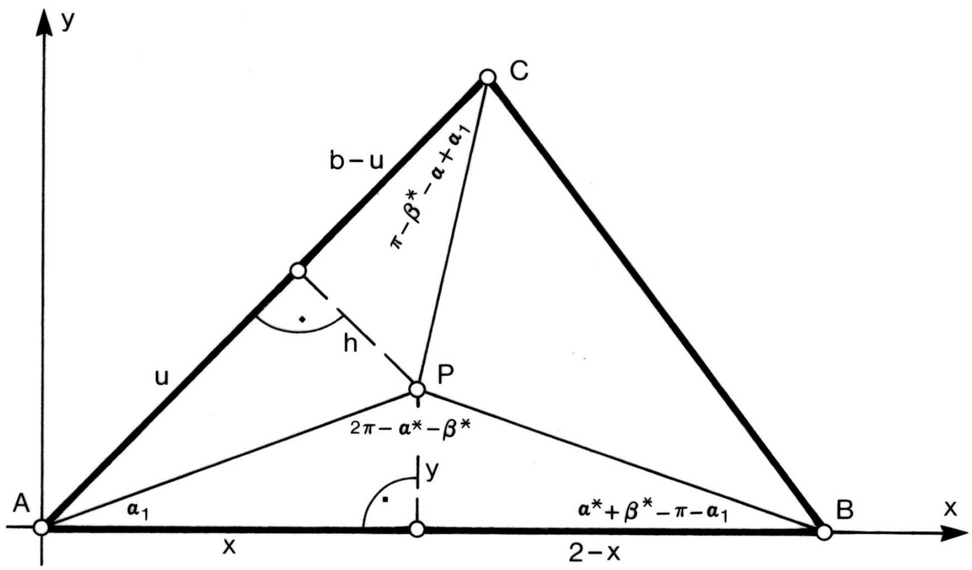

Figure 3: Illustration to the proof of Proposition 1.1

Conversely we can express the Cartesian coordinates x and y in terms of α^* and β^*:

Proposition 1.2 *We have*

$$\left\{\begin{aligned}
x &= \quad 2b\,\frac{\sin(\alpha^*-\alpha)(2\sin\alpha^*-a\sin(\alpha^*+\beta^*)\cos(\beta^*-\beta))}{(a\sin(\alpha^*+\beta^*)-2\sin\alpha^*\cos(\beta^*-\beta))^2+4\sin^2\alpha^*\sin^2(\beta^*-\beta)}\,, \\
y &= -2ab\,\frac{\sin(\alpha^*-\alpha)\sin\alpha^*+\beta^*)\sin(\beta^*-\beta)}{(a\sin(\alpha^*+\beta^*)-2\sin\alpha^*\cos(\beta^*-\beta))^2+4\sin^2\alpha^*\sin^2(\beta^*-\beta)}\,.
\end{aligned}\right.$$

Proof: Prop. 1.1 implies

$$\left\{\begin{array}{l}
x^2+y^2-2x+2y\cot(\alpha^*+\beta^*)=0\,, \\
x^2+y^2-bx(\cos\alpha+\cot\beta^*\sin\alpha)-by(\sin\alpha-\sin\beta^*\cos\alpha)=0\,.
\end{array}\right.$$

Hence we get the intersection of two circles. One of the points of intersection is $(0,0)$. From this it is clear that the other point of intersection has coordinates which are rational functions of $\cot(\alpha^*+\beta^*)$, $\sin\beta^*$ and b. By using $b\sin\alpha = a\sin\beta$ the formulae can easily be verified. $\square$

Although — from the geometric point of view — these formulae make only sense for points P which lie inside the given triangle we shall desist from this restriction in the following investigations.

For later purposes we study the mapping properties of the map $(x,y)\mapsto(\alpha^*,\beta^*)$ in Prop. 1.1. It maps the Euclidean plane (the three circles throught the vertices A, B and C with diameters a, b and c are omitted) onto a dense subset of $\mathbb{R}^2$:

Proposition 1.3 *Let k_1, k_2 and k_3 be the circles defined by the equations* $x^2+y^2-2x=0$, $x^2+y^2-b(x\cos\alpha+y\sin\alpha)=0$ *and* $x^2-2x+y^2+2y\cot(\alpha+\beta)=0$, *respectively. Let h denote the hyperbola $uv-u\cot\alpha+v\cot\alpha+1=0$, g_1 the straight line $u=\tan(\alpha+\beta)$ and g_2 the straight line $v=\tan\beta$. Then the map*

$$\begin{aligned} j:\ \mathbb{R}^2\setminus(k_1\cup k_2\cup k_3) &\rightarrow \mathbb{R}^2\setminus(h\cup g_1\cup g_2\cup\{(0,0)\}),\\ (x,y)\ \mapsto\ j(x,y) &= \left(-\frac{2y}{x^2+y^2-2x},\ b\,\frac{x\sin\alpha-y\cos\alpha}{x^2+y^2-b(x\cos\alpha+y\sin\alpha)}\right). \end{aligned}$$

is an homeomorphism.

Proof: First we prove that the range of j lies in $\mathbb{R}^2\setminus(h\cup g_1\cup g_2\cup\{(0,0)\})$: Assume that $(x,y)\notin k_1\cup k_2\cup k_3$,

$$u=-\frac{2y}{x^2+y^2-2x}\quad\text{and}\quad v=b\,\frac{x\sin\alpha-y\cos\alpha}{x^2+y^2-b(x\cos\alpha+y\sin\alpha)}\,.$$

If we had $(u,v)\in h\cup g_1\cup g_2$ we would easily get $(x,y)\in k_3$. Obviously the range of j does not include $(0,0)$.
For $(u,v)\notin h\cup g_1\cup g_2$, $(u,v)\neq(0,0)$ we put

$$B(v):=bv\cos\alpha+b\sin\alpha-2v\quad\text{and}\quad C(u,v):=2v+buv\sin\alpha-bu\cos\alpha.$$

Because of $v\neq\tan\beta$ we get $B(v)\neq 0$ and $(uB(v),C(u,v))\neq(0,0)$. Hence the functions

$$x(u,v)=2C(u,v)\,\frac{B(v)+C(u,v)}{u^2B^2(v)+C^2(u,v)},\qquad y(u,v)=-2uB(v)\,\frac{B(v)+C(u,v)}{u^2B^2(v)+C^2(u,v)}$$

are continuous. We put $g(u,v):=(x(u,v),\ y(u,v))$. As $(u,v)\notin h$, we get $B(v)+C(u,v)\neq 0$. Hence $x(u,v)^2+y(u,v)^2-2x(u,v)\neq 0$. As $u\neq\tan(\alpha+\beta)$ we get

$$\begin{aligned} x(u,v)^2+y(u,v)^2-b(x(u,v)\cos\alpha+y(u,v)\sin\alpha)&\neq 0\,,\\ x(u,v)^2+y(u,v)^2-2x(u,v)+2y(u,v)\cot(\alpha+\beta)&\neq 0. \end{aligned}$$

By using Mathematica it is easily seen that g and j are inverse to each other. $\square$

2. The Balaton-curve f

For a given triangle with angles α and β the remarkable points orthocentre H, in-centre I, circumcentre U, Torricelli's point T_1, first isodynamic point D_1 (for a detailed discussion see [3, 4]) have angle coordinates (α^*, β^*) which obey the linear law

$$\begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \frac{2\pi}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} \alpha - \frac{\pi}{3} \\ \beta - \frac{\pi}{3} \end{pmatrix}, \quad t \in \mathbb{R},$$

namely with $t = -1$, $t = \frac{1}{2}$, $t = 2$, $t = 0$, and $t = 1$, respectively. The cases $t = -1$, $t = \frac{1}{2}$, $t = 2$ and $t = 0$ are illustrated in Fig. 4. This linear law represents a curve f in Cartesian coordinates — the *Balaton-curve* of the triangle, as it was first presented by the first named author at a congress on the Lake Balaton in Hungary in 1995 ([3]). We notice that f represents a line in the angle coordinates only in the case $(\alpha, \beta) \neq (\frac{\pi}{3}, \frac{\pi}{3})$. Otherwise — if the triangle is equilateral — the curve degenerates to the centre of the triangle. As this case is of no interest we exclude it in the following.

There is a vast literature in connection with triangles on similar locus problems. In place of other papers we only mention [1] and [2]. See also the references there.

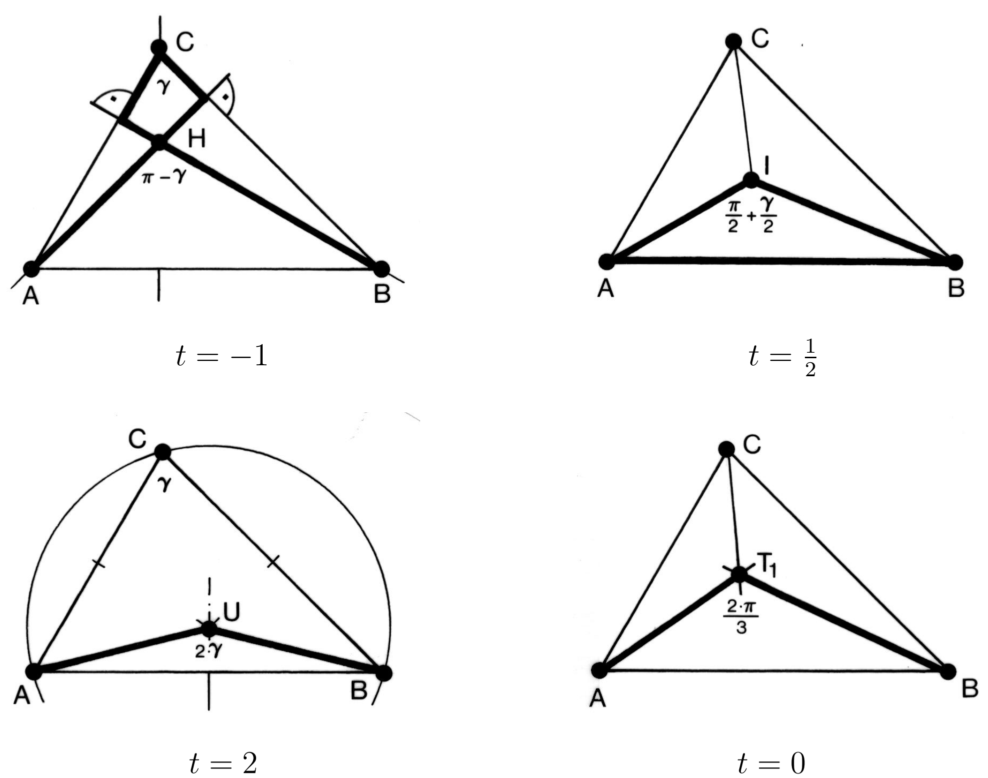

Figure 4: The angle coordinates of H, I, U, and T_1

$\beta = \frac{\pi}{3}$ implies $\beta^* = \frac{2\pi}{3}$ and hence Prop. 1.1 results in the equation

$$x^2 + y^2 - b(x \cos\alpha + y \sin\alpha) + \frac{b}{\sqrt{3}}(x \sin\alpha - y \cos\alpha) = 0.$$

This is a circle with centre $\left(\frac{b}{2}\left(\cos\alpha - \sin\alpha/\sqrt{3}\right), \frac{b}{2}\left(\sin\alpha + \cos\alpha/\sqrt{3}\right)\right)$ and radius $b/\sqrt{3}$. From now on we exclude this case.

Henceforth let
$$\beta \neq \frac{\pi}{3} \quad \text{and} \quad \theta := \frac{3\alpha-\pi}{3\beta-\pi}\,.$$
Note that $\alpha - \frac{\pi}{3}$ (resp. $\beta - \frac{\pi}{3}$) is the deviation (excess or defect) of the angle α (resp. β) from the most natural angle $\frac{\pi}{3}$.

Theorem 2.1 *Let Δ be the triangle with angles α and β and side-lengths a, b and $c = 2$. Then, with $\alpha^*(t) := \frac{2\pi}{3} + \theta(t - \frac{2\pi}{3})$ or $\alpha^*(t) := \frac{2\pi}{3} + t(\alpha - \frac{\pi}{3})$ and $\beta^*(t) := \frac{2\pi}{3} + t(\beta - \frac{\pi}{3})$, respectively, we have the following parameter representations of the Balaton-curve of Δ:*

$$\begin{cases} x(t) &=& 2b\,\dfrac{\sin(\alpha^*(t)-\alpha)(2\sin\alpha^*(t) - a\sin(\alpha^*(t)+t)\cos(t-\beta))}{(a\sin(\alpha^*(t)+t) - 2\sin\alpha^*(t)\cos(t-\beta))^2 + 4\sin^2\alpha^*(t)\sin^2(t-\beta)} \\ y(t) &=& -2ab\,\dfrac{\sin(\alpha^*(t)-\alpha)\sin(\alpha^*(t)+t)\sin(t-\beta)}{(a\sin(\alpha^*(t)+t) - 2\sin\alpha^*(t)\cos(t-\beta))^2 + 4\sin^2\alpha^*(t)\sin^2(t-\beta)}\,, \end{cases}$$

$$\begin{cases} x(t) &=& 2b\,\dfrac{\sin(\alpha^*(t)-\alpha)(2\sin\alpha^*(t) - a\sin(\alpha^*(t)+\beta^*(t))\cos(\beta^*(t)-\beta))}{(a\sin(\alpha^*(t)+\beta^*(t)) - 2\sin\alpha^*(t)\cos(\beta^*(t)-\beta))^2 + 4\sin^2\alpha^*(t)\sin^2(\beta^*(t)-\beta)} \\ y(t) &=& -2ab\,\dfrac{\sin(\alpha^*(t)-\alpha)\sin(\alpha^*(t)+\beta^*(t))\sin(\beta^*(t)-\beta)}{(a\sin(\alpha^*(t)+\beta^*(t)) - 2\sin\alpha^*(t)\cos(\beta^*(t)-\beta) + 4\sin^2\alpha^*(t)\sin^2(\beta^*(t)-\beta)}\,. \end{cases}$$

Proof: This follows immediately from Prop. 1.2. □

In the following we always assume that $\alpha^*(t) = \frac{2\pi}{3} + \theta(t - \frac{2\pi}{3})$.

Theorem 2.1 requires a supplementary explanation in the cases where the denominator of $x(t)$ or $y(t)$ vanishes. For this and later purposes we introduce the following functions:

$$\begin{aligned} Z_1(t) &:= b\sin(\alpha^*(t)-\alpha)(2\sin\alpha^*(t) - a\sin(\alpha^*(t)+t)\cos(t-\beta)), \\ Z_2(t) &:= -ab\sin(\alpha^*(t)-\alpha)\sin(\alpha^*(t)+t)\sin(t-\beta), \\ N(t) &:= (a\sin(\alpha^*(t)+t) - 2\sin\alpha^*(t)\cos(t-\beta))^2 + 4\sin^2\alpha^*(t)\sin^2(t-\beta). \end{aligned}$$

Note that $x(t) = 2Z_1(t)/N(t)$ and $y(t) = 2Z_2(t)/N(t)$. Before we investigate the zeros of the denominator of the general curve let us consider some special cases which fit into the general consideration often only afterwards.

(1) $\alpha = \frac{\pi}{3}$. Then $\alpha^*(t) = \frac{2\pi}{3}$ and hence by Prop. 1.1 we have, on the one hand

$$-\frac{2y}{x^2+y^2-2x} = \frac{\tan t - \sqrt{3}}{1+\sqrt{3}\tan t}\,,$$

and on the other hand

$$\tan t = b\,\frac{x\sqrt{3}-y}{2(x^2+y^2) - b(x+y\sqrt{3})}\,.$$

This implies

$$\begin{aligned} &-\sqrt{3}(x^2-2x+y^2-\tfrac{2}{\sqrt{3}}y) + (x^2-2x+y^2+2y\sqrt{3})\tan t = \\ &= 2y(1+\sqrt{3}\tan t) + (x^2+y^2-2x)(\tan t - \sqrt{3}) = 0 \end{aligned}$$

and hence

$$\begin{aligned} 0 &= -\sqrt{3}(x^2-2x+y^2-\tfrac{2}{\sqrt{3}}y)(x^2+y^2-b\tfrac{x}{2}-b\tfrac{\sqrt{3}}{2}y) + \tfrac{b}{2}(x\sqrt{3}-y)(x^2-2x+y^2+2y\sqrt{3}) \\ &= -\sqrt{3}(x^2+y^2)(x^2+y^2-x(b+2)-y\tfrac{b+2}{\sqrt{3}}+2b), \end{aligned}$$

i.e.,

$$0 = x^2 + y^2 - x(b+2) - \frac{y}{\sqrt{3}}(b+2) + 2b = \left(x - \frac{b+2}{2}\right)^2 + \left(y - \frac{b+2}{2\sqrt{3}}\right)^2 - \frac{b^2-2b+4}{3}\,.$$

This is the equation of a circle with radius $a/\sqrt{3}$.

(2) $\gamma = \frac{\pi}{3}$. Then $\alpha+\beta = \frac{2\pi}{3}$ and hence $\alpha^*(t) + t = \frac{4\pi}{3}$. As $\tan\frac{4\pi}{3} = \sqrt{3}$ we get in this case from Prop. 1.1 the curve $x^2 - 2x + y^2 + \frac{2}{\sqrt{3}}y = 0$, and hence the circle $(x-1)^2 + (y+\frac{1}{\sqrt{3}})^2 = \frac{4}{3}$.

(3) $\alpha = \beta$. In this case the remarkable points lie on the perpendicular bisector $x = 1$ of $\overline{AB}$ of the side c and one can convince oneself easily with the help of Theorem 2.1 that this perpendicular bisector is the Balaton-curve of this isosceles triangle.

(4) $\alpha = \gamma$. In this case we get the perpendicular bisector of the side b. It has the equation $y = (2-x)\cot\alpha$.

(5) $\beta = \gamma$. In this case the Balaton-curve is the perpendicular bisector of the side a. It has the equation $y = x\cot\beta$.

In the cases (1) to (5) θ has the values $0,\ -1,\ 1,\ -\frac{1}{2}$, and -2, respectively. We exclude these cases in case of need in the following considerations.

Lemma 2.1 *If $N(t_0) = 0$, then there are integers m and n such that either $(\alpha^*(t_0),\ t_0) = (\alpha + m\pi,\ \beta + n\pi)$ or $(\alpha^*(t_0),\ t_0) = (m\pi,\ n\pi)$. Furthermore there are integers p and q such that $\theta = p/q$, $\gcd(p,q) = 1$, $q > 0$, and $p \equiv q \pmod 3$.*

Proof: This is an exercise in elementary analysis and left to the reader. □

In the first case of Lemma 2.1 the numerators and the denominator of $x(t)$ and $y(t)$ vanish of order 2 and hence both functions remain bounded in a neighbourhood of t_0. This follows from the proof of the following Proposition:

Proposition 2.1 *Let m and n be integers, $t_0 := \beta + n\pi$ and $\alpha^*(t_0) = \alpha + m\pi$. Then*

$$N(t_0) = 0,\ x(t_0) = \frac{2\theta b(\theta b - a\cos(\alpha+\beta))}{a^2 - 2ab\theta\cos(\alpha+\beta) + \theta^2 b^2)} \text{ and } y(t_0) = -\frac{2\theta ab\sin(\alpha+\beta)}{a^2 - 2\theta ab\cos(\alpha+\beta) + \theta^2 b^2}\,.$$

Proof: We have

$$\begin{aligned}\lim_{t\to t_0} \frac{a\sin(\alpha^*(t)+t) - 2\sin\alpha^*(t)\cos(t-\beta)}{t-t_0} &= (-1)^{m+n}(a(\theta+1)\cos(\alpha+\beta) - 2\theta\cos\alpha) = \\ &= \frac{2(-1)^{m+n}}{\sin(\alpha+\beta)}((\theta+1)\sin\alpha\cos(\alpha+\beta) - \theta\sin(\alpha+\beta)\cos\alpha) = \\ &= \frac{2(-1)^m}{\sin(\alpha+\beta)}(\sin\alpha\cos(\alpha+\beta) - \theta\sin\beta)\end{aligned}$$

and

$$\lim_{t\to t_0} 2\,\frac{\sin\alpha^*(t)\sin(t-\beta)}{t-t_0} = 2(-1)^{m+n}\sin\alpha.$$

Hence

$$\begin{aligned}\lim_{t\to t_0}\frac{N(t)}{(t-t_0)^2} &= \frac{4}{\sin^2(\alpha+\beta)}\left(\sin\alpha\cos(\alpha+\beta) - \theta\sin\beta\right)^2 + 4\sin^2\alpha = \\ &= \frac{4\left(\sin^2\alpha - 2\theta\sin\alpha\sin\beta\cos(\alpha+\beta) + \theta^2\sin^2\beta\right)}{\sin^2(\alpha+\beta)} = a^2 - 2\theta ab\cos(\alpha+\beta) + \theta^2 b\,.\end{aligned}$$

Furthermore

$$\lim_{t\to t_0} b\,\frac{\sin(\alpha^*(t)-\alpha)}{t-t_0} = (-1)^m b\theta$$

and

$$\begin{aligned}\lim_{t\to t_0}\frac{2\sin\alpha^*(t) - a\sin(\alpha^*(t)+t)\cos(t-\beta)}{t-t_0} &= (-1)^m(a(\theta+1)\cos(\alpha+\beta) - 2\theta\cos\alpha) = \\ &= \frac{2(-1)^m}{\sin(\alpha+\beta)}\left(\theta\left(\sin\alpha\cos(\alpha+\beta) - \cos\alpha\sin(\alpha+\beta)\right) + \sin\alpha\cos(\alpha+\beta)\right) = \\ &= (-1)^m\left(a\cos(\alpha+\beta) - \theta b\right)\end{aligned}$$

and hence

$$\lim_{t\to t_0}\frac{Z_1(t)}{(t-t_0)^2} = \theta b(\theta b - a\cos(\alpha+\beta)).$$

Finally we have

$$\lim_{t\to t_0} a\,\frac{\sin(\alpha^*(t)+t)\sin(t-\beta)}{t-t_0} = a(-1)^m\sin(\alpha+\beta)$$

and hence

$$\lim_{t\to t_0}\frac{Z_2(t)}{(t-t_0)^2} = -\theta ab\sin(\alpha+\beta).$$ □

The second case, namely $(\alpha^*(t_0),\ t_0) = (m\pi,\ n\pi)$, gives rise to a point at infinity. We discuss this case in more detail in the next chapter.

3. The algebraic Balaton-curves

In this chapter we prove that the Balaton-curves are algebraic for rational θ and show that their equations are irreducible.

Let integers p and q be chosen such that $\theta = p/q,\ \ q>0$ and $\gcd(p,q)=1$. It is clear that $x(t)$ and $y(t)$ have period $q\pi$. Conversely, assume that

$$(x(t_0),\ y(t_0)) = (x(t_1),\ y(t_1)) \notin \{(0,0),\ (2,0),\ (b\cos\alpha,\ b\sin\alpha)\}.$$

Then, by Prop. 1.1 we get $\tan t_0 = \tan t_1$ and in addition $\tan(\alpha^*(t_0)+t_0) = \tan(\alpha^*(t_1)+t_1)$ and hence for some integers m and n $\ t_0 = t_1 + n\pi$ and $\alpha^*(t_0) = \alpha^*(t_1) + m\pi$. This implies $\frac{2\pi}{3} + \theta(t_0 - \frac{2\pi}{3}) = \frac{2\pi}{3} + \theta(t_1 - \frac{2\pi}{3}) + m\pi$, and hence $p(t_0 - t_1) = mq\pi$. Therefore $pn = qm$ and hence $q|n$. As a corollary $q\pi$ is the primitive period of the parameter representation $t \mapsto (x(t),\ y(t))$, $t\in\mathbb{R}$.

Lemma 3.1 *Let n and m be integers, $t_0 := n\pi$ and $\alpha^*(t_0) = m\pi$. Then:*

(1) $N(t_0) = N'(t_0) = 0$, $N''(t_0) = 2a^2(\theta+1)^2 - 8a\theta(\theta+1)\cos\beta + 8\theta^2 \neq 0$.

(2) $Z_1(t_0) = 0$ *and* $Z_1''(t_0) = 2b\theta\cos\alpha(\theta b\cos\alpha - a\cos\beta) + 2a^2(\theta+1)\sin^2\beta$.

(3) $Z_2(t_0) = 0$, $Z_2'(t_0) \neq 0$ *and* $Z_2''(t_0) = 2ab(\theta+1)(\theta\cos\alpha\sin\beta + \sin\alpha\cos\beta)$.

Proof: (1) As $\sin\alpha^*(t)$ vanishes at t_0 exactly of order 1 and as $4\sin^2\alpha^*(t)\sin^2(t-\beta) \le N(t)$, we get the first two assertions and the last one.
We have $N(t) = a^2\sin^2(\alpha^*(t)+t) - 4a\sin\alpha^*(t)\sin(\alpha^*(t)+t)\cos(t-\beta) + 4\sin^2\alpha^*(t)$ and hence

$$\begin{aligned}\frac{1}{2}N''(t_0) &= \lim_{t\to n\pi}\frac{N(t)}{(t-n\pi)^2} = a^2\lim_{t\to n\pi}\frac{\sin^2(\alpha^*(t)+t)}{(t-n\pi)^2} - \\ &\quad - 4a(-1)^n\cos\beta\lim_{t\to n\pi}\frac{\sin\alpha^*(t)}{t-n\pi}\cdot\lim_{t\to n\pi}\frac{\sin(\alpha^*(t)+t)}{t-n\pi} + 4\lim_{t\to n\pi}\frac{\sin^2\alpha^*(t)}{(t-n\pi)^2} = \\ &= a^2(\theta+1)^2 - 4a(-1)^n\cos\beta\cdot\theta(-1)^m\cdot(\theta+1)(-1)^{n+m} + 4\theta^2.\end{aligned}$$

(2) The first assertion is trivial. We have

$$\begin{aligned} Z_1'(t) = \quad & b\theta\cos(\alpha^*(t)-\alpha)\,(2\sin\alpha^*(t) - a\sin(\alpha^*(t)+t)\cos(t-\beta)) + b\sin(\alpha^*(t)-\alpha)\cdot \\ & \cdot(2\theta\cos\alpha^*(t) - a(\theta+1)\cos(\alpha^*(t)+t)\cos(t-\beta) + a\sin(\alpha^*(t)+t)\sin(t-\beta)) \end{aligned}$$

and hence

$$\begin{aligned} Z_1''(n\pi) \quad &= 2b\theta\cos(m\pi-\alpha)\,(2\theta\cos m\pi - a(\theta+1)\cos((m+n)\pi)\cos(n\pi-\beta)) + \\ &\quad + 2ab(\theta+1)\sin(n\pi-\alpha)\cos((m+n)\pi)\sin(n\pi-\beta) = \\ &= 2b\theta\cos\alpha(2\theta - a(\theta+1)\cos\beta) + 2ab(\theta+1)\sin\alpha\sin\beta = \\ &= 2b\theta\cos\alpha(b\theta\cos\alpha - a\cos\beta) + 2a^2(\theta+1)\sin^2\beta. \end{aligned}$$

(3) The first assertion is trivial. As $\sin(\alpha^*(t)+t)$ vanishes at $n\pi$ of the first order and the function $(a\sin(\alpha^*(t)+\beta) - 2\sin\alpha^*(t)\sin(t-\beta)$ does not vanish at $n\pi$, we get the second one. We have

$$\begin{aligned} Z_2'(t) = \quad & -ab\theta\cos(\alpha^*(t)-\alpha)\sin(\alpha^*(t)+t)\sin(t-\beta) - ab(\theta+1)\sin(\alpha^*(t)-\alpha)\cdot \\ & \cdot\cos(\alpha^*(t)+t)\sin(t-\beta) - -ab\sin(\alpha^*(t)-\alpha)\sin(\alpha^*(t)+t)\cos(t-\beta). \end{aligned}$$

This results in

$$\begin{aligned} Z_2''(n\pi) \quad &= -2ab\theta(\theta+1)\cos(m\pi-\alpha)\cos((m+n)\pi)\sin(n\pi-\beta) - 2ab(\theta+1)\sin(m\pi-\alpha)\cdot \\ &\cdot\cos((m+n)\pi)\cos(n\pi-\beta) = 2ab(\theta+1)(\theta\cos\alpha\sin\beta + \sin\alpha\cos\beta). \qquad \square \end{aligned}$$

Theorem 3.1 *The Balaton-curve of the triangle Δ is bounded in $\mathbb{R}^2$ if and only if $p \not\equiv q \pmod 3$. Otherwise it has exactly one asymptote with the equation*

$$(a(\theta+1)\sin\beta)\,x + (a\cos\beta - \theta b\cos\alpha)y + \frac{2a^2(\theta+1)\sin\beta(2\theta\cos\beta - a(\theta+1))}{a^2(\theta+1)^2 - 4a\theta(\theta+1)\cos\beta + 4\theta^2} = 0.$$

Proof: Assume first that the curve is unbounded. Then there is a $t_0 \in \mathbb{R}$, such that $N(t_0) = 0$ and by Lemma 2.1 and Prop. 2.1 $p \equiv q \pmod 3$ and $(\alpha^*(t_0),\ t_0) = (m\pi,\ n\pi)$ for some $m, n \in \mathbb{Z}$.

As x, y have period $q\pi$ we may assume that $0 \le n < q$. As $\frac{3m-2}{3n-2} = \frac{p}{q}$, we get $q|3n-2$. As there is exactly one n in the half-open interval $[0, q)$ with this property, the curve has at most one asymptote.

We prove that, with the abbreviation

$$k := \frac{a(\theta+1)\sin\beta}{2\theta - a(\theta+1)\cos\beta},$$

the limit $d := \lim_{t\to n\pi}(y(t) - cx(t))$ exists and is equal to

$$\frac{2ac(2\theta\cos\beta - a(\theta+1))}{a^2(\theta+1)^2 - 4a\theta(\theta+1)\cos\beta + 4\theta^2}.$$

(The proof in the case $2\theta = a(\theta+1)\cos\beta$ has to be dealt with separately and is left to the reader.) The equation of the asymptote is then $y = kx + d$.

We have

$$y(t) - kx(t) = -2b\,\frac{\sin(\alpha^*(t_0)-\alpha)}{N(t)}\,(a\sin(\alpha^*(t)+t)(\sin(t-\beta) - k\cos(t-\beta)) + 2k\sin\alpha^*(t)).$$

The limit $\lim_{t\to n\pi}(y(t)-kx(t))$ exists if and only if the factor in the brackets vanishes at $n\pi$ at least of order two, i.e., its derivative vanishes. We have to choose k such that

$$0=(\theta+1)a\cos((m+n)\pi)\,(\sin(n\pi-\beta)-k\cos(n\pi-\beta))+2k\theta\cos m\pi.$$

This results in the above formula for k.

By de l'Hôspital's rule

$$d=2\,\frac{Z_2''(n\pi)-kZ_1''(n\pi)}{N''(n\pi)}.$$

Using Lemma 3.1 we get the formula for the asymptote.

Conversely assume now that $p\equiv q \pmod 3$, where we do not assume that p and q are coprime (but $3\nmid q$). Then we may assume that $p\equiv q\equiv 1 \pmod 3$, for otherwise we replace p by $2p$ and q by $2q$. We put $n=\frac{q+2}{3}$ and $m=\frac{p+2}{3}$. Then $m\pi-\frac{2\pi}{3}=\theta(n\pi-\frac{2\pi}{3})$, and hence with $t_0:=n\pi$, $\alpha^*(t_0)=m\pi$. As by Lemma 3.1 Z_2 vanishes at t_0 exactly of order 1 and N of order 2, we get $\lim_{t\to t_0}y(t)=\infty$. □

Let us define recursively a sequence $(R_k)_{k\ge1}$ of rational functions and two sequences $(P_k)_{k\ge1}$, $(Q_k)_{k\ge1}$ of polynomials, both with coefficients in $\mathbb{Q}$ by

$$R_1=X,\ R_{k+1}=\frac{R_k+X}{1-XR_k},\ P_1=X,\ Q_1=1,\ P_{k+1}=P_k+XQ_k,\ Q_{k+1}=Q_k-XP_k.$$

Obviously $R_k=P_k/Q_k$. Then it is easily seen by induction on k that for every real number t $R_k(\tan t)=\tan kt$.

Proposition 3.1 *For $k\ge1$ we have*

(1) $P_k=-\frac{i}{2}(1+iX)^k+\frac{i}{2}(1-iX)^k=\Im\left((1+iX)^k\right)=\sum_{2\nmid t\le k}\binom{k}{t}(-1)^{\frac{t-1}{2}}X^t.$

(2) $Q_k=\frac{1}{2}(1+iX)^k+\frac{1}{2}(1-iX)^k=\Re\left((1+iX)^k\right)=\sum_{2\mid t\le k}\binom{k}{t}(-1)^{\frac{t}{2}}X^t.$

Proof: Each of the first assertions follows by induction on k, the third follow from the Binomial Theorem. □

Proposition 3.2 *For all $t\in\mathbb{R}$ holds $q\left(\alpha^*(t)+t\right)-(p+q)t=\frac{2\pi}{3}(q-p)$.*

Proof: The left hand side is equal to $q(\frac{2\pi}{3}+\frac{p}{q}(t-\frac{2\pi}{3})+t)-(p+q)t=\frac{2\pi}{3}q-\frac{2\pi}{3}p$. □

In the next theorem we find a polynomial $F\in\mathbb{R}[X,Y]$ which is satisfied by the given Balaton-curve. It can be rewritten in a rather catchy way when we use complex variables. Let us write the complex variable as $z=x+iy$, where x and y are real. The vertices $A=0$, $B=2$ and $C=w$ are interpreted as complex numbers. For an integer p and $z\in\mathbb{C}$ let

$$s_p(z):=\begin{cases}\bar z^{p} & p\ge0\\ z^{-p} & p\le 0\,.\end{cases}$$

Then we have

Theorem 3.2 *Tripole-equation: Let $\zeta=-\frac{1}{2}+i\frac{\sqrt3}{2}$. Then the Balaton-curve of the triangle Δ with the vertex w $(=C)$ in the upper half plane satisfies the equation*

$$\Im\left(\zeta^{p-q}s_p(z)\cdot s_q(z-2)\cdot s_{p+q}(\bar z-\bar w)\right)=0.$$

Proof: Let $s := \operatorname{sgn}(p+q)$ and $\varepsilon \in \{-1,0,1\}$ be chosen such that $p-q \equiv \varepsilon \pmod 3$. We have $\tan \frac{2\pi}{3}(q-p) = \varepsilon\sqrt{3}$ and hence by Prop. 3.2

$$\varepsilon\sqrt{3} = \tan\left(q(\alpha^*(t)+t) - s|p+q|t\right)$$

which implies

$$R_q(\tan(\alpha^*(t)+t)) - sR_{|p+q|}(\tan t) = \tan(q(\alpha^*(t)+t)) - s\tan(|p+q|t) =$$
$$\varepsilon\sqrt{3}(1+s\tan(q(\alpha^*(t)+t))\tan(|p+q|t)) = \varepsilon\sqrt{3}(1+sR_q(\tan(\alpha^*(t)+t))R_{|p+q|}(\tan t)),$$

hence

$$\begin{aligned}
0 &= P_q(\tan(\alpha^*(t)+t))Q_{|p+q|}(\tan t) - sP_{|p+q|}(\tan t)Q_q(\tan(\alpha^*(t)+t)) - \\
&\quad \varepsilon\sqrt{3}\,Q_q(\tan(\alpha^*(t)+t))Q_{|p+q|}(\tan t) - s\varepsilon\sqrt{3}\,P_q\left(\tan(\alpha^*(t)+t)\right)P_{|p+q|}(\tan t) = \\
&= \Im\left(1+i\tan(\alpha^*(t)+t)\right)^q \Re(1+i\tan t)^{|p+q|} - \\
&\quad - s\,\Im(1+i\tan t)^{|p+q|}\,\Re\left(1+i\tan(\alpha^*(t)+t)\right)^q - \\
&\quad - \varepsilon\sqrt{3}\,\Re\left(1+i\tan(\alpha^*(t)+t)\right)^q \Re(1+i\tan t)^{|p+q|} - \\
&\quad - s\varepsilon\sqrt{3}\,\Im\left(1+i\tan(\alpha^*(t)+t)\right)^q \Im(1+i\tan t)^{|p+q|} = \\
&= \Im\left((1+i\tan(\alpha^*(t)+t))^q(1-is\tan t)^{|p+q|}\right) - \\
&\quad - \varepsilon\sqrt{3}\,\Re\left((1+i\tan(\alpha^*(t)+t))^q(1-is\tan t)^{|p+q|}\right).
\end{aligned}$$

The formula $\Im(\zeta^{p-q}z) = \frac{1}{2}\left(\sqrt{3}\,\varepsilon\Re(z) - \Im(z)\right)$ for $\varepsilon \neq 0$ and $= \Im z$ for $\varepsilon = 0$ results in

$$\begin{aligned}
0 &= \Im\left(\zeta^{p-q}(1+i\tan(\alpha^*(t)+t))^q(1-is\tan t)^{|p+q|}\right) = \\
&= \Im\left(\zeta^{p-q}\left(1-\frac{2iy}{x^2+y^2-2x}\right)^q\left(1-isb\,\frac{x\sin\alpha - y\cos\alpha}{x^2+y^2-b(x\cos\alpha+y\sin\alpha)}\right)^{|p+q|}\right).
\end{aligned}$$

Multiplication with $(x^2+y^2-2x)^q(x^2+y^2-b(x\cos\alpha+y\sin\alpha))^{|p+q|}$ gives

$$\Im\left(\zeta^{p-q}(x^2+y^2-2x-2iy)^q(x^2+y^2-(x-isy)be^{is\alpha})^{|p+q|}\right) = 0.$$

If $p+q>0$, that is if $s=1$, the polynomial can be written in the form $\Im f(z)$, where

$$f(z) = \zeta^{p-q}(z\bar{z}-2z)^q(z\bar{z}-\bar{z}w)^{p+q} = \zeta^{p-q}z^q\bar{z}^{p+q}(\bar{z}-2)^q(z-w)^{p+q}.$$

If $p>0$ it has the real factor $(z\bar{z})^q$, if $p<0$ it has the real factor $(z\bar{z})^{p+q}$.
If finally $p+q<0$, that is in the case $s=-1$, the polynomial in Theorem 3.2 has the form $\Im f(z)$, where this time

$$f(z) = \zeta^{p-q}(z\bar{z}-2z)^q(z\bar{z}-z\bar{w})^{-p-q} = \zeta^{p-q}z^{-p}(\bar{z}-2)^q(\bar{z}-\bar{w})^{-p-q}. \qquad \square$$

- Assume for a moment that $p \equiv q \pmod 3$. If we write the polynomial

$$s_p(x+iy)\cdot s_q(x+iy-2)\cdot s_{p+q}(x-iy-\bar{w})$$

as a sum of homogenous polynomials, then the one with the highest degree is

$$(x-iy)^p(x-iy)^q(x+iy)^{p+q}, \quad (x+iy)^{-p}(x-iy)^q(x+iy)^{p+q},$$
$$\text{or } (x+iy)^{-p}(x-iy)^q(x-iy)^{-p-q}$$

respectively, according to whether $p>0$, $0<-p<q$ or $p+q<0$. In any case this polynomial is real. Hence the Tripole-equation has a degree $< |p|+q+|p+q|$.

- If $p \not\equiv q \pmod 3$, its degree is $\leq |p|+q+|p+q|$.

Proposition 3.3 *Let s_A be the multiplicity of the Balaton-curve f at point $A=(0,0)$, s_B the multiplicity at point $B=(2,0)$ and s_C that at $C=b(\cos\alpha,\ \sin\alpha)$. Then*

$$s_A=\begin{cases}|p| & \\ |p|-1 & \end{cases}\text{for}\ \begin{matrix}p\not\equiv q\pmod 3\\ p\equiv q\pmod 3,\end{matrix}\qquad s_B=\begin{cases}q & \\ q-1 & \end{cases}\text{for}\ \begin{matrix}p\not\equiv q\pmod 3\\ p\equiv q\pmod 3,\end{matrix}$$

$$s_C=\begin{cases}|p+q| & \\ |p+q|-1 & \end{cases}\text{for}\ \begin{matrix}p\not\equiv q\pmod 3\\ p\equiv q\pmod 3.\end{matrix}$$

Proof: We prove the first two assertions and leave the rest of the proof to the reader.

(1) Let t_0 be chosen such that $x(t_0)=y(t_0)=0$. By Lemma 2.1 and Prop. 2.1 we have $N(t_0)\neq 0$. If we had $\sin(\alpha^*(t_0)-\alpha)=0$, we would get $\sin(\alpha^*(t_0)+t_0)=0$ (and hence $\sin\alpha^*(t_0)=0$, contrary to $N(t_0)\neq 0$) or $\sin(t_0-\beta)=0$; but then $a\sin(\alpha^*(t_0)+\beta)-2\sin\alpha^*(t_0)=0$, contrary to our assumption.
Therefore $\sin(\alpha^*(t_0)-\alpha)=0$ and hence for some integer m $\alpha^*(t_0)=\alpha+m\pi$. Let us denote these parameter values by t_m. We have $\frac{2\pi}{3}+\theta(t_m-\frac{2\pi}{3})=\alpha+m\pi$, and hence $t_m=\beta+\frac{\pi}{3}+\frac{\pi}{\theta}(m-\frac{1}{3})$. We have to find the number of integers m, such that t_m lies in a given interval of length $q\pi$ and such that $t_m\notin\beta+\pi\mathbb{Z}$ (for otherwise $N(t_0)=0$). In the case that p is positive we choose the interval $[0,\ q\pi)+\beta+\frac{\pi}{3}(1-\frac{1}{\theta})$, and if p is negative, we choose the interval $(-q\pi,\ 0]+\beta+\frac{\pi}{3}(1-\frac{1}{\theta})$. Then the condition is equivalent with $qm\pi/|p|\in[0,\ q\pi)$ and $\frac{1}{3}+\frac{q}{p}(m-\frac{1}{3})\notin\mathbb{Z}$.
If $p\not\equiv q\pmod 3$, the second condition is automatically satisfied and the first one is equivalent with $m\in[0,\ |p|)$. Hence the result.
If $p\equiv q\pmod 3$, then from $|p|$ we have to subtract the number of $m\in\mathbb{Z}\cap[0,\ |p|)$, for which $\frac{1}{3}+\frac{q}{p}(m-\frac{1}{3})\in\mathbb{Z}$, that is, $p+q(3m-1)\in 3p\mathbb{Z}$. As the congruence $\frac{p-q}{3}+qm\equiv 0\pmod{|p|}$ has exactly one solution mod $|p|$, the proof is complete.

(2) Let t_0 be chosen such that $x(t_0)=2$ and $y(t_0)=0$. Note that by Lemma 2.1 and Prop. 2.1 $N(t_0)\neq 0$. If we had $\sin(t_0-\beta)\neq 0$, we would either get $\sin(\alpha^*(t_0)-\alpha)=0$ (and hence $x(t_0)=0$), or $\sin(\alpha^*(t_0)+t_0)=0$, and hence

$$2=4b\,\frac{\sin(\alpha^*(t_0)-\alpha)\sin\alpha^*(t_0)}{4\sin^2\alpha^*(t_0)},$$

that is

$$2\sin\alpha^*(t_0)=b\sin(\alpha^*(t_0)-\alpha)=2\sin\alpha^*(t_0)-a\sin(\alpha^*(t_0)+\beta),$$

which implies $\sin(\alpha^*(t_0)+\beta)=0$. But then $t_0-\beta\in\pi\mathbb{Z}$ and so $\sin(t_0-\beta)=0$, contrary to our assumption.
Therefore $\sin(t_0-\beta)=0$. There is an $n\in\mathbb{Z}$ such that $t_0=\beta+\pi n$. Let us denote these parameter values by t_n. We have to find the number of n such that $t_n\in\beta+[0,\ q\pi)$ and to subtract those n for which $\alpha^*(t_n)\in\alpha+\pi\mathbb{Z}$. We have $t_n\in\beta+[0,\ q\pi)$ if and only if $0\le n<q$, and $\alpha^*(t_n)\in\alpha+\pi\mathbb{Z}$ if and only if

$$\frac{2\pi}{3}+\frac{p}{q}\left(\beta+n\pi-\frac{2\pi}{3}\right)\in\alpha+\pi\mathbb{Z},$$

that is $p\equiv q\pmod 3$ and (as $\frac{p}{q}(\beta-\frac{\pi}{3})=\alpha-\frac{\pi}{3}$), $pn+\frac{q-p}{3}\in q\mathbb{Z}$. In the case $p\equiv q\pmod 3$ there is exactly one such n. □

Lemma 3.2 *Let v_x for every $x\in\mathbb{R}$ be the multiplicity with which the Balaton-curve intersects the $x-$axis at x. Then $\sum_{x\in\mathbb{R}}v_x\ge s_A+s_B+s_C$.*

Proof: Let $y(t_0) = 0$. Assume first that $x = x(t_0) \notin \{0, 2\}$. By Lemma 2.1 and Prop. 2.1 $N(t_0) \neq 0$. Then we have $\sin(\alpha^*(t_0) + t_0) = 0$, and, as

$$b \sin(\alpha^*(t_0) - \alpha) = 2 \sin \alpha^*(t_0) - a \sin(\alpha^*(t_0) + \beta),$$

we get

$$x = 2\, \frac{(a \sin(\alpha^*(t_0) + \beta) - 2 \sin \alpha^*(t_0))\,(-2 \sin \alpha^*(t_0))}{4 \sin^2 \alpha^*(t_0)} = 2 - a\, \frac{\sin(\alpha^*(t_0) + \beta)}{\sin \alpha^*(t_0)}$$

and $\alpha^*(t_0) + t_0 = k\pi$ for some $k \in \mathbb{Z}$. Let us denote these parameter values by t_k. Then we have

$$\left(\frac{p}{q} + 1\right) t_k + \frac{2\pi}{3}\left(1 - \frac{p}{q}\right) = k\pi\,,$$

that is

$$t_k = \frac{kq\pi}{p+q} + \frac{2\pi}{3}\, \frac{p-q}{p+q}\,.$$

There are $|p+q|$ such k for which t_k lies in $[0,\ q\pi) + \frac{2\pi}{3}\frac{p-q}{p+q}$. From $|p+q|$ we have to subtract the number of those k for which $\sin(\alpha^*(t_k) + \beta) = 0$, or $\sin \alpha^*(t_k) = \sin \alpha$ or $\sin \alpha^*(t_k) = 0$ (two of these cases cannot occur simultaneously).

Assume that $\alpha^*(t_k) + \beta = m\pi$ for some $m \in \mathbb{Z}$. There is at most one such t_k in a half-open interval of length $q\pi$, for if $m\pi - \beta + t_k = k\pi$ and $m'\pi - \beta + t_{k'} = k'\pi$, we would get $t_k - t_{k'} \equiv 0 \pmod{\pi}$ and hence $\frac{q(k-k')}{p+q}\pi \equiv 0 \pmod{\pi}$, which implies $q(k - k') \equiv 0 \pmod{|p+q|}$ and hence $k \equiv k' \pmod{|p+q|}$. Therefore $k \neq k'$ implies $|t_k - t_{k'}| \geq q\pi$.

Furthermore we have $\alpha^*(t_k) + t_k = k\pi$, that is $t_k - \beta = (k - m)\pi$. Hence $\sin(t_k - \beta) = 0$. The function $y(t)$, $t \in \mathbb{R}$, then has a double zero at t_k and hence $v_B \geq s_B + 1$.

Similarily there is at most one k such that $\alpha^*(t_k) - \alpha \in \pi\mathbb{Z}$ and t_k lies in a given interval of length $q\pi$. Furthermore $a \sin(\alpha^*(t_k) + \beta) - 2 \sin \alpha^*(t_k) = 0$. $y(t)$, $t \in \mathbb{R}$, has then an at least double zero at t_k. We get $v_A \geq s_A + 1$.

Finally we have to investigate under which conditions the case $\sin \alpha^*(t_k) = 0$ can occur. Then $kq + \frac{2}{3}(p - q) = n(p + q)$ for some $n \in \mathbb{Z}$ and therefore $p \equiv q \pmod{3}$ and in that case there is indeed exactly one such k in the interval $[0,\ |p+q|)$. Therefore Prop. 3.3 implies the result. □

Note that for rational θ all Balaton-curves satisfy an irreducible polynomial $F \in \mathbb{R}[X, Y]$. For if F is a polynomial such that for all t in an open interval I of lenght $q\pi$ $F\,(x(t), y(t)) = 0$ and if $F = F_1\, \ldots\, F_k$ is a decomposition into irreducible factors, then

$$F_1\,(x(t), y(t))\, \ldots\, F_k\,(x(t), y(t)) = 0$$

and, as every function $F_i\,(x(t), y(t))$, $t \in I$, is real-analytic (if I is suitable chosen), there is some factor F_i such that $F_i(x(t), y(t)) = 0$.

Fig. 5 shows the algebraic Balaton-curve f of the triangle with the angles $\alpha = \frac{\pi}{2}$ and $\beta = \frac{4\pi}{9}$, that is with $\theta = \frac{3}{2}$. Note that $3 \not\equiv 2 \pmod{3}$. The degree of this curve is 10.

Theorem 3.3 *Assume that $p \not\equiv q \pmod{3}$. Then the Tripole-equation*

$$\Im\left(\zeta^{p-q} s_p(x + iy) s_q(x + iy - 2) s_{p+q}(x - iy - \bar{w})\right) = 0$$

of the Balaton-curve of the triangle with third vertex w is irreducible and of degree $|p| + q + |p + q|$.

Proof: Let $F(x, y) = 0$ be the Tripole-equation and let $G \in \mathbb{R}[X, Y]$ be an irreducible polynomial such that $G(x(t), y(t)) = 0$. Then $G|F$ and by Lemma 3.2 we have $\deg G \geq |p| + q + |p + q| \geq \deg F$. This implies both assertions. □

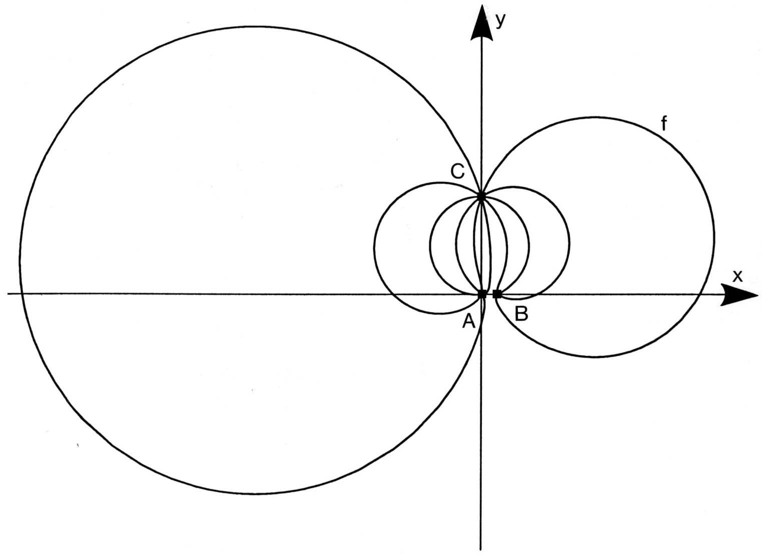

Figure 5: An algebraic curve f with $\theta = \frac{3}{2}$, degree 10 and no real point at infinity

The Tripole-equation is reducible in the case $p \equiv q \pmod 3$. In order to prove this we introduce the polynomial

$$k_U(X,Y) := X^2 + Y^2 - 2X - \frac{b - 2\cos\alpha}{\sin\alpha} Y \in \mathbb{R}[X,Y].$$

$k_U(x,y) = 0$ is the equation of the circumcircle of the given triangle.

Fig. 6 shows the algebraic Balaton-curve of the triangle with angles $\alpha = \frac{3\pi}{5}$ and $\beta = \frac{\pi}{4}$, that is with $\theta = -\frac{16}{5}$, and its circumcircle k_U, which is drawn in a broken line. The degree of the curve is 29.

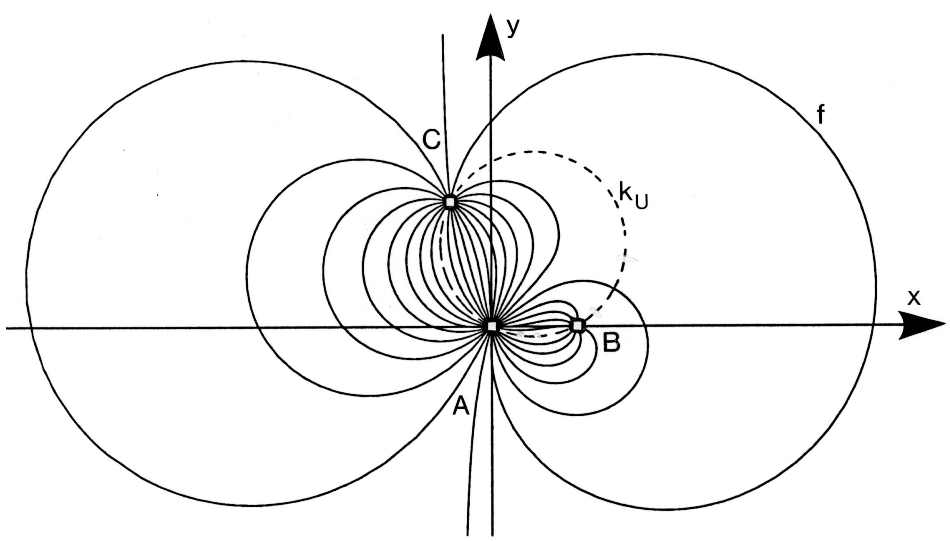

Figure 6: An algebraic curve f with $\theta = -\frac{16}{5}$ of degree 29
and exactly one real point at infinity

Theorem 3.4 *Assume that $p \equiv q \pmod 3$ and let*

$$F(X,Y) = \Im\left(s_p(X+iY)s_q(X+iY-2)s_{p+q}(X-iY-\bar{w})\right)$$

be the Tripole-polynomial of the Balaton-curve of the triangle with third vertex w. Then $k_U|F$ and $F(x,y)/k_U(x,y) = 0$ is the irreducible equation of the Balaton-curve; its degree is $|p| + q + |p+q| - 3$.

Proof: We have $(p+q)(\alpha-\beta) - (p-q)(\alpha+\beta) = 2(\alpha q - \beta p)$. As $\dfrac{\alpha - \pi/3}{\beta - \pi/3} = \dfrac{p}{q}$, we get $\alpha q - \beta p \in \mathbb{Z}$ and hence $e^{i(p+q)(\alpha-\beta)} = e^{i(\alpha+\beta)(p-q)}$.
Now let z be a point of the circumcircle. Then $z\bar{z} - z - \bar{z} - \frac{1}{2i}(z-\bar{z})d = 0$, where $d := \dfrac{b}{\sin\alpha} - 2\cot\alpha$. Hence

$$\bar{z} = z\,\frac{2-id}{2z-2-id}, \quad \bar{z} - 2 = -(z-2)\frac{2+id}{2z-2-id} \quad \text{and} \quad \bar{z} - \bar{w} = \frac{2z - zid - 2z\bar{w} + 2\bar{w} + id\bar{w}}{2z-2-id}.$$

As w is a point of the circumcircle we get in particular

$$2w + 2\bar{w} - 2b^2 - idw + id\bar{w} = 0 \quad \text{and hence} \quad 2\bar{w} + id\bar{w} = 2b^2 + idw - 2w.$$

Substituting this into the numerator of $\bar{z} - \bar{w}$ we get

$$\bar{z} - \bar{w} = (z-w)\,\frac{2-id-2\bar{w}}{2z-2-2id}.$$

Note that all the numbers

$$\frac{2-id}{2z-2-id}, \; -\frac{2+id}{2z-2-id} \quad \text{and} \quad \frac{2-id-\bar{w}}{2z-2-id}$$

have absolute value 1 and that for such numbers v and for all integers k $\;s_k(v) = v^{-k}$. Therefore for z on the circumcircle, we get

$$\begin{aligned}
&2i\Im\left(s_p(z)s_q(z-2)s_{p+q}(\bar{z}-\bar{w})\right) = s_p(z)s_q(z-2)s_{p+q}(\bar{z}-\bar{w}) - s_p(\bar{z})s_q(\bar{z}-2)s_{p+q}(z-w) = \\
&= s_p(z)s_q(z-2)s_{p+q}(z-w)\left(s_{p+q}\left(\frac{2-id-2\bar{w}}{2z-2-id}\right) - s_p\left(\frac{2-id}{2z-2-id}\right)s_q\left(-\frac{2+id}{2z-2-id}\right)\right) = \\
&= s_p(z)s_q(z-2)s_{p+q}(z-w)(2z-2-id)^{p+q}\left[(2-id-2\bar{w})^{-p-q} - (2-id)^{-p}(-2-id)^{-q}\right]
\end{aligned}$$

hence, in order to prove the first assertion, it is enough to prove that the term in the bracket is 0.
Now

$$2 - id = \frac{2ie^{-i(\alpha+\beta)}}{\sin(\alpha+\beta)} \quad \text{and therefore} \quad 2 - id - 2\bar{w} = \frac{2ie^{-i(\alpha-\beta)}}{\sin(\alpha+\beta)}.$$

Hence it is enough to prove that

$$(ie^{-i(\alpha-\beta)})^{-p-q} = (ie^{-i(\alpha+\beta)})^{-p}(ie^{i(\alpha+\beta)})^{-q}.$$

This follows from the discussion at the beginning of the proof. The rest of the proof follows the same lines as that of Theorem 3.3. □

Note that by Theorem 3.3 and Theorem 3.4 the degree remains unaltered if we interchange two of the angles α, β or γ of the given triangle.

Theorems 3.3 and 3.4 enable to investigate the Balaton-curves as subsets of the projective complex plane and to determine their geometric properties, e.g., the singular points of the curve, etc.

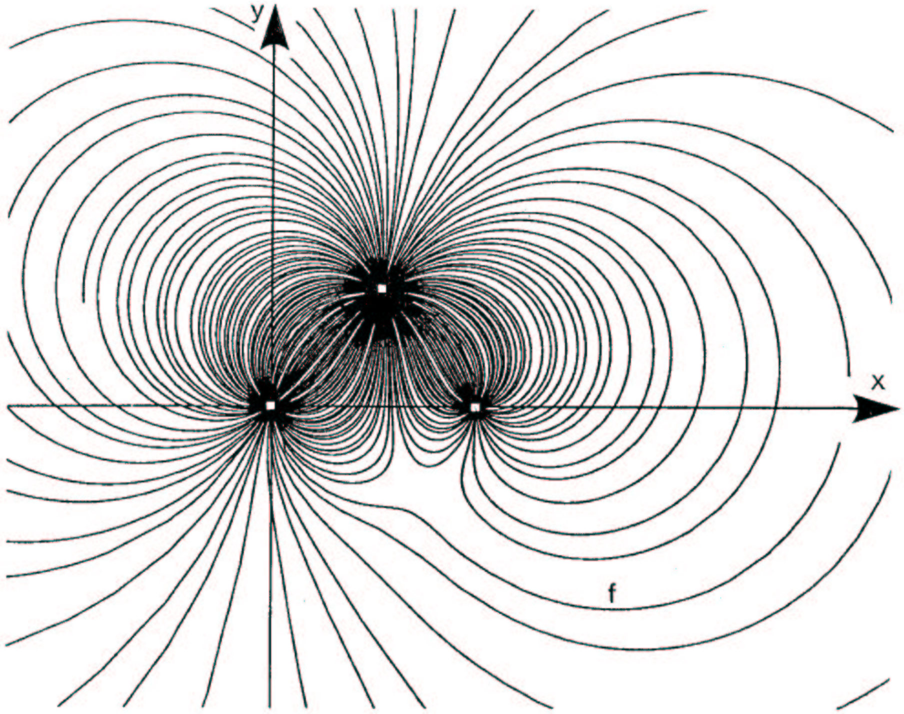

Figure 7: A transcendental curve f with $\theta = \frac{1+\sqrt{5}}{2}$; the white squares represent the vertices A, B, C

4. The transcendental Balaton-curves

Fig. 7 shows a part of the Balaton-curve of the triangle with angles $\alpha = \frac{\pi}{4}$ and $\beta = \pi\frac{9-\sqrt{5}}{24}$ that is $\theta = \frac{1+\sqrt{5}}{2}$. The curve is dense in $\mathbb{R}^2$ by Theorem 4.1 below.

For real x let $\{x\} = x - [x]$ be the *fractional part* of x. In the proof of the following Theorem we use KRONECKER's Approximation Theorem in the following form: if μ, ν are real numbers such that $1, \mu, \nu$ are linearly independent over the rationals and if δ and ε are arbitrary real numbers, then the sequence $(\{n\mu + \delta\},\ \{n\nu + \varepsilon\})_{n\geq 1}$ is dense in the unit square $[0,1] \times [0,1]$.

Let $F \subseteq \mathbb{R}^2$ be finite, $\Gamma := \pi\mathbb{Z}^2$ and let $f:\ \mathbb{R}^2 \setminus (F+\Gamma) \to \mathbb{R}^2$ be a continuous function with period lattice Γ and whose range is dense in $\mathbb{R}^2$. Assume further that $g:\ \mathbb{R} \to \mathbb{R}^2$, $g(t) = (u_1 t + v_1,\ u_2 t + v_2)$ is the parameter representation of a straight line with the irrational slope u_2/u_1. Then the range of $f \circ g$ is dense in $\mathbb{R}^2$.

This follows from KRONECKER's Theorem. Assume that $\varepsilon > 0$ and that $(x, y) \in \mathbb{R}^2$ is given. Then, as u_1/u_2 is irrational, there is a real number γ such that $1,\ \gamma u_1/\pi,\ \gamma u_2/\pi$ are linearly independent over $\mathbb{Q}$ and $g(n\gamma) \in F + \Gamma$ for only finitely many n (choose $\gamma \notin \mathbb{Q}(u_1, u_2, \pi)$). Hence there are integers n, p, q such that

$$g(n\gamma) \notin F + \Gamma, \quad \left|\frac{n\gamma u_1}{\pi} + \frac{v_1}{\pi} - \frac{x}{\pi} - p\right| < \frac{\varepsilon}{\pi}, \quad \left|\frac{n\gamma u_2}{\pi} + \frac{v_2}{\pi} - \frac{y}{\pi} - q\right| < \frac{\varepsilon}{\pi}$$

or what is the same, $|g(n\gamma) - (x, y) - \pi(p, q)| < \varepsilon\sqrt{2}$. As f is continuous, $f(g(n\gamma))$ can be made by an appropriate integer n arbitrarily close to $f(x, y)$. As f has a dense range, the result follows.

We apply this consideration to $F = \{(0,0),\ (\alpha, \beta)\}$, the map $f:\ (\alpha^*, \beta^*) \mapsto (x, y)$ in Prop. 1.2 and to $g(t) = \big((\alpha - \frac{\pi}{3})t + \frac{2\pi}{3},\ (\beta - \frac{\pi}{3})t + \frac{2\pi}{3}\big)$. Function f has a dense range due to

Prop. 1.3 and is continuous on $\mathbb{R}^2 \setminus (F+\Gamma)$. Hence we have

Theorem 4.1 *Let* $\theta = \dfrac{3\alpha-\pi}{3\beta-\pi}$ *be irrational. Then the Balaton-curve of the triangle* Δ *with angles* α *and* β *is dense in* $\mathbb{R}^2$*. In particular the curve is transcendental.*

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References

[1] Z. Čerin: *Isocentroidal triangles and regular hexagons*. Radovi Matematički **9**, 224–239 (1999).

[2] Z. Čerin: *Locus Properties of the Neumann Cubic*. J. Geom. **63**, 39–56 (1998).

[3] H. Dirnböck: *Torricelli-Punkte, isodynamische Punkte, Balaton-Kurven*. Unpublished manuscript (1995).

[4] R.A. Johnson: *Advanced Euclidean Geometry (Modern Geometry). An Elementary Treatise on the Geometry of the Triangle and the Circle*. Dover Publications, Inc. 1960.

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