Journal for Geometry and Graphics
Volume 8 (2004), No. 1, 17–22.

An Elementary Proof of "the Most Elementary Theorem" of Euclidean Geometry

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Abstract. We give a fairly elementary proof of the fact that if ABB' and $AC'C$ are triples of collinear points with the lines BC and $B'C'$ intersecting at D, then $d(AB) + d(BD) = d(AC') + d(C'D)$ if and only if $d(AB') + d(B'D) = d(AC) + d(CD)$, where $d(XY)$ denotes the length of the line segment joining X and Y. The "only if" part of this theorem is attributed to Urquhart, and referred to by Dan Pedoe as the most elementary theorem of Euclidean Geometry. We also give a simple proof of a variant of Urquhart's theorem that was discovered by Pedoe.

Key Words: Geometry of quadrangles, Urquhart's theorem

MSC 2000: 51M04

1. Introduction

"The most elementary theorem" referred to in the title states that *if ABB' and $AC'C$ are triples of collinear points with the line segments BC and $B'C'$ intersecting at D and if the distances obey $d(AB)+d(BD) = d(AC')+d(C'D)$, then $d(AB')+d(B'D) = d(AC)+d(CD)$* (see Fig. 1). The origin and some history of this theorem are discussed in [1], where the author attributes the theorem to the late L.M. Urquhart (1902–1966) who *"discovered it when considering some of the fundamental concepts of the theory of special relativity"*, and where he asserts that *"the proof by purely geometric methods is not elementary"*, giving variants and equivalent forms of the theorem and citing references where proofs can be found. In this note, we give an elementary proof that is also conceptual and fairly free of computations. The proof, however, does involve circles together with rather unconventional arguments, and as such it may not satisfy D. Pedoe's curiosity regarding the existence of a circle-free proof of Urquhart's Theorem.

ISSN 1433-8157/\$ 2.50

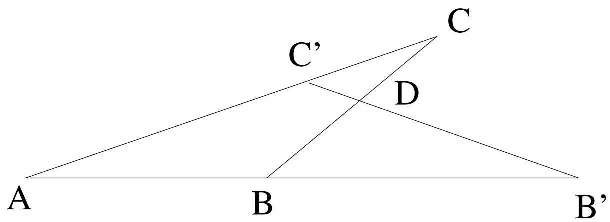

Figure 1: Illustration of “the most elementary theorem”

2. Some preparatory remarks

We fix in the Euclidean plane two reference *rays* Ax and Ay. If t is a positive number, and if X and Y are the points on Ax and Ay with $d(AX) = d(AY) = t$, then there is a unique circle Ω_t that touches Ax at X and Ay and Y. The shorter arc joining X and Y along Ω_t is denoted by Γ_t or by $\widehat{XY}$, and it is called *the arc* XY (see Fig. 2). The term *arc* will always stand for an arc of this type. Similarly, the term *segment* will always stand for a line segment that joins a point on Ax with a point on Ay, this time not requiring that these points are equidistant from A.

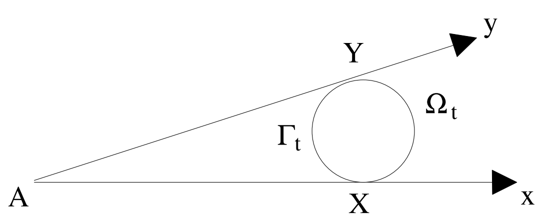

Figure 2: The arc Γ_t and circle Ω_t

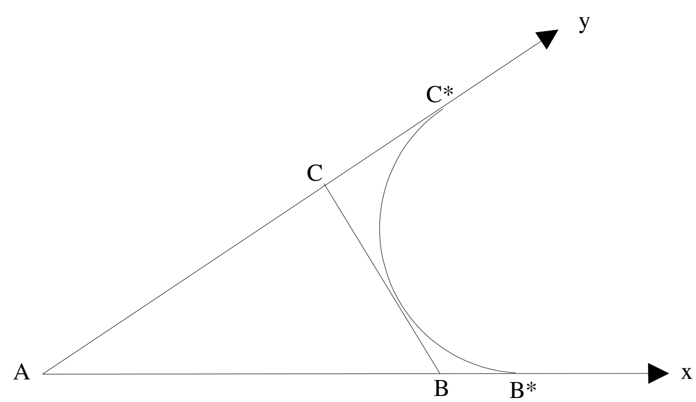

Figure 3: Ex-arc of BC

If BC is a segment, and if the A-excircle of ABC touches Ax at B^* and Ay at C^*, then the arc $\widehat{B^*C^*}$ is called the *ex-arc* of BC (see Fig. 3). It is useful to observe that $d(AB^*) = d(AC^*) = p(ABC)/2$, where $p(ABC)$ denotes the perimeter of ABC.

It is intuitively obvious that different arcs do not interset each other. To see this, let α be the angle between Ax and the angle bisector Az of xAy, let Aw be a ray making an angle $\beta \leq \alpha$ with Az and meeting the arc $\Gamma_t = \widehat{XY}$ at W, as shown in Fig. 4, and let $\rho = \cos\beta/\cos\alpha$. Let O be the center of Ω_t, r be its radius, and s be the length of the perpendicular OP to Aw. Then

$$\begin{aligned}
d(AW) &= d(AP) - d(PW) = d(AO)\cos\beta - \sqrt{r^2 - s^2} \\
&= d(AO)\cos\beta - \sqrt{(d(AY)\tan\alpha)^2 - (d(AO)\sin\beta)^2} \\
&= t\sec\alpha\cos\beta - \sqrt{t^2\tan^2\alpha - t^2\sec^2\alpha\sin^2\beta} \\
&= t\left(\rho - \sqrt{\sec^2\alpha - 1 - \sec^2\alpha\sin^2\beta}\right) \\
&= t\left(\rho - \sqrt{\rho^2 - 1}\right),
\end{aligned}$$

showing that $d(AW)$ increases with t and that different arcs do not intersect.

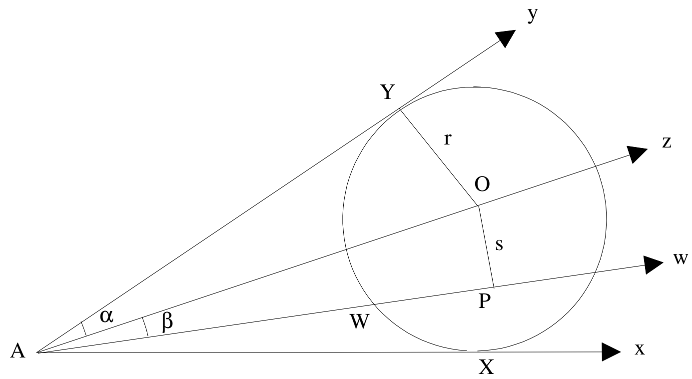

Figure 4: Different arcs do not intersect

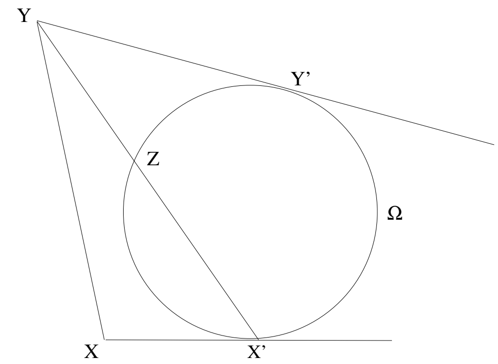

Figure 5: Proving eq. (1)

The *triangle determined by an arc* is the curvilinear triangular region enclosed between the reference rays Ax, Ay and that arc. Similarly, the *triangle determined by a segment* is the ordinary triangular region enclosed in the reference rays and that segment. A segment (or arc) is said to *lie above* another segment (or arc) if the triangle determined by the first is contained in the one determined by the second. We reiterate that due to the fact that different arcs do not intersect, this *lying-above* partial order on the set of segments and arcs restricts to a total order on the set of arcs.

We shall also make use of the again intuitively obvious fact that if a line XY does not intersect a circle Ω and if x and y are the lengths of the tangent lines from X and Y to Ω, then

$$d(YX) + x > y. \tag{1}$$

To see this, note that Y cannot lie in the region enclosed by the tangent lines from X to Ω. Therefore, the point Y and Ω lie on the same side of one of the tangents XX' to Ω (see Fig. 5). Similarly, the point X and Ω lie on the same side of one of the tangents YY' to Ω. Since Y, X, X' are not collinear, YX' is not tangent to Ω and therefore crosses it at a point Z that lies between Y and X'. From $d(YY')^2 = d(YZ)\,d(YX')$, it follows that $d(YX') > d(YY')$ and hence $d(YX) + d(XX') > d(YX') > d(YY')$, as desired. It is worth mentioning that (1) still holds under the weaker assumption that Y does not lie in the one quarter Q determined by the two tangent lines from X to Ω and not bordering on Ω.

3. The proof

We are now ready to prove a stronger version of Urquhart's Theorem, together with its converse. Note that Urquhart's Theorem states that in Theorem 1 the second equality of (i) implies the third equality. It is curious why its converse is not mentioned at all in [1].

Theorem 1. *Let BC and $B'C'$ be two segments that intersect at D.*

(i) *If the ex-arcs of BC and $B'C'$ coincide, then*

$$p(ABC) = p(AB'C'), \quad p(ABD) = p(AC'D), \quad p(AB'D) = p(ACD). \tag{2}$$

(ii) *If the ex-arc of BC lies strictly above the ex-arc of $B'C'$, then*

$$p(ABC) < p(AB'C'), \quad p(ABD) < p(AC'D), \quad p(AB'D) < p(ACD). \tag{3}$$

(iii) *If any one of the three equalities in (2) holds, then the remaining two hold also, and the ex-arcs of BC and $B'C'$ coincide.*

Proof. Suppose that the ex-arc of BC lies above (or on) the ex-arc of $B'C'$. Then the segment BC lies above the ex-arc of $B'C'$. We draw the ex-arc $\widehat{UV}$ of $B'C'$, and we let BC^* be the segment tangent to $\widehat{UV}$ (see Fig. 6). We let D^* be the point where BC^* intersects $B'C'$, and we let X and X' be the points where BC^* and $B'C'$ touch $\widehat{UV}$.

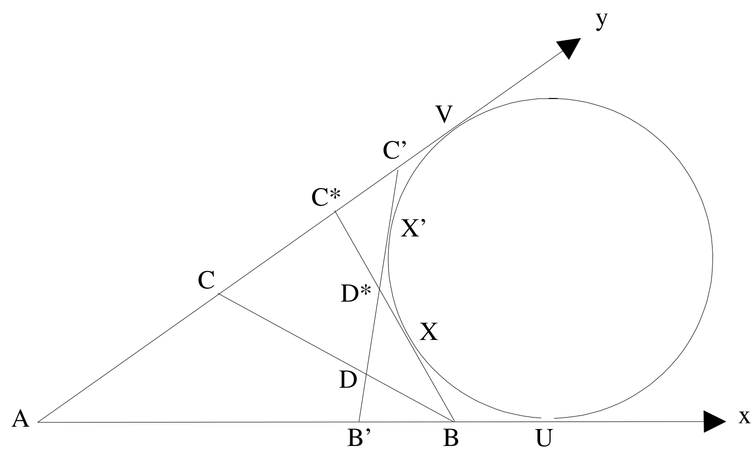

Figure 6: Illustrating the proof of Theorem 1

Noting that

$$p(ABD) + p(ACD) = p(ABC) + 2\,d(AD), \tag{4}$$

we see that any two of the equalities in (2) imply the third. It is trivial that $p(ABC^*) = p(AB'C')$, each being equal to $2\,d(AU)$. Also,

$$\begin{aligned}
p(ABD^*) - p(AC'D^*) &= (d(AB) + d(BD^*)) - (d(AC') + d(C'D^*)) \\
&= (d(AB) + d(BX) + d(XD^*)) - (d(AC') + d(C'X') + d(X'D^*)) \\
&= (d(AB) + d(BU) + d(XD^*)) - (d(AC') + d(C'V) + d(X'D^*)) \\
&= (d(AU) - d(AV)) + (d(XD^*) - d(X'D^*)) = 0,
\end{aligned}$$

and therefore $p(ABD^*) = p(AC'D^*)$. Hence

$$p(ABC^*) = p(AC'B'), \quad p(ABD^*) = p(AC'D^*), \quad p(AB'D^*) = p(AC^*D^*).$$

Thus if BC coincides with BC^*, then (2) holds. This proves (i). Otherwise, we have

$$\begin{aligned}
p(AB'C') &= p(ABC^*) > p(ABC), \\
p(AC'D) &> p(AC'D^*) = p(ABD^*) > p(ABD), \\
p(ACD) - p(AB'D) &= (d(AC) + d(CD)) - (d(AB') + d(B'D)) \\
&= (d(AC) + d(CD) + d(DD^*) + d(D^*X')) \\
&\quad - (d(AB') + d(B'D) + d(DD^*) + d(D^*X')) \\
&= d(AC) + (d(CD) + d(DX')) - (d(AB') + d(B'X')) \\
&\overset{(1)}{>} d(AC) + d(CV) - (d(AB') + d(B'U)) \\
&= 0.
\end{aligned}$$

This proves (ii).

In order to prove (iii), note that if any of the equalities of (2) holds, then (ii) would imply that the ex-arcs of BC and $B'C'$ coincide. The rest follows from (i). □

We end this note by rephrasing and giving a simple proof of an equivalent of URQUHART's Theorem that was obtained by D. PEDOE by *applying the method of reciprocal polars, invoking the contentious shades of Poncelet, Gergonne, Plücker and even Möbius. Without venturing to call this "the most elementary theorem of circle geometry"*, D. PEDOE asserts that *it is clear that this is not a trivial theorem.* The theorem states that

if $BACD$ is a parallelogram and if a circle touches the sides AB and AC and intersects BC in the points E and F, then there exists a circle which touches (the extensions of) DB and DC and passes through E and F.

We make the observation that since ABC and DCB are glide reflections of each other, statements about DCB can be *glide-reflected* into appropriate statements about ABC, rendering DCB redundant. Thus the theorem can be restated as follows.

Theorem 2. *A variant of* PEDOE*'s equivalent of* URQUHART*'s Theorem: If ABC is a triangle and if a circle touches the sides AB and AC and intersects BC in the points E and F, then there exists a circle which touches the extensions of AB and AC and passes through the reflections E^* and F^* of E and F about the midpoint of BC.*

Proof. We place ABC in our reference frame xAy so that BC is a segment, and our circle is Ω_t for some t. Letting $x = x(t) = d(BE)$ and $y = y(t) = d(CF)$, we see that (x, y) is the necessarily unique solution of the system

$$x(a-y) = (c-t)^2, \quad y(a-x) = (b-t)^2.$$

This system is clearly invariant under the simultaneous substitutions $(x \leftrightarrow y)$ and $(t \to b+c-t)$. Therefore, letting $s = b+c-t$, we see that the circle Ω_s has the desired property.

It is worth mentioning that the domain of definition of $x(t)$ and $y(t)$ is the closed interval $J = [(-a+b+c)/2,\ (a+b+c)/2]$, where the endpoints are the values of t which correspond to the incircle and excircle. □

Finally, I find it rather curious that not much, if any, research was generated by the appearance of [1], in spite of its intriguing title, its interesting content and the provocatively simple questions it raises, and in spite of being written by the author of such an attractive book as [2].

Acknowledgments

This work is supported by a research grant from Yarmouk University. The author would like to express his thanks to Yarmouk University for that grant, to his son Ahmad HAJJA for drawing the figures, and to the referees and editors for their valuable comments.

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Received March 28, 2004; final form June 1, 2004