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The Inspherical Gergonne Center of a Tetrahedron

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Abstract. With the traditional definition of the Gergonne center of a triangle in mind, it is natural to consider, for a given tetrahedron, the intersection of the cevians that join the vertices to the points where the insphere touches the faces. However, these cevians are concurrent for the limited class of what we call *inspherical tetrahedra*. Another approach is to note that the cevians through any point inside a triangle divide the sides into 6 segments, and that the Gergonne center is characterized by the requirement that every two segments sharing a vertex are equal. Similarly, the cevians through any point inside a tetrahedron divide the faces into 12 subtriangles, and one may define the Gergonne center as the point for which every two subtriangles that share an edge are *equal in area*. This was done by the authors in [8], where the existence and uniquenes of such a point is established. It turns out that tetrahedra whose Gergonne center has the stronger property that every two subtriangles that share an edge are *congruent* (resp. *skew-congruent*) are precisely the inspherical (resp. equifacial) ones. This is proved in Section 4. In Sections 2 and 3, we give a characterization of the inspherical tetrahedra and we outline a method for constructing them.

Key Words: Barycentric coordinates, Ceva's theorem, Gergonne point, Fermat-Torricelli point, isosceles or equifacial tetrahedron, isogonal tetrahedron.

MSC 2000: 51M04, 51M20, 52B10

1. Introduction and terminology

The Gergonne center of a triangle ABC is defined to be the point of intersection of the cevians $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$, where A', B', and C' are the points where the incircle touches the sides

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$\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, as in Fig. 1. Such cevians are concurrent by Ceva's Theorem since A', B', C' can be equivalently defined by the algebraic relations

$$AB' = AC', \quad BC' = BA', \quad CA' = CB',$$

where XY denotes the length of the line segment $\overline{XY}$. It is then quite natural to consider the point of intersection (if any) of the cevians $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, and $\overline{DD'}$ of a tetrahedron $ABCD$, where A', B', C', and D' are the points where the insphere touches the faces. We shall see that such cevians are concurrent for a limited class of tetrahedra and we shall describe such a class. We call the resulting center (when it exists) the *inspherical Gergonne center*, or simply the *inspherical center* to distinguish it from the *Gergonne center* defined in [8], and we shall call a tetrahedron with an inspherical center an *inspherical tetrahedron*. The relation between these two centers is discussed in Section 4.

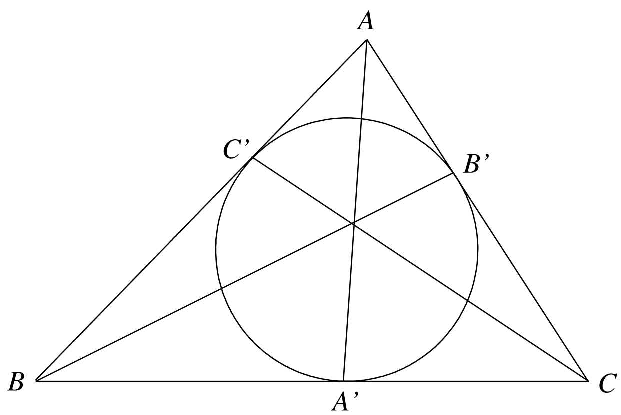

Figure 1:

Throughout the rest of this article, the term *tetrahedron* stands for a *non-degenerate tetrahedron*. An *equifacial* or *isosceless* tetrahedron is a tetrahedron whose faces are congruent, or equivalently have the same area; see [10, Chapter 9, pages 90–97], [12, Theorem 4.4, page 156], [2, Cor. 307, page 108] or [7]. If $A_1A_2A_3A_4$ is a tetrahedron, then by its j-th face we shall mean the face opposite to the j-th vertex A_j. Two edges of a tetrahedron will be called *opposite* if they have no vertex in common. Two triangles ABC and $A'B'C'$ are called *congruent* if there is an isometry that carries A, B, and C to A', B', and C' in the *same order*. Finally, we call two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ *equivalent*, and we write $(x_1, \dots, x_n) \equiv (y_1, \dots, y_n)$ if one of them is a positive multiple of the other. Thus two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are barycentric coordinates of the same point if and only if they are equivalent.

2. A characterization of inspherical tetrahedra

In Theorem 1 below, we prove that a tetrahedron is inspherical if and only if its insphere touches the faces at their Fermat-Torricelli points. However, to see how large the class of such tetrahedra is and to describe how they can be constructed, a stronger and more technical statement is needed and is provided by Theorem 2.

For ease of reference and to be self-contained, we start with two fairly well-known lemmas. The first describes the Fermat-Torricelli point of a triangle and the second is a three-dimensional version of Ceva's Theorem. Both will be used freely.

Lemma 1. *Let ABC be a non-degenerate triangle. Then there exists a unique point F whose distances from the vertices have a minimum sum. If the angles of ABC are less than $120°$ each, then F is interior and is characterized by the equi-angular property*

$$\angle AFB = \angle BFC = \angle CFA = 120°.$$

Otherwise, F is the vertex that holds the largest angle.

In all cases, F is called the Fermat-Torricelli point of ABC.

Proof. See [6], or [1] and [13] for higher-dimensional versions. □

Lemma 2. *Let $T = A_1A_2A_3A_4$ be a tetrahedron and let D_j, $1 \leq j \leq 4$, be a point on its j-th face. Then the cevians $\overline{A_jD_j}$, $1 \leq j \leq 4$, are concurrent if and only if there exist M_1, M_2, M_3 and M_4 such that for all arrangements (i,j,k,t) of $\{1,2,3,4\}$, (M_i, M_j, M_k) are barycentric coordinates of D_t relative to $A_iA_jA_k$. In this case, (M_1, M_2, M_3, M_4) are barycentric coordinates of the point of concurrence relative to $A_1A_2A_3A_4$.*

Proof. See [14]. □

Theorem 1. *Let $A_1A_2A_3A_4$ be a tetrahedron and let D_j, $1 \leq j \leq 4$, be the point where the insphere touches the j-th face. Then the cevians $\overline{A_jD_j}$, $1 \leq j \leq 4$, are concurrent if and only if D_j, $1 \leq j \leq 4$, is the Fermat-Torricelli point of the j-th face.*

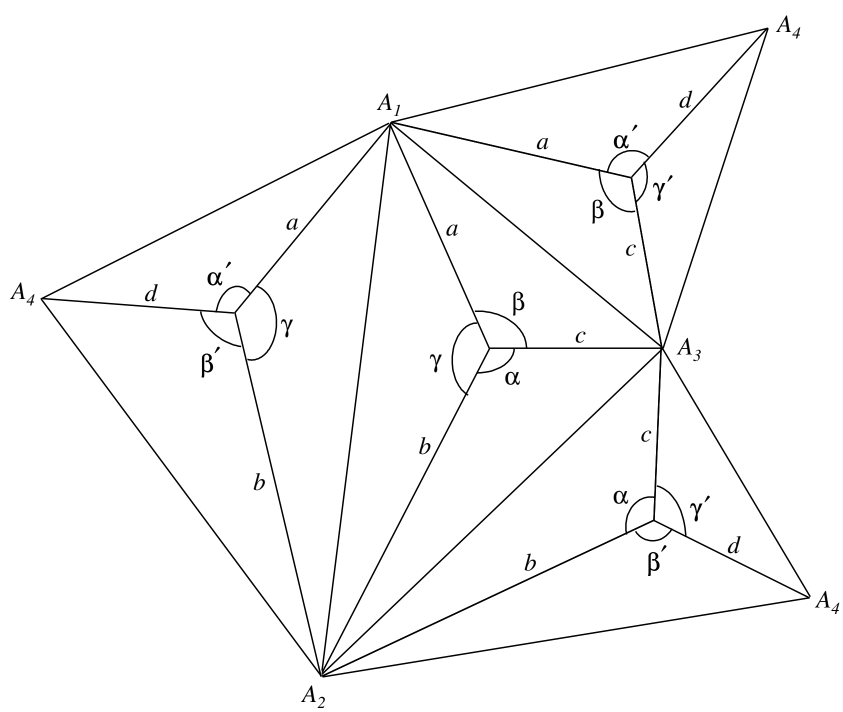

Figure 2:

Proof. Following the argument in [10, page 93], we first note that the line segments $\overline{A_1D_2}$, $\overline{A_1D_3}$, and $\overline{A_1D_4}$ have equal lengths since they are tangent lines to the same sphere from the same point A_1. We denote their common length by a, and we define b, c, d similarly. This is shown in Fig. 3 where our tetrahedron is flattened. It follows that the triangles $A_1A_2D_3$

and $A_1A_2D_4$ are congruent and hence $\angle A_1D_3A_2 = \angle A_1D_4A_2$. We denote the measure of this angle by γ, and we define α, β, α', β', γ' similarly, as in Fig. 2. The equations

$$\alpha + \beta + \gamma = \alpha + \beta' + \gamma' = \alpha' + \beta + \gamma' = \alpha' + \beta' + \gamma = 360^\circ \tag{1}$$

lead immediately to the conclusion

$$\alpha = \alpha', \ \beta = \beta', \ \gamma = \gamma'. \tag{2}$$

Now assume that the cevians $\overline{A_jD_j}$, $1 \leq j \leq 4$, are concurrent. If (M_1, M_2, M_3, M_4) are barycentric coordinates of their point of intersection relative to $A_1A_2A_3A_4$, then by Lemma 2, (M_1, M_2, M_3) are barycentric coordinates of D_4 relative to $A_1A_2A_3$. Thus

$$\begin{aligned} (bc\sin\alpha, ca\sin\beta, ab\sin\gamma) &\equiv (M_1, M_2, M_3) \\ (bd\sin\beta, da\sin\alpha, ab\sin\gamma) &\equiv (M_1, M_2, M_4). \end{aligned}$$

Therefore

$$\frac{bc\sin\alpha}{ca\sin\beta} = \frac{bd\sin\beta}{ad\sin\alpha} \quad \left(= \frac{M_1}{M_2}\right)$$

and hence $\alpha = \beta$. Similarly, $\alpha = \gamma$ and therefore

$$\alpha = \beta = \gamma = 120^\circ.$$

Hence D_1 is the Fermat-Torricelli point of the face $A_1A_2A_3$. Similarly for the other faces. Thus if our cevians are concurrent, then the D_j's are the Fermat-Torricelli points of the faces.

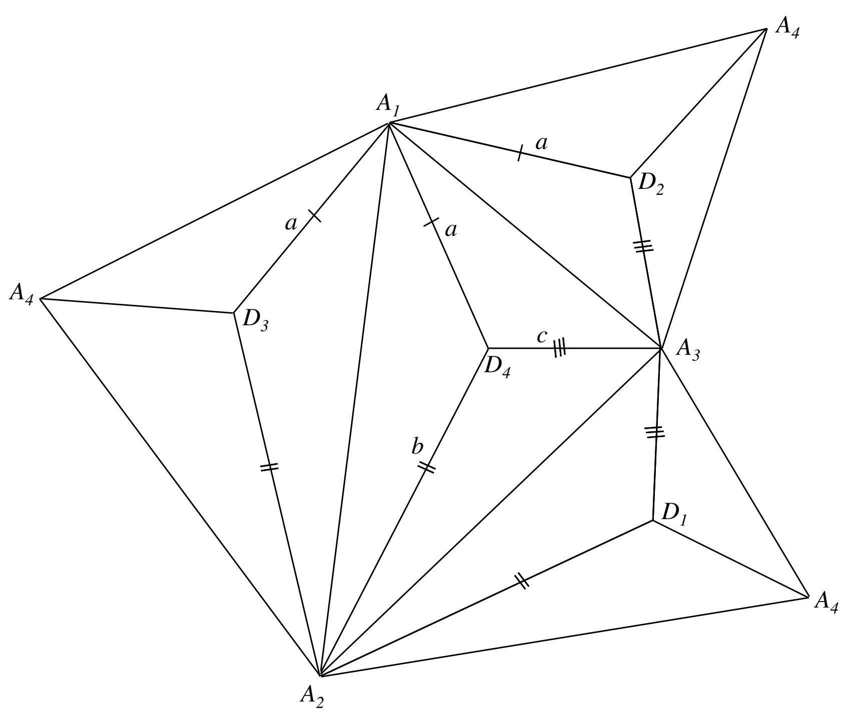

Figure 3:

Conversely, suppose that the D_j's are the Fermat-Torricelli points of the faces. It is clear that the barycentric coordinates of D_4 with respect to $A_1A_2A_3$ are

$$(bc\sin 120^\circ, ca\sin 120^\circ, ab\sin 120^\circ) \equiv \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

Similar statements hold for the other D_j's. Lemma 2 now implies that the cevians $\overline{A_jD_j}$ are concurrent (and that they concur at the point whose barycentric coordinates relative to $A_1A_2A_3A_4$ are $(1/a, 1/b, 1/c, 1/d)$). □

3. Construction of general inspherical and non-inspherical tetrahedra

Theorem 1 describes how to construct a non-inspherical tetrahedron. Start with any triangle ABC and any point P inside ABC that is different from the Fermat-Torricelli point of ABC. Place a small sphere S so that it touches the plane of ABC at P and draw three planes through the sides of ABC that are tangent to S. If D is the point where these three planes intersect, then $ABCD$ would be a non-inspherical tetrahedron. To ensure that these planes intersect within the halfspace of the plane of ABC which contains S, we take the radius of S sufficiently small. For example, it is sufficient (but by no means necessary) to take the radius of S to be less than each of the distances of P from the sides of ABC.

On the other hand, to describe how to construct inspherical tetrahedra, we need the following theorem, which will also be needed in proving Theorem 4.

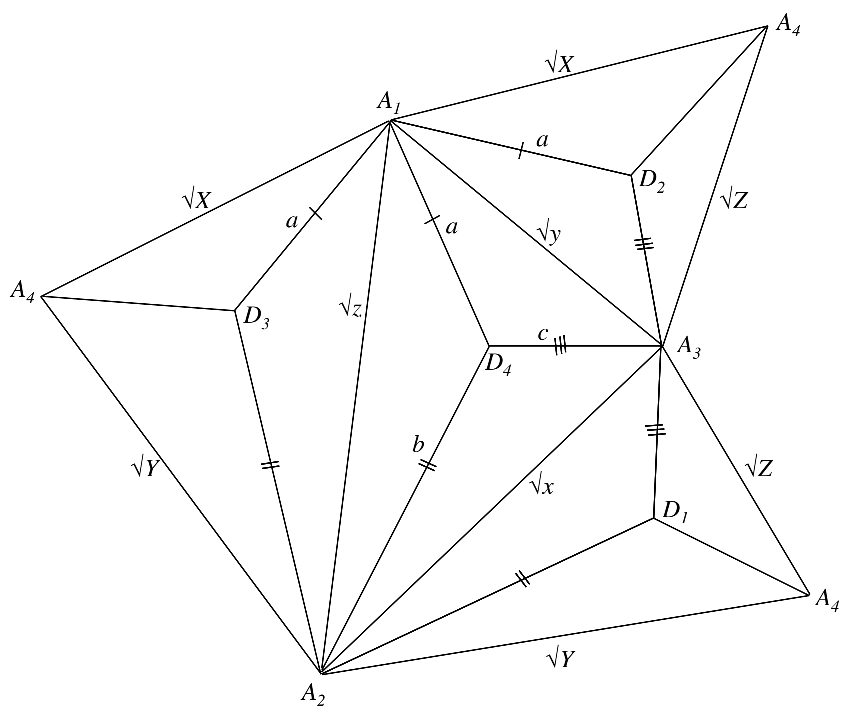

Figure 4:

Theorem 2. *Let $A_1A_2A_3A_4$ be a tetrahedron whose faces have interior Fermat-Torricelli points D_1, D_2, D_3 and D_4. If $A_iD_j = A_iD_k = A_iD_t$ for all arrangements (i, j, k, t) of $\{1, 2, 3, 4\}$, then our tetrahedron is inspherical.*

Proof. Let a denote the common length of the line segments $\overline{A_1D_2}$, $\overline{A_1D_3}$, and $\overline{A_1D_4}$, and define b, c, d similarly, as shown in Fig. 4. Let I be the incenter and r the inradius of our tetrahedron, and let I_j be the orthogonal projection of I on the j-th face. To prove that $I_j = D_j$ for all j, it is sufficient by symmetry to prove that $I_4 = D_4$. Equivalently, it is sufficient to prove that the distances from I_4 to A_1, A_2, and A_3 are a, b, and c, respectively. Again, it is sufficient by symmetry to prove that $I_4A_1 = a$, or equivalently

$$\alpha^2 - r^2 = a^2, \tag{3}$$

where $\alpha = IA_1$. Without loss of generality, we may assume that I is the origin.

To accomplish this, we note that all the elements of our tetrahedron can be computed in terms of a, b, c, and d. We find r (in terms of a, b, c, d) by first finding the lengths of the edges (using the Law of Cosines), then the areas F_j of the faces and the volume V of the tetrahedron, and then using the formula $V = (r/3)(F_1+F_2+F_3+F_4)$. Then we find $\alpha = \|A_1\|$ by solving a system of 10 linear equations in the unknowns $A_i \cdot A_j$, $i \le j$. We then check that (3) is indeed satisfied.

We find it more convenient to denote the lengths of the edges by $\sqrt{x}, \sqrt{X}, \ldots$ as shown in Fig. 4. Thus we have

$$\begin{aligned} x = b^2 + c^2 + bc, \quad y = c^2 + a^2 + ca, \quad z = a^2 + b^2 + ab, \\ X = a^2 + d^2 + ad, \quad Y = b^2 + d^2 + bd, \quad Z = c^2 + d^2 + cd. \end{aligned} \tag{4}$$

The area F_j of the j-th face is given by $F_j = (\sqrt{3}/4)f_j$, where

$$f_1 = bc + cd + db, \quad f_2 = ac + cd + da, \quad f_3 = ab + bd + da, \quad f_4 = ab + bc + ca. \tag{5}$$

The volume V of the tetrahedron is given by

$$\begin{aligned} 144V^2 &= (xX + yY + zZ)(x + X + y + Y + z + Z) - (xyz + xYZ + XyZ + XYz) \\ &= -2(x^2X + X^2x + y^2Y + Y^2y + z^2Z + Z^2z). \end{aligned} \tag{6}$$

Using (4), this simplifies into

$$16V^2 = abcdS, \tag{7}$$

where

$$S = ab + bc + ca + ad + bd + cd = \frac{f_1 + f_2 + f_3 + f_4}{2}. \tag{8}$$

Since

$$V = \frac{r}{3}(F_1 + F_2 + F_3 + F_4) = \frac{r}{3}\frac{\sqrt{3}}{4}(f_1 + f_2 + f_3 + f_4) = \frac{r}{3}\frac{\sqrt{3}}{4}(2S) = \frac{r\sqrt{3}}{6}S,$$

it follows that

$$16V^2 = \frac{4r^2S^2}{3}. \tag{9}$$

From (7) and (9), it follows that

$$r^2 = \frac{3abcd}{4S}. \tag{10}$$

It remains to compute the distances from the incenter I to the vertices. The barycentric coordinates of I with respect to $A_1A_2A_3A_4$ are (f_1, f_2, f_3, f_4). Therefore

$$I = 0 = f_1A_1 + f_2A_2 + f_3A_3 + f_4A_4.$$

Taking the scalar product with each A_j, and letting $A_{ij} = A_i \cdot A_j$, we obtain the first four linear equations represented by (11) below. Next, every edge gives rise to an equation in the manner that

$$z = \|A_2A_3\|^2 = (A_2 - A_3) \cdot (A_2 - A_3) = A_{22} + A_{33} - 2A_{23}.$$

The six equations thus obtained are the last 6 equations of the system (11).

The system

$$\begin{bmatrix} f_1 & 0 & 0 & 0 & f_2 & f_3 & f_4 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 & f_1 & 0 & 0 & f_3 & f_4 & 0 \\ 0 & 0 & f_3 & 0 & 0 & f_1 & 0 & f_2 & 0 & f_4 \\ 0 & 0 & 0 & f_4 & 0 & 0 & f_1 & 0 & f_2 & f_3 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{44} \\ A_{12} \\ A_{13} \\ A_{14} \\ A_{23} \\ A_{24} \\ A_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x \\ y \\ z \\ X \\ Y \\ Z \end{bmatrix} \tag{11}$$

has a unique solution. Using (4) and (5), we obtain the values of the A_{ij} in terms of a, b, c, and d. In particular, we obtain

$$\alpha^2 = A_{11} = a^2 + \frac{3abcd}{4S}.$$

This together with (10) implies that $\alpha^2 - r^2 = a^2$, as desired in (3), and completes the proof. $\square$

Theorem 2 above describes completely how to construct a general inspherical tetrahedron. We draw three vectors $\overrightarrow{OA_1}$, $\overrightarrow{OA_2}$, and $\overrightarrow{OA_3}$ that make an angle of 120° with each other and that have arbitrary lengths a, b, and c (see Fig. 5). In other words, we take a triangle $A_1A_2A_3$ with Fermat-Torricelli point O. By valley-folding along the edges of $A_1A_2A_3$, we create three more triangles $B_1A_2A_3$, $A_1B_2A_3$, and $A_1A_2B_3$ that are identical with $A_1A_2A_3$. Let O_1, O_2, and O_3 be the Fermat-Torricelli points of these triangles, respectively. For each j, take a point C_j on the line segments O_jB_j in such a way that O_1C_1, O_2C_2, and O_3C_3 have the same length, d say. Now, we cut along the edges A_1C_2, C_2A_3, A_3C_1, C_1A_2, A_2C_3, and C_3A_1 and we valley-fold along the sides of $A_1A_2A_3$ until the points C_1, C_2, and C_3 coincide and occupy the same position, A_4 say. The resulting tetrahedron $A_1A_2A_3A_4$ satisfies the conditions of Theorem 2, and thus is inspherical.

It is quite legitimate to question the claim, implicitly made above, that *totally arbitrary* positive numbers a, b, c, d do give rise to a feasible tetrahedron. After all, a very similar situation was met when constructing an equifacial tetrahedron. Trying to get the vertices of a given triangle to coincide and occupy the same position by valley-folding along the three segments that join, two by two, the mid-points of the sides, one soon discovers that this is possible only if the given triangle is acute-angled. Theorem 3 below assures us that the construction described in the previous paragraph is valid for all choices of positive a, b, c, d. Its proof makes use of the following lemma.

Lemma 3. *Let x, y, z, X, Y, Z be given positive numbers. Then there exists a tetrahedron such that the lengths of the sides of one of its faces are $\sqrt{x}, \sqrt{y}, \sqrt{z}$ and the lengths of the opposite sides are $\sqrt{X}, \sqrt{Y}, \sqrt{Z}$, respectively, if and only if the right hand side of (6) is positive and all the relevant triangle inequalities pertaining to the six faces are satisfied.*

Proof. See [9] or [3]. □

Theorem 3. *Let a_1, a_2, a_3, a_4 be arbitrary positive numbers. Then there exists a tetrahedron $A_1A_2A_3A_4$ whose faces have interior Fermat-Torricelli points D_1, D_2, D_3, D_4, such that*

$$A_iD_j = A_iD_k = A_iD_t = a_i \text{ for all arrangements } (i, j, k, t) \text{ of } \{1, 2, 3, 4\}.$$

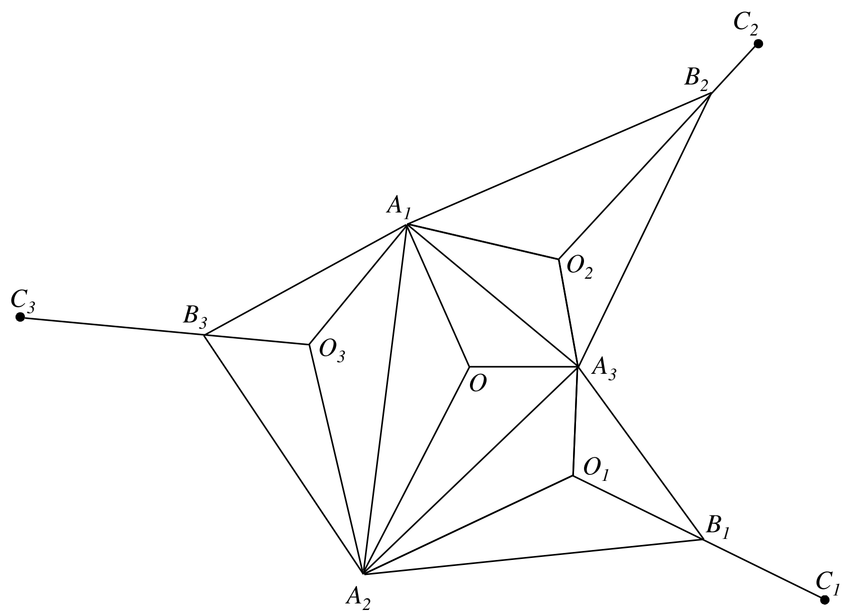

Figure 5:

Proof. Let a_1, a_2, a_3, a_4 be renamed as a, b, c, d and let $x, X, \ldots$ be defined as in (4). In view of the construction exhibited in Fig. 5, and in view of Lemma 3, we need only show that the right hand side of (6) is positive. This holds since the right hand side of (6) simplifies into the positive quantity $abcdS$ given in (7). □

4. Relation to other Gergonne centers

We end this note by discussing possible alternative definitions of the Gergonne center. Observe that any cevians $\overline{A_1D_1}$, $\overline{A_2D_2}$, and $\overline{A_3D_3}$ of a triangle $A_1A_2A_3$ divide the sides into six segments $\overline{A_iD_j}$, $i \neq j$, and that these cevians meet at the Gergonne center if and only if any two of these segments that share a vertex are equal, or equivalently

$$A_iD_j = A_iD_k \text{ for every arrangement } (i, j, k) \text{ of } \{1, 2, 3\}.$$

Similarly, any cevians $\overline{A_1D_1}$, $\overline{A_2D_2}$, $\overline{A_3D_3}$, and $\overline{A_4D_4}$ of a tetrahedron $A_1A_2A_3A_4$ divide the faces into 12 triangles $A_iA_jD_k$, i, j, k are pairwise distinct. In [8], it was proved that there exist unique concurrent cevians such that the twelve triangles into which they divide the faces have the property that any two which share an edge have equal area, or equivalently

$$A_iA_jD_s \text{ and } A_iA_jD_t \text{ have the same area for every arrangement } (i,j,s,t) \text{ of } \{1,2,3,4\}. \tag{12}$$

The above discussion shows that the inspherical center is obtained by replacing (12) with the stronger requirement that

$$A_iA_jD_s \text{ and } A_iA_jD_t \text{ are congruent for every arrangement } (i,j,s,t) \text{ of } \{1,2,3,4\}. \tag{13}$$

This shows in particular that if the inspherical center exists, then it is the Gergonne center itself. Thus in a sense, the two approaches to defining the Gergonne center of a tetrahedron led to the same center. Also, any reference to the insphere is made obsolete in view of (13).

It is natural to also investigate the result of replacing (13) by the similar condition

$$A_iA_jD_s \text{ and } A_jA_iD_t \text{ are congruent for every arrangement } (i,j,s,t) \text{ of } \{1,2,3,4\} \tag{14}$$

and to see whether this skew-congruence condition, too, is equivalent to some natural geometric requirement. The next theorem provides the answer.

Theorem 4. *Let $T = A_1A_2A_3A_4$ be a tetrahedron and let $\overline{A_jD_j}$, $1 \le j \le 4$, be the unique concurrent cevians (guaranteed in [8]) that satisfy (12). Then (13) holds if and only if T is inspherical (in which case the D_j's are the Fermat-Torricelli points of the faces), and (14) holds if and only if T is equifacial (in which case the D_j's are the centroids of the faces.)*

Proof. If (13) holds, then the first paragraph in the proof of Theorem 1 shows that the D_j's are the Fermat-Torricelli points of the faces. By Theorem 2, T is inspherical. Conversely, if T is inspherical, then the Fermat-Torricelli points satisfy (13) and hence (12). By their uniqueness, the D_j's are the Fermat-Torricelli points and hence they satisfy (13).

If (14) holds, then letting a, b, c and $\alpha, \alpha', \dots$ be as shown in Fig. 2, we see that (1) holds and leads to the conclusion (2). Thus the faces of T are congruent and T is equifacial. Conversely, if T is equifacial, then the centroids of the faces satisfy (12). By their uniqueness, the D_j's are the centroids and hence they satisfy (14). □

Epilogue and acknowledgments

After writing this article, it came to our attention that the problem of characterizing inspherical tetrahedra was considered by Edwin Koźniewski and Renata A. Górska in [11], where an alternative characterization is achieved. We are grateful to E. Koźniewski for sending us a copy of [11] and for drawing our attention to the fact that our Theorem 1 is already known to Aleksey Zaslavsky as announced in [15]. We are also grateful to Zaslavsky for writing and telling us that the material in [15] will appear (in Russian) in the March 2004 issue of *Mathematicheskoe Prosveshenije* and that the contents are contributions by scholars D. Kosov (Moscow), M. Isaev (Barnaul) and V. Filimonov (Ekaterinburg). Later, we found out that Theorem 1 is more than one century old, as implied by Footnote 1 of [4, page 373], where the term *isogonal* is used to mean inspherical. The same footnote states that the three angles subtended by the sides of a face of a tetrahedron at the point where the insphere touches that face are the same for all faces. Thus if the insphere of a tetrahedron touches a face ABC at X, then the measures of the three angles $\angle AXB$, $\angle BXC$, and $\angle CXA$

are independent of that particular face. This beautiful, yet obscure, theorem is also stated in [5, 2nd paragraph, page 174], and it clearly implies that the condition, in Theorem 1, that *all* points of contact of the insphere of $ABCD$ with the faces are the Fermat-Torricelli points of the respective faces is equivalent to the seemingly weaker condition that *one* of these points is.

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