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A Voronoi Poset

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Abstract. Given a set S of n points in general position, we consider all k-th order Voronoi diagrams on S, for $k = 1, \dots, n$, simultaneously. We recall symmetry relations for the number of cells, number of vertices and number of circles of certain orders. We introduce a poset $\Pi(S)$ that consists of the k-th order Voronoi cells for all $k = 1, \dots, n$, that occur for some set S. We prove that there exists a rank function on $\Pi(S)$ and moreover that the number of elements of odd rank equals the number of elements of even rank of $\Pi(S)$, provided that n is odd.

Key words: k-th order Voronoi diagrams, k-sets, posets, point configurations

MSC 2000: 52B55, 68U05

1. Introduction

The dynamics of Voronoi diagrams in the plane is well understood. When $n-1$ points are fixed and one point is moving continuously inside the convex hull, combinatorial changes of the Voronoi diagram correspond to changes in the configuration of empty circles, see [11], Chapter 1 and [1]. Changes in the configuration of non-empty circles correspond to combinatorial changes of higher order Voronoi diagrams. Here the k-th order Voronoi diagram associates to each subset of size k of generating sites that region in the plane that consist of points closest to these k sites.

We consider all k-th order Voronoi diagrams simultaneously for k between 1 and n. We do so by introducing the Voronoi poset $\Pi(S)$ of a set S of n distinct sites in the plane. The poset consists of all sets of labels that correspond to a subset of sites that defines some non-empty Voronoi cell in some k-th order Voronoi diagram.

Higher order Voronoi diagrams have been investigated by numerous people. Many results are published in an article by D.T. Lee, [10]. A survey is given in Edelsbrunner's book, [8], on algorithms in combinatorial geometry. Let S be a set of n points in general position in $\mathbb{R}^3$. A subset A of k points of S is a *k-set* if it can be separated from its complementary set $B = S \setminus A$ by a plane V_A. There is a close connection between higher order Voronoi diagrams and k-sets established by a lifting transformation ψ that changes the point inside circle relation in $\mathbb{R}^2$ into a point below hyperplane relation in $\mathbb{R}^3$. It turns out that these

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circles containing points serve as a 'building block' for higher order Voronoi diagrams as we discuss in full detail in Section 2. As a consequence, formulas counting k-sets in $\mathbb{R}^3$ can be applied in the counting of vertices, edges and cells of higher order Voronoi diagrams. Instead of considering circles that contain a fixed number of, say, k points, one can also consider circles that contain at most k points. This is done in [9].

Let T be a set of n points in $\mathbb{R}^3$ in general position that are the vertices of a convex polytope. Sharir, [13], Lemma 4.4 and Clarkson and Shor, [5], Theorem 3.5 prove that the number of k-sets of T is given by $2(k+1)(n-k-2)$. They prove this formula using probabilistic methods that we do not discuss here.

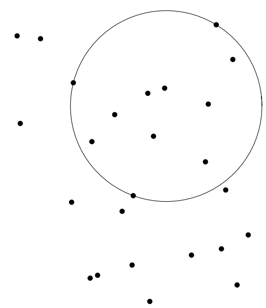

Figure 1: An invariant for circle configurations

This formula can also be derived in the context of k-th order Voronoi diagrams from Lee's results as has been observed by several people, [5, 2]. We give this derivation explicitly and state in Theorem 4.3 that

$$c_i + c_{n-i-3} = 2(i+1)(n-i-2),$$

where c_i denotes the number of circles defined by a set S of n points in general position in the plane, containing exactly i points of S. For an illustration, see Fig. 1. Moreover we explicitly derive similar formulas for the number of cells f_k in the k-th order Voronoi diagram $V_k(S)$, see Lemma 4.1, and the number of vertices v_k, see Lemma 4.2, in $V_k(S)$.

$$\begin{aligned} f_k + f_{n-k+1} &= 2k(n-k+1) + 1 - n, \\ v_k + v_{n-k} &= 4k(n-k). \end{aligned}$$

These 'symmetry relations' are independent of the particular position of the sites in S, provided S is in general position: while the number of cells in some k-th order Voronoi diagram may change, depending on the configuration, the sum of the number of cells in the k-th order diagram and the number of cells in the $(n-k+1)$-th diagram remains constant.

In Section 3 we introduce the Voronoi poset mentioned above and prove that $\Pi(S)$ has a rank function. As an application of the symmetry relations we prove in Theorem 5.1 that the number of elements of odd rank in $\Pi(S)$ equals the number of even rank, provided that n is odd.

The Voronoi poset of a set S of n moving points seems a natural object to study as changes of the poset occur exactly at those moments where S is not in general position. As there are tight connections between higher order Voronoi diagrams, k-levels in certain arrangements in $\mathbb{R}^3$ and certain k-sets in $\mathbb{R}^3$, the study of the Voronoi poset may have applications in these areas as well.

2. Higher order Voronoi diagrams

2.1. Definition of k-th order Voronoi diagram

Let $S = \{p_1, \ldots, p_n\}$ be a set of n distinct points in $\mathbb{R}^2$ in general position. Let $0 \le k \le n$. For every point p in the plane we ask for the k nearest points from S. That is, we look for a subset $A \subset S$, such that

$$|A| = k, \qquad \forall x \in A, \quad \forall y \in S - A: \quad d(p,x) \quad \le \quad d(p,y).$$

For two points in $\mathbb{R}^2$, we define a half-plane $h(x,y) := \{p \in \mathbb{R}^2 \mid d(x,p) \le d(y,p)\}$. The *Voronoi cell* of $A \subset S$ of *order* $|A|$ is the intersection of half-planes

$$V(A) \quad := \bigcap_{x \in A,\ y \in S-A} h(x,y),$$

whenever this intersection is not empty. As an intersection of half-planes, $V(A)$ is a convex polygon.

The *k-th order Voronoi diagram* is the subdivision of $\mathbb{R}^2$, induced by the set of Voronoi cells of order k. For later purposes, we identify the k-th order Voronoi diagram with the set of non empty k-th order Voronoi cells.

$$V_k(S) \quad := \quad \{V(A) \mid A \subset S,\ |A| = k,\ V(A) \ne \emptyset\}.$$

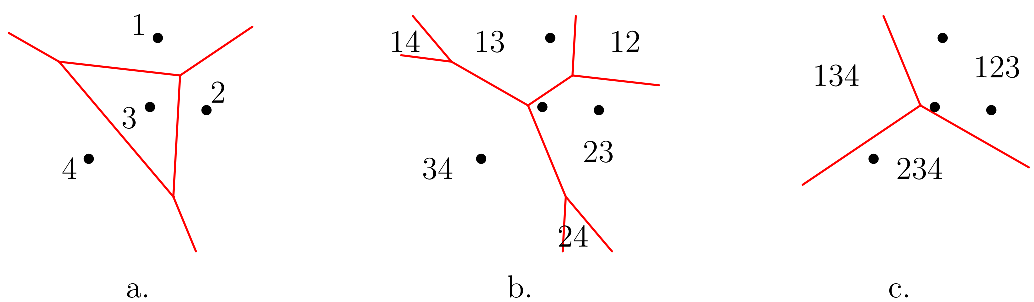

Figure 2: A first, second and third order Voronoi diagram.

Example 2.1 Let $S = \{p_1, p_2, p_3, p_4\}$, with $p_1 = (45, 86)$, $p_2 = (76, 40)$, $p_3 = (40, 42)$ and $p_4 = (1, 9)\}$. Fig. 2 shows the first, second, and third order Voronoi diagram of S. In every non-empty Voronoi cell the corresponding point labels are displayed.

Remark 2.2 A planar graph that represents a point-face dual of the k-th order Voronoi diagram can be constructed as follows, cf. [3]. Write down for every $A \subset S$ with $|A| = k$ and $V_k(A) \ne \emptyset$ its *centroid* $c(A)$, defined by $c(A) = (1/k) \sum_{p \in A} p$. Two centroids $C(A)$ and $c(B)$ are connected by an edge exactly iff $V_k(A)$ and $V_k(B)$ share an edge.

2.2. Circles and higher order Voronoi diagrams

In this section, we state some elementary properties of higher order Voronoi diagrams. Every edge in $V_k(S)$ is part of some bisector $B(a,b)$, with $a, b \in S$. The Voronoi vertices are exactly those points that are in the centers of the circles determined by three points from S. Therefore, under our general position assumption, every Voronoi vertex has valency three. The following theorem describes the local situation around a Voronoi vertex. The symbol $\bigodot_{a,b,c}$ denotes the circle passing through the points a, b, and c.

Theorem 2.3 *Let x be the center of $\odot_{a,b,c}$, for $a, b, c \in S$, let*

$$H \quad = \quad \{ \ z \in S \mid d(x,z) < d(x,a) \ \},$$

and let $k = |H|$. Then x is a Voronoi vertex of $V_{k+1}(S)$ and $V_{k+2}(S)$. The Voronoi edges and cells that contain x are given in Fig. 3. Moreover, all Voronoi vertices are of this form.

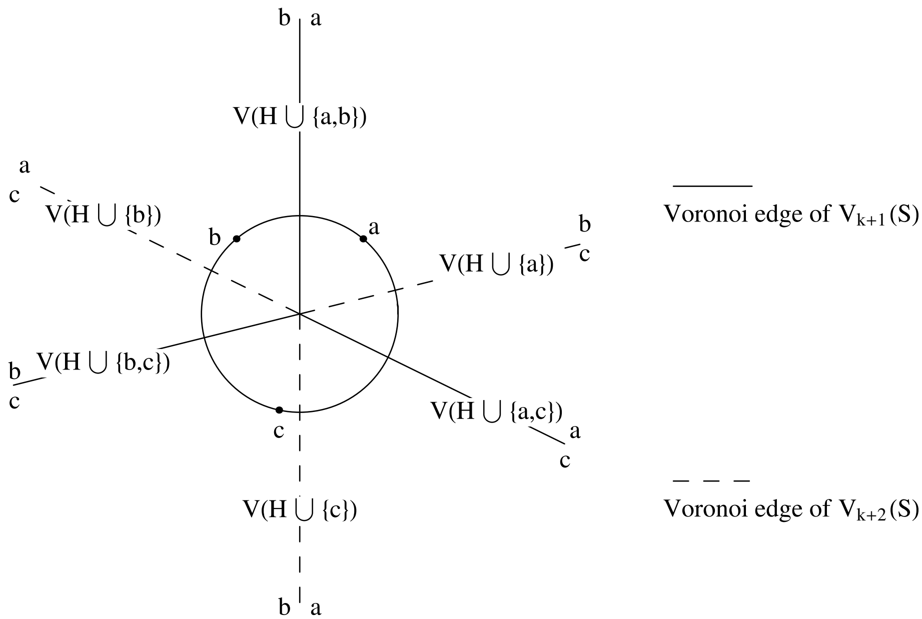

Figure 3: The Voronoi diagram around x

Proof: [7], Theorem 1 and Theorem 2. □

Let a, b, c and H be as defined in Theorem 2.3. We define the *order* of a circle $\odot_{a,b,c}$ as $|H|$. Notation: $|\odot_{a,b,c}| \ := \ |H|$. An *order k Voronoi circle* $\odot_{a,b,c}$ is a circle through three points a, b and c from S that contains exactly k points from $S - \{a, b, c\}$. In fact, from all $\binom{n}{3}$ Voronoi circles $\odot_{a,b,c}$ and all sets $H_{a,b,c}$, compare Theorem 2.3, almost enough information is provided to construct all k-th order Voronoi diagrams $V_k(S)$ for $k = 1, \ldots, n-1$.

Algorithm 2.4 Voronoi diagrams of all orders.

Input: set S of n points in general position.
Output: all k-th order Voronoi diagrams $V_k(S)$ for $k = 1, \ldots, n-1$.

1. Compute all circles $\odot_{a,b,c}$ defined by S.
2. Compute all sets $H_{a,b,c}$ defined by S.
3. Take all circles of order $k-1$ and order $k-2$. The centers of these circles are exactly the vertices of $V_k(S)$.
4. Theorem 2.3 gives for every vertex the three incident edges and the three incident cells.
5. Two vertices are connected by an edge iff the two vertices have two incident cells in common. Skip the edge if it is used.
6. Edges that are not skipped are unbounded edges. Their direction and orientation still have to be computed. The direction is simply the direction of the bisector containing the edge. The orientation follows from Fig. 3.

Theorem 2.3 shows that for dynamic point sets topological changes in the family of k-th order Voronoi diagrams for $k = 1, \ldots n-1$ correspond to changes in the configuration of Voronoi circles. More on this topic can be found in [11].

Remark 2.5 Denote the number of circles of order k by c_k and the number of vertices in a k-th order Voronoi diagram by v_k. Define $c_{-1} = 0$. As a consequence of Theorem 2.3 we get

$$v_k = c_{k-1} + c_{k-2}. \tag{1}$$

2.3. Counting vertices, edges and cells

The following theorem shows that the total number of vertices, edges and Voronoi cells does not depend on the positions of the points in S, assuming general position.

Theorem 2.6 *Let v_k, e_k, and f_k denote the number of vertices, edges and cells in $V_k(S)$ for some set S of size n in general position. The total number of vertices, edges and cells in the Voronoi diagram of all orders are as follows.*
(i) $\sum_{k=1}^{n} v_k = \frac{1}{3}n(n-1)(n-2)$.
(ii) $\sum_{k=1}^{n} e_k = \frac{1}{2}n(n-1)^2$.
(iii) $\sum_{k=1}^{n} f_k = \frac{1}{6}n(n^2+5)$.

Proof: We prove the three claims.
(i) Every circle center defined by three distinct sites from S is a Voronoi vertex in some k-th and $(k+1)$-th order Voronoi diagram. As there are $\binom{n}{3}$ distinct circles, the first claim follows.
(ii) Consider the arrangement of bisectors $\mathcal{A}(S)$. Fix one bisector $B(a,b)$. As S is in general position, we may assume that the bisector $B(a,b)$ is divided into $n-1$ line segments by the Voronoi circle centers $abx_3, abx_4, \ldots, abx_n$, where we write $S = \{a, b, x_3, \ldots, x_n\}$. Every line segment is an edge in some k-th order Voronoi diagram. As there are $\binom{n}{2}$ distinct bisectors, claim *(ii)* follows.
(iii) The Euler formula, $v_k - e_k + f_k = 1$, holds for every order. Therefore

$$\sum_{k=1}^{n} f_k = n + \sum_{k=1}^{n} e_k - \sum_{k=1}^{n} v_k,$$

which completes the proof. □

The number of vertices, edges and cells in $V_k(S)$ depends on the configuration of S as the ordinary Voronoi diagram shows. The following theorem gives expressions for these numbers. Let f_k^∞ denote the number of unbounded cells in the k-th order Voronoi diagram. By definition $f_0^\infty := 0$.

Theorem 2.7 *Let S be in general position. Then the number of vertices, edges and cells in the k-th order Voronoi diagram can be expressed as follows.*
(i) $v_k = 2(f_k - 1) - f_k^\infty$.
(ii) $e_k = 3(f_k - 1) - f_k^\infty$.
(iii) $f_k = (2k-1)n - (k^2-1) - \sum_{i=1}^{k} f_{i-1}^\infty$.

Proof: [8, 10]. □

Note that $f_n = 1$. Substituting $k = n$ in the expression for f_k in Theorem 2.7 yields the following equation for the total number of unbounded cells:

$$\sum_{i=1}^{n} f_{i-1}^{\infty} \quad = \quad n(n-1). \tag{2}$$

The unbounded cells in the k-th order Voronoi diagram can be characterized as follows: let $\overline{pq}$ denote the line segment with endpoints p and q and l_{pq} the line through p and q.

Property 2.8 *A cell $V(A)$ of the k-th order Voronoi diagram $V_k(S)$ is unbounded if and only if one of the following two conditions holds.*
(i) There exists a line l that separates A from $S - A$.
(ii) There exist two consecutive points p and q, with $p, q \in S - A$, on $\delta CH(S - A)$ such that the points in $A - \overline{pq}$ are in the open half plane defined by l_{pq} opposite to $CH(S - A)$.

Proof: [12], Property OK4. □

Under the general position assumption, we only need to consider condition *(i)* in Property 2.8. It is clear that in this case the following symmetry holds:

$$f_k^{\infty} \quad = \quad f_{n-k}^{\infty}. \tag{3}$$

3. The Voronoi poset

3.1. Definition and examples

Fix a labeling of the sites in S and identify a set of sites $A \subset S$ that defines a non-empty Voronoi cell $V(A)$ with the set of labels $L(A) \subset [n]$ of the sites in A. A subset L of $[n]$ may or may not correspond to some Voronoi cell $V(A_L)$. For $k = 1$ we retain the ordinary Voronoi diagram, which implies the correspondence

$$V_1(S) \;\leftrightarrow\; \{\{1\}, \{2\}, \ldots, \{n\}\}.$$

We define $V_0(S) = \{\emptyset\}$. The set $\{\{1, \ldots, n\}\}$ corresponds to $V_n(S)$. We consider the set of all Voronoi cells that appear for a given set S of points and call the set of corresponding labels the *Voronoi poset* $\Pi(S)$ of S:

$$\Pi(S) \;:=\; \bigcup_k \{\; L(A) \mid V(A) \in V_k(S) \;\}.$$

This definition also makes sense when we drop the general position assumption.

We order the elements in the poset by set inclusion of the sets $L(A)$. This yields a partially ordered set. For more on partially ordered sets consult [15]. The poset is bounded since we have the empty set as $\hat{0}$, the unique minimal element, and the set $[n]$ as $\hat{1}$, the unique maximal element. In general, a poset is called *graded* if it is bounded and if every maximal chain has equal length. We show that $\Pi(S)$ is graded. Below we give an example showing that $\Pi(S)$ is in general not a lattice.

Property 3.1 $\Pi(S)$ *is graded.*

Proof: We show that $r(L(A)) = |L(A)|$ is a *rank function* for $\Pi(S)$. A rank function maps an element x from a poset to a unique level in such a way that the level corresponds to the length of any maximal chain from x to $\hat{0}$. Let $L(A) \in \Pi(S)$, with $|L(A)| = k$. Every point $x \in V(A)$ has all k points from A as its k nearest neighbors. Order those points with respect to their distance to x. As we assumed general position it is always possible to change the choice of x in such a way that this order is strict. By removing at each step the furthest point still available, we obtain a chain of length k that descends to $\hat{0}$. $\square$

Example 3.2 For $n = 3$ there is only one Voronoi poset,

$$\Pi_3(S) = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\},$$

while for $n = 4$ there are two essentially distinct posets:

$$\begin{aligned} \Pi_4(S_1) &= \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 24, 34, 123, 124, 134, 234, 1234\}, \\ \Pi_4(S_2) &= \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 124, 134, 234, 1234\}. \end{aligned}$$

These two posets correspond to the two configurations shown in Fig. 4. It can be easily verified that these two are the only two posets up to relabeling. Note that $\Pi_4(S_1)$ shows that the Voronoi poset is in general not a lattice. A lattice requires that every two elements of the poset have a unique minimal upper bound. In this example, the elements 2 and 3 have two minimal upper bounds, namely 123 and 234.

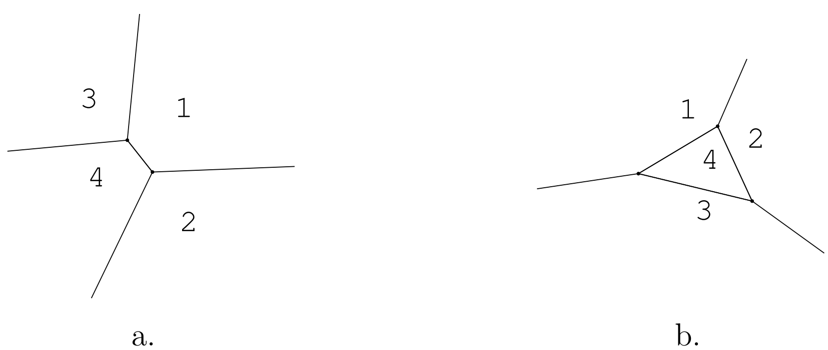

Figure 4: The two distinct first order Voronoi diagrams

3.2. The order complex of the Voronoi poset

The standard way to associate a topological space to a finite poset $(P, \leq)$ is by means of the *order complex* $\Delta(P)$ of the poset, see [4, 14]. The order complex is the simplicial complex of all nonempty chains of P. A *chain* of P of *length* k is a totally ordered subset $x_0 < x_1 < x_2 < \ldots < x_k$ of elements $x \in P$. The well-known *geometric realization* associates a topological space with a simplicial complex.

As a Voronoi poset on a set of n points $p_1, \ldots, p_n$ always has a unique maximal element $\{1, \ldots, n\}$, the geometric realization of the order complex is a cone and therefore contractible. This shows that the topological space that we have associated with S is homotopy equivalent with a point, and therefore not very interesting.

More promising is to consider the complement, that is the *anti Voronoi poset* $aP(S)$, consisting of those subsets of $\{1, \ldots, n\}$ that are not in the Voronoi poset. Another possibility is to consider the arrangement of bisectors.

4. Symmetry relations

Given a set S of sites, we count for every order k the number of vertices v_k, the number of edges e_k and the number of non empty Voronoi cells f_k. The *f-vector* of $\Pi(S)$ is the vector $\{f_1, f_2, \ldots, f_n\}$. The c- and e-vector are defined analogously. Note that the f-vector of $\Pi(S)$ may change if the position of the sites in S changes.

4.1. Symmetry in the number of cells

It turns out that a symmetry exists in the f-vectors.

Lemma 4.1 *Consider the f-vector of $\Pi(S)$, where $|S| = n$. Then $f_k + f_{n-k+1}$ is a constant independent of the position of the points in S. More precisely,*

$$f_k + f_{n-k+1} = 2k(n-k+1) + 1 - n. \tag{4}$$

Proof: We apply Theorem 2.7 to f_k and f_{n-k+1}:

$$\begin{aligned} f_k + f_{n-k+1} &= 2k-1)n - k^2 + 1 - \textstyle\sum_{i=1}^{k} f_{i-1}^{\infty} \\ &+ \quad (2(n-k+1)-1)n - (n-k+1)^2 + 1 - \textstyle\sum_{i=1}^{n-k+1} f_{i-1}^{\infty}, \\ &= 2kn - 2k^2 + 2k + 1 - n + n(n-1) - (\textstyle\sum_{i=1}^{k} f_{i-1}^{\infty} + \sum_{i=1}^{n-k+1} f_{i-1}^{\infty}). \end{aligned}$$

We join the two sums by applying Symmetry Equation 3 and evaluate the result by using Equation 2.

$$\sum_{i=1}^{k} f_{i-1}^{\infty} + \sum_{i=1}^{n-k+1} f_{i-1}^{\infty} = \sum_{i=1}^{n} f_{i-1}^{\infty} = n(n-1).$$

The lemma follows from combining the two equations above. □

4.2. Symmetry in the number of vertices

A similar equation holds for the number of vertices of a collection of Voronoi diagrams $V_k(S)$, for $k = 1, \ldots, n-1$.

Lemma 4.2 *Let S be a set of n points in general position. Let v_k denote the number of vertices in the k-th order Voronoi diagram. Then:*

$$v_k + v_{n-k} = 4k(n-k) - 2n. \tag{5}$$

Proof: Using Theorem 2.7 we write $v_k + v_{n-k}$ in terms of numbers of cells. Next we regroup and apply Symmetry Equation 3. After applying Theorem 2.7 we combine using symmetry again. Finally, using $\sum_{i=1}^{n} f_{i-1}^{\infty} = n(n-1)$ completes the proof.

$$\begin{aligned} v_k + v_{n-k} &= 2(f_k - 1) - f_k^{\infty} + 2(f_{n-k} - 1) - f_{n-k}^{\infty}, \\ &= 2(f_k + f_{n-k} - 2 - f_k^{\infty}), \\ &= 2(n^2 - 2n + 2kn - 2k^2 - (\textstyle\sum_{i=1}^{k} f_{i-1}^{\infty} + \sum_{i=1}^{n-k} f_{i-1}^{\infty} + f_k^{\infty})), \\ &= 2(n^2 - 2n + 2kn - 2k^2 - \textstyle\sum_{i=1}^{n} f_{i-1}^{\infty}), \\ &= -2n + 4kn - 4k^2 = 4k(n-k) - 2n. \end{aligned}$$

□

4.3. Symmetry in the number of Voronoi circles

Recall that the order of a Voronoi circle equals the number of points of S contained in its interior. The following theorem states that for n arbitrary points in general position, the number of circles c_i containing exactly i points on their inside plus the number of circles c_{n-i-3} containing exactly i points on their outside is constant. We prove this by applying above results and the *lifting transformation* ψ defined by

$$\begin{array}{rccl} \psi: & \mathbb{R}^2 & \to & \mathbb{R}^3, \\ & (x,y) & \mapsto & (x,y,x^2+y^2). \end{array}$$

This transformation changes the point-inside-circle relation in 2-dimensional space in a point-below-plane relation in 3-dimensional space, [6, 11].

Theorem 4.3 *Let S be a set of n points in general position. Let c_i denote the number of Voronoi circles containing exactly i points. Then*

$$c_i + c_{n-i-3} = 2(i+1)(n-2-i) = 2i(n-i-3) + 2(n-2). \tag{6}$$

Proof: We prove the theorem by induction.
[i=0]. We use the lifting transformation. As every circle defined by S in the plane contains only three points from S, every hyperplane defined by $\psi(S)$ contains only three points from $\psi(S)$ as well. The number c_0 of empty circles of S in the plane equals the number of facets of the lower hull of $\psi(S)$ in three dimensions. At the same time, the number c_{n-3} of circles that contain all other points of S equals the number of facets of the upper hull of $\psi(S)$. All images of points in S under ψ are part of the convex hull of $\psi(S)$. Since the convex hull of a point set of n points consists of $2n-4$ facets, if every facet is a triangle, see [6], Theorem 11.1, the claim follows.
[induction step]. We deduce the expression for $c_k + c_{n-k-3}$ by applying Equation 1, followed by combining Lemma 4.2 and the induction hypothesis:

$$\begin{aligned} c_k + c_{n-k-3} &= c_{k-1} + c_k + c_{n-k-3} + c_{n-k-2} - (c_{k-1} + c_{n-k-2}), \\ &= v_{k+1} + v_{n-(k+1)} - (c_{k-1} + c_{n-k-2}), \\ &= 2(2(k+1)-1)(n-(k+1)) - 2(k+1) \\ &\quad -(2(k-1+1)(n-2-(k-1))), \\ &= 2(k+1)(n-2-k). \end{aligned}$$

□

Remark 4.4 Computer calculations did not suggest any symmetry relation for the number of edges similar to Equalities 4 or 6.

4.4. Relations between cells and circles

Write $\tilde{f}_k := f_k + f_{n-k+1}$ and $\tilde{c}_i := c_i + c_{n-i-3}$.

Corollary 4.5 *Let $1 \leq i \leq \lceil \frac{n}{2} \rceil$. Then $\tilde{f}_i = \tilde{f}_1 + \tilde{c}_{i-2} = \tilde{c}_{i-2} + n + 1$.*

Proof: This follows directly from Lemma 4.1 and Theorem 4.3. □

Property 4.6 *Let f_i^∞ denote the number of unbounded cells in the i-th order diagram and let c_i denote the number of circles of order i:*

$$f_i^\infty + (c_{i-1} - c_{i-2}) = 2(n-i). \tag{7}$$

Proof: We prove the property by induction.
[i=1]. c_{-1} is zero by definition. The number of vertices v_1 in the first order Voronoi diagram equals the number of circles of order zero, c_0. The claim follows from applying Theorem 2.7:

$$f_1^\infty + (c_0 - c_{-1}) \;=\; f_1^\infty + v_1 \;=\; f_1^\infty + 2(f_1 - 1) - f_1^\infty \;=\; 2(n-1).$$

[induction step]. Assume we have proved that $f_i^\infty + (c_{i-1} - c_{i-2}) \;=\; 2(n-i)$. We rewrite this, using induction, as

$$c_{i-1} \;=\; 2ni - i(i+1) - \sum_{k=1}^{i+1} f_{k-1}. \tag{8}$$

Evaluate $c_i - c_{i-1}$:

$$c_i - c_{i-1} \;=\; (c_i + c_{i-1}) - 2c_{i-1} \;=\; v_{i+1} - 2c_{i-1} \;=\; 2(f_{i+1} - 1) - f_{i+1}^\infty - 2c_{i-1}.$$

Substituting this expression for $c_i - c_{i-1}$ and applying Theorem 2.7 and Equation 8 proves the claim:

$$f_{i+1}^\infty + (c_i - c_{i-1}) \;=\; 2(f_{i+1} - 1 - c_{i-1}) \;=\; 2(n - i - 1).$$ □

Corollary 4.7 *The c-vector totally determines the f-vector. The correspondence is given by $f_k \;=\; n - k + 1 + c_{k-2}$.*

Proof: Applying Equation 7 we get $\sum_{i=1}^{k} f_{i-1}^\infty = (k-1)(2n-k) - c_{k-2}$. The claim follows from evaluating Theorem 2.7 using the expression above. □

5. Even versus odd order cells

Given a grading on a set of objects, it is common to consider the *Poincaré polynomial* $P(t)$ of the grading. The i-th coefficient of this polynomial equals the number of objects of grade i. In our case, the objects are the elements of the Voronoi poset $\Pi(S)$, while the grading is given by the rank function on the poset. Recall that the rank of an element x in $\Pi(S)$ is just the order k of the Voronoi diagram in which x occurs as a cell. The i-th coefficient of the Poincaré polynomial $P(t)$ is given by f_i, as f_i gives the number of cells in the i-th order diagram $V_i(S)$. So, the Poincaré polynomial $P(t)$ of $\Pi(S)$ with respect to our rank function is given by

$$P(t) \;=\; f_0 + f_1 t + f_2 t^2 + \ldots + f_n t^n.$$

As an application of the symmetry relations we compare the number of cells in the even order Voronoi diagrams with the number of cells in the odd order diagrams. In terms of the Poincaré polynomial $P(t)$ of above, the following result can also be formulated as

$$P(-1) \;=\; 0.$$

Theorem 5.1 *Let S be a set of n points in general position with $n \geq 3$. Assume n is odd. In this case, the number of cells in the even order Voronoi diagrams equals the number of cells in the odd order Voronoi diagrams.*

Proof: Write $\tilde{f}_i = f_i + f_{n-i+1}$. We show that $A = 0$, where $A = -f_0 + f_1 - f_2 + \ldots - f_{n-1} + f_n$. So A is the number of cells in the odd order diagrams minus the number of cells in the even order diagrams:

$$A = -f_0 + \tilde{f}_1 + \frac{1}{2}\tilde{f}_{\frac{n+1}{2}} + t_n,$$

where

$$t_n := \sum_{i=2}^{\frac{n-1}{2}} (-1)^{i+1} \tilde{f}_i.$$

Clearly, $f_0 = 1$, as f_0 counts the empty set. $\tilde{f}_1$ is the number of points in S plus the number of cells in $V_n(S)$, so $\tilde{f}_1 = n + 1$. Applying Equation 4 gives:

$$\tilde{f}_{\frac{n+1}{2}} = -(-1)^{\frac{n+1}{2}} \frac{n^2+3}{4}.$$

Straightforward calculations show that:

$$t_n = (-1)^{\frac{n+1}{2}} \frac{n^2+3}{4} - n,$$

from which it follows that:

$$A = -1 + n + 1 - (-1)^{\frac{n+1}{2}} \frac{n^2+3}{4} + (-1)^{\frac{n+1}{2}} \frac{n^2+3}{4} - n = 0. \qquad \square$$

The claim of Theorem 5.1 does not hold when n is even. However, the following result does hold.

Lemma 5.2 *Let S be a set of n points in general position, with $n \geq 3$. Assume n is even. Let $A(S)$ denote the number of cells in the odd order Voronoi diagrams minus the number of cells in the even order diagram. Then:*

$$n \equiv 0(4) \Rightarrow A(S) \text{ odd.}$$
$$n \equiv 2(4) \Rightarrow A(S) \text{ even.}$$

Proof: Similar computations as in the proof of Theorem 5.1. $\square$

Note that as $v_k = c_{k-1} + c_{k-2}$ it follows immediately that:

$$\sum_{k=1}^{n-1} (-1)^{k+1} v_k = 0,$$

for all n, where v_k denotes the number of vertices in the k-th order Voronoi diagram.

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