

Geometry of Roofs from the View Point of Graph Theory

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Abstract. Roofs discussed in this article are defined as polyhedral surfaces on the basis of two assumptions: (1) all eaves of a roof form a planar (simply connected or k -connected) polygon called the *base of the roof*, (2) every hipped roof end makes the same angle with the (horizontal) plane which contains the base.

Thus every roof, and equivalently the orthographic projection of this roof onto a plane, is uniquely defined by its base. Namely, each ridge of a roof can be obtained as a line segment of the bisectrix of the angle formed by two appropriate edges of the base; if these axes are parallel, then the ridge is the axis of symmetry. Disregarding the metric properties of a roof, we can treat such roofs as planar graphs. Usually, i.e., if the vertices of the base of a given roof are in general position, these are 3-regular graphs. For such graphs (with a simply connected or k -connected base of the roof) we formulate and prove a new Euler formula (Euler formula for regular roofs), and the so-called equations of a regular roof.

Key Words: geometry of roofs, generalized polygon, k -connected generalized polygon, planar graphs, connected graphs, regular graphs, Euler formula for regular roofs

MSC 2000: 51N05, 52B05, 05C90, 68U05

1. Introduction

This article is the first part of a paper which is concerned with geometrical properties of roofs of buildings. Roofs, as a special class of polyhedral surfaces, are examined in many aspects:

- a complete algorithmic definition,
- a graph-theoretical characterization of the skeleton of a roof,
- the existence of all possible shapes of roofs,
- the existence of a supposed shape of a roof.

The results of the presented articles may be interesting from many points of view:

- physically plausible engineering examples of 3-regular graphs,
- a new example of an Euler formula for special polyhedral surfaces,
- computational geometry for special polyhedral surfaces,
- computer graphics for the design and visualization of roofs of buildings,
- and consequently, computer aided architectural design.

The geometry of roofs is also an interesting example of an application of Descriptive Geometry methods for the complete geometric design of a roof (shape of a roof, dihedral angle between two adjacent hipped roof ends, true size of a hipped roof end).

Roofs considered in this article are defined on the basis of two assumptions:

- (1) all eaves form a planar (simply connected or k -connected) polygon called the *base of the roof*,
- (2) every hipped roof end makes the same angle with the (horizontal) plane which contains the base (cf. [5, 10]).

Therefore a roof and equivalently the orthographic projection of this roof onto a plane is uniquely defined by its base. Namely, each roof ridge can be obtained as a line segment of the bisectrix of the angle formed by two appropriate edges (more precisely: by half-lines containing these edges) of the base. When these edges are parallel, then the roof ridge is the axis of symmetry (mirror line) of these edges (half-lines). Due to an elementary triangle property the three bisectrices of any triangle have exactly one common point.

Disregarding the metric properties of a roof, we can treat such roofs as graphs. More precisely, the orthographic projection of the skeleton of a given roof can be seen as a possible geometric realization of a graph. Usually, i.e., if the vertices of the base of a given roof are in general position, then we obtain 3-regular graphs (cf. [1, 2]). For such graphs (for a simply connected or k -connected base of a roof) we formulate and prove a new Euler formula, “Euler’s formula for regular roofs”. This formula can be applied for finding and checking a solution of a roof by a computer program. It is well known that in computer graphics the classic Euler formula for polyhedra and its generalisation for polyhedra with holes and tunnels is used (cf. [3, 4, 6, 8]). There are also Euler formulas for other special classes of graphs (cf. [9]). Such formulas allow to check the number of vertices, edges, and faces of a polyhedron in a computer program which creates and visualizes geometric objects.

2. Generalized polygons

Let us consider the set $\{A_0, A_1, A_2, \dots, A_n\}$ of $n + 1$ points, $n \geq 2$, in the Euclidean plane π . By a *broken line* $\mathbf{C}(A_0, A_1, A_2, \dots, A_n)$ ¹ we mean the union

$$\bigcup_{i=1}^n \langle A_{i-1}A_i \rangle \text{ of line segments } \langle A_{i-1}A_i \rangle.$$

A broken line $\mathbf{C}(A_0, A_1, A_2, \dots, A_n)$ with $n \geq 3$ is said to be *closed* if $A_0 = A_n$.

Using the symbol $\mathbf{L}(A_iA_jA_k)$ for the collinearity of the points A_i, A_j, A_k ([7]), a broken line $\mathbf{C}(A_0, A_1, A_2, \dots, A_n)$ with $n \geq 3$ is said to be *ordinary* if the following conditions are

¹In Graph Theory such an object is called a *polygonal arc*.

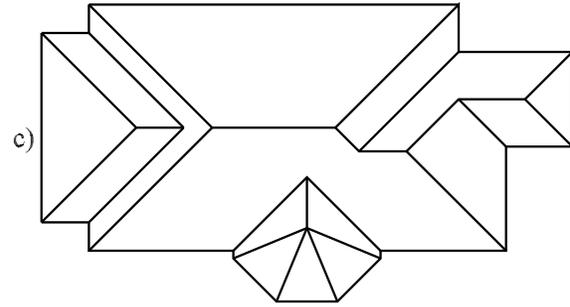
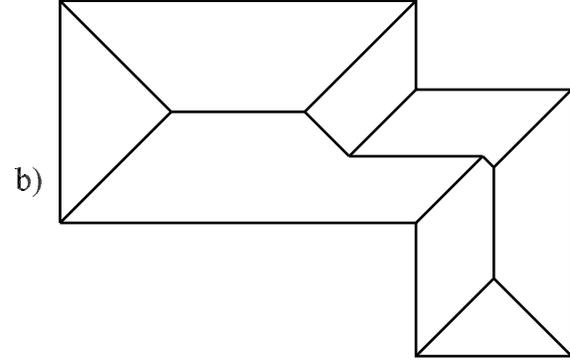
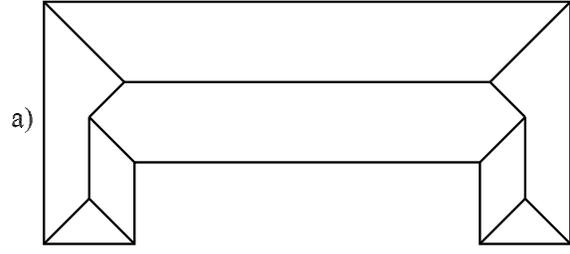


Figure 1: Different shapes of roofs together with the roof skeleton. a) the building of the secondary school in Augustow in Poland (built 1922) with a regular roof, b) a detached house with a regular roof, c) a detached house with an irregular roof

satisfied:

$$\begin{aligned}
 & \neg \mathbf{L}(A_{i-1}A_iA_{i+1}), \quad i = 1, 2, \dots, n-1, \\
 & A_0 = A_n \implies \neg \mathbf{L}(A_0A_1A_{n-1}), \\
 & |i-j| \neq 0, 1, n-1 \implies \langle A_{i-1}A_i \rangle \cap \langle A_{j-1}A_j \rangle = \emptyset \\
 & \quad \wedge (\langle A_0A_1 \rangle \cap \langle A_{n-1}A_n \rangle = \emptyset \vee A_0 = A_n),
 \end{aligned} \tag{1}$$

where $i = 1, \dots, n$ and $j = 3, \dots, n$.

It is well known that every ordinary closed broken line $\mathbf{C}(A_0, A_1, A_2, \dots, A_n)$ divides the plane π into two disjoint regions; the first of which is bounded, the second unbounded ([2]).

The bounded region will be denoted by \mathbf{C}^{int} and the unbounded region by \mathbf{C}^{ext} . Then

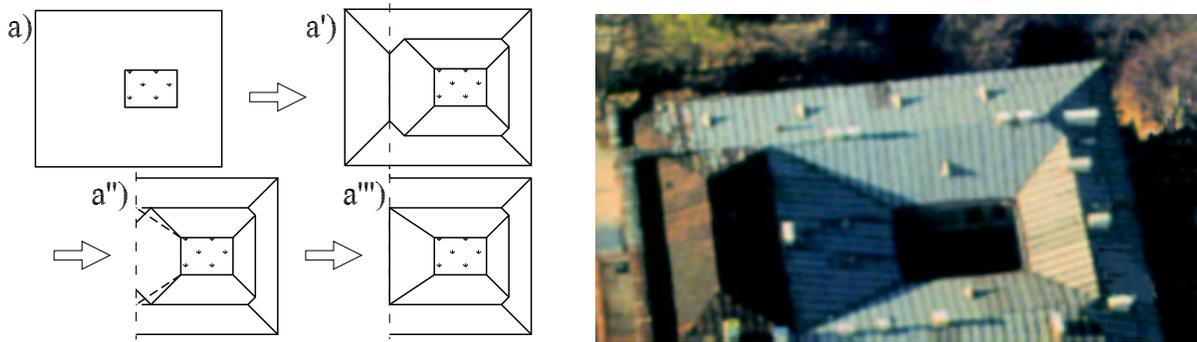


Figure 2: The partly-regular roof of the building of the Białystok University, Department of Economy (built in the 19th century), and the orthographic projection of a part of its skeleton with a way of an architectural design modification from a regular roof to a partly-regular roof ($a \rightarrow a' \rightarrow a'' \rightarrow a'''$). At the top of the figure the fragment of the town with atrial buildings (i.e., with court-yards) is shown. At the top-right of the photograph the atrial building of the Białystok University is displayed, at the bottom the block of atrial buildings of the Theological Seminary (built 1985) (phot.: W. WOLKOW)

$\mathbf{C}^{int} \cup \mathbf{C}^{ext} \cup \mathbf{C} = \pi$. Let $cl(X)$ denote a *closure* of X in the topological sense. It is easy to see that $cl(\mathbf{C}^{int}) = \mathbf{C}^{int} \cup \mathbf{C}$.

Let $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k$ with $1 \leq k \in \mathbb{N}$ be a finite sequence of ordinary closed broken lines such that $cl(\mathbf{C}_1^{int}) \subset \mathbf{C}_1^{int}$ and $cl(\mathbf{C}_i^{int}) \cap cl(\mathbf{C}_j^{int}) = \emptyset$ for $i \neq j$, $i, j = 2, 3, \dots, k$. By a *simple generalized polygon* $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ we mean the set

$$\bigcup_{i=1}^k \left(\mathbf{c}_i \cup \left(\mathbf{c}_1^{int} \cap \left(\bigcap_{i=2}^k \mathbf{c}_i^{ext} \right) \right) \right).$$

From the topological point of view the polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ is a *k-connected* set in the Euclidean plane ([7]). Every line segment which belongs to \mathbf{C}_i for any $i = 1, 2, \dots, k$ will be called an *edge of the polygon*. Every sum

$$\mathbf{P} = \bigcup_{j=1}^k \mathbf{P}_j = \{\mathbf{P}^j\}_{j=1,2,\dots,k} = \{\mathbf{P}_j(\mathbf{C}_{j1}, \mathbf{C}_{j2}, \dots, \mathbf{C}_{j,l_j})\}_{j=1,2,\dots,k}$$

of k separable simple generalized polygons $\mathbf{P}_j(\mathbf{C}_{j1}^j, \mathbf{C}_{j2}^j, \dots, \mathbf{C}_{j,l_j}^j)$ with $k \geq 2$ will be called a *composed generalized polygon*. We call every simple generalized polygon or composed generalized polygon a *generalized polygon*. For $k = 1, l_j > 1$, we have a simple generalized polygon. For $k = 1, l_j = 1$, we have a polygon in an elementary sense. In the case $k = 1$ we omit the index j and we write $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_{l_1})$ instead of $\mathbf{P}_1(\mathbf{C}_{11}, \mathbf{C}_{12}, \dots, \mathbf{C}_{1,l_1}) = \{\mathbf{P}_j(\mathbf{C}_{j1}, \mathbf{C}_{j2}, \dots, \mathbf{C}_{j,l_j})\}_{j=1}$.

3. Geometric algorithmic determination of the roof

3.1. Roof terminology

First, we describe the terminology of elements of the class of roofs discussed in this paper (see Fig. 3). The bottom horizontal ridges of a given roof called *eaves* form a planar simple generalized polygon called the *base of the roof*. The polygons which constitute the roof will be called *hipped roof ends*. Each eaves belongs to one and only one hipped roof end. The vertices of the base of a roof will be called *corner vertices*. The remaining vertices of a roof, and simultaneously the remaining vertices of all hipped roof ends, will be called *top points* of a roof. The ridges of a roof which emerge from corner vertices will be called *corner ridges* (*roof hips ridges*); the remaining ridges do not meet any edge of the base and are called *disappearing ridges*. All disappearing ridges form the *line of disappearing ridges* of the roof. Disappearing ridges which are parallel to the base of a roof will be called *roof ridges*.

In the Descriptive Geometry literature the precise geometrical definitions of roofs and the general construction of a roof (the so-called *general algorithmic determination of a roof* over a given base) cannot be found. However, in many monographs on Descriptive Geometry the geometry of roofs is extensively discussed (cf. [5, 10]).

In our further considerations every roof will be represented by its skeleton, i.e., the set of all ridges and vertices of the polyhedral surfaces. However, for the sake of simplicity we use the term “*roof*” instead of the “*skeleton of the roof*”. Moreover, the orthographic projection of geometric objects (points, lines, line segments of straight lines) will be displayed on the plane of projection (in this paper the plane π) by symbols (Greek or Latin) with primes; when an element belongs to π we omit the prime (see Figs. 4, 5). In order to obtain an algorithmic characterization of the roofs, a topographic section method (see Fig. 6) will be used. In the sequel an algorithmic solution will be presented for the determination of the roof over a given base.

3.2. Some useful notations

Let us consider a simple generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ where

$$\mathbf{C}_i = \mathbf{C}(A_{i0}, A_{i1}, \dots, A_{i,l_i}), \quad A_{i0} = A_{i,l_i}, \quad k \geq 1, \quad i = 1, 2, \dots, k; \quad l_i \geq 3.$$

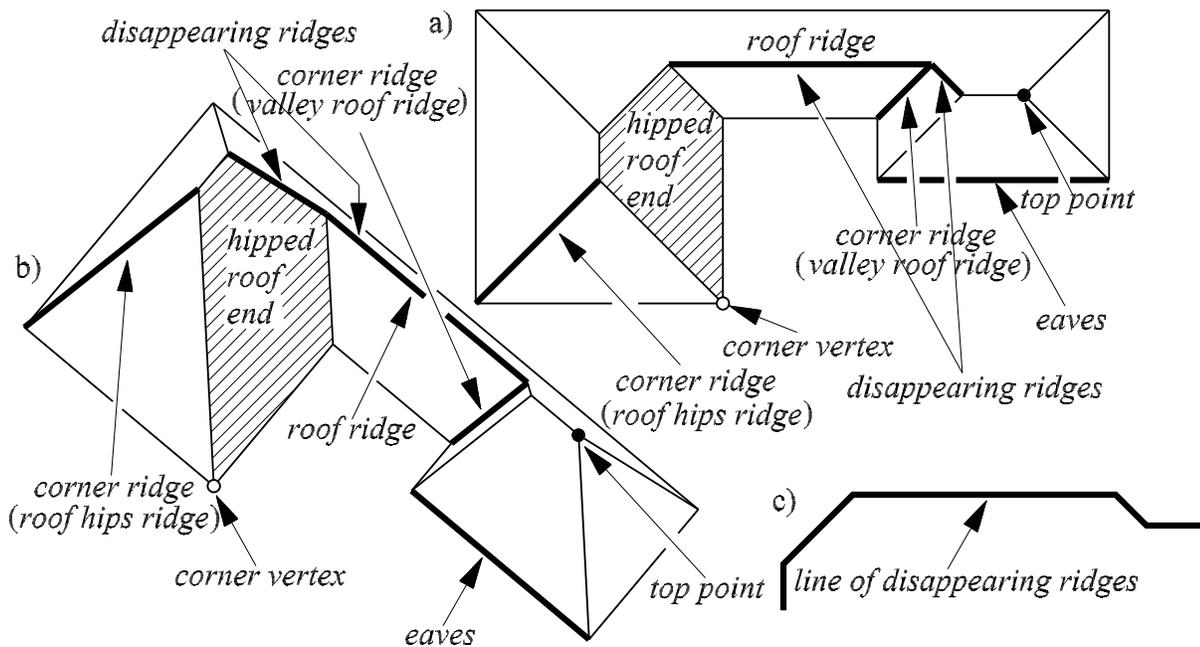


Figure 3: The elements of a roof: a) a roof with its elements in orthographic projection, b) the roof with its elements in a military axonometry, c) the line of disappearing ridges of the roof in orthographic projection

Such a polygon has $\sum_{i=1}^k l_i$ vertices A_{ij} and the same number of sides $\langle A_{i,j-1}A_{ij} \rangle$, $j = 1, 2, \dots, l_i$, $i = 1, 2, \dots, k$. The set of vertices of the above polygon will be denoted by \mathcal{V} . Let us denote by $\mathcal{R}(\mathbf{P}(C_1, C_2, \dots, C_k)) = \mathcal{R}(\mathbf{P})$ in brief — the roof which we want to construct starting from the polygon $\mathbf{P}(C_1, C_2, \dots, C_k)$.

In order to determine the roof $\mathcal{R}(\mathbf{P})$, it suffices to construct two basic sets,

- the set \mathcal{T} of top points of the roof $\mathcal{R}(\mathbf{P})$,
- the set \mathcal{R} of ridges of the roof, and
- two index sets \mathbb{T} and \mathbb{R} associated suitably with them.

In principle, it is sufficient to construct the set \mathcal{T} only, however under the assumption of creating an appropriate index set \mathbb{T} which defines the manner of adjacency of the elements of \mathcal{T} .

Moreover, we will construct three useful auxiliary sets,

- the set \mathcal{Q} of intersection points of all possible triples of half-planes which contain the hipped roof ends,
- the set \mathcal{B} of common parts (called *b-half-lines*) of all possible pairs of half-planes as mentioned above, and
- a *running set* $\mathcal{A} \subset \mathcal{B}$ of b-half-lines which are active during the process of construction of the roof $\mathcal{R}(\mathbf{P})$ — together with their index sets \mathbb{Q} , \mathbb{B} , \mathbb{A} , respectively.

Let us denote by c_{ij} the line segment $\langle A_{i,j-1}A_{ij} \rangle$, by \vec{c}_{ij} the normalized vector $\overrightarrow{A_{i,j-1}A_{ij}}$, by \vec{c}_{ij}^\perp the unit vector orthogonal to \vec{c}_{ij} and pointing to the interior of \mathbf{P} (see Fig. 6b). Using the ‘modulo- l_i -rule’ with the modification that 0 is replaced by l_i , we denote with c_{i1} the line segment $\langle A_{i,l_i}A_{i1} \rangle$. For the simplicity of notation, we will use the same letter for a line segment, a b-half-line, a straight line, i.e., by c_{ij} (and generally by the symbols a, b, ... with indices or without indices) we denote suitable b-half-lines and lines determined by the line

segment c_{ij} for appropriate i, j .

Moreover, we introduce some definitions which associate the sets of elements (points, b-half-lines (line segments, ridges)) with their index sets, and other convenient notations:

Note that the point $X_{u_1, v_1; u_2, v_2; u_3, v_3}$ can be identified with its three-element set of indices (briefly *three-set*) $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ and the b-half-line (line segment, ridge) $x_{w_1, z_1; w_2, z_2}$ can be identified with its two-element set of indices (briefly *two-set*) $\{(w_1, z_1), (w_2, z_2)\}$. We assume that the point $X_{u_1, v_1; u_2, v_2; u_3, v_3}$ is incident with a b-half-line (line segment, ridge) $y_{w_1, z_1; w_2, z_2}$ if and only if $\{(w_1, z_1), (w_2, z_2)\} \subset \{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$. We assume briefly that the three-set $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ is incident with a two-set $\{(w_1, z_1), (w_2, z_2)\}$ if and only if $\{(w_1, z_1), (w_2, z_2)\} \subset \{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$. Similarly, we assume that a half-line (line segment, ridge) $x_{y_1, z_1; y_2, z_2}$ connects two points $Y_{u_1, v_1; u_2, v_2; u_3, v_3}, Z_{u_4, v_4; u_5, v_5; u_6, v_6}$ if and only if $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\} \cap \{(u_4, v_4), (u_5, v_5), (u_6, v_6)\} = \{(y_1, z_1), (y_2, z_2)\}$. For such points we assume that they are connectable.

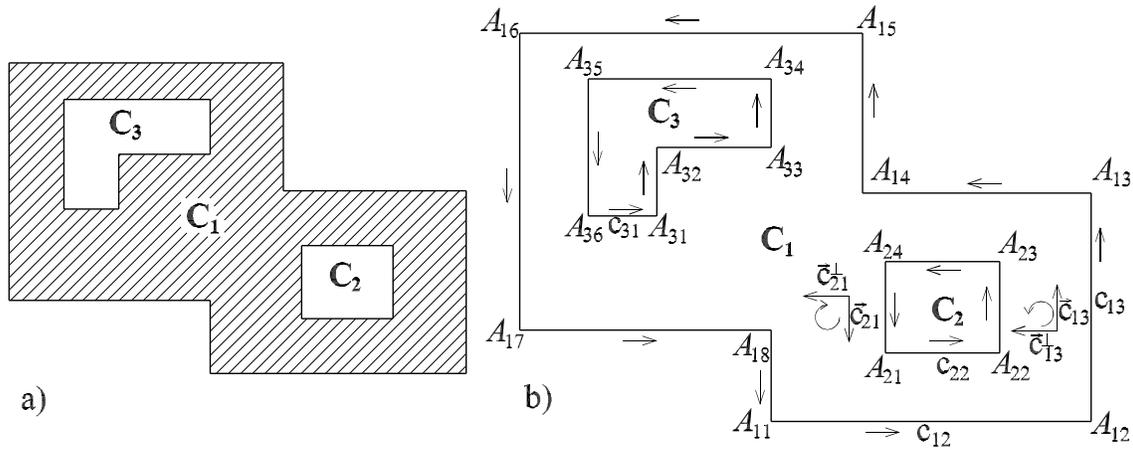


Figure 4: Base of the roof is the generalized polygon $\mathbf{P}(C_1, C_2, \dots, C_k)$ with $k = 3$, $l_1 = 8$, $l_2 = 4$, $l_3 = 6$: a) contour, b) initial elements of the constructed roof

We can treat the considered polygon $\mathbf{P}(C_1, C_2, \dots, C_k)$ as a plan of a building (horizontal projection) and simultaneously as a contour of the roof of this building. As noted above, we call this polygon the *base of the roof*. We suppose that all hipped roof ends make the same angle of slope with the plane of the roof base; the angle measure lies between 0° and 90° .

The roof $\mathcal{R}(\mathbf{P})$ can be constructed in two ways:

- in the plane π (*planar* manner), and
- in the three-dimensional space (*spatial* manner) where the plane π is embedded.

The spatial method seems to be convenient and also more natural; therefore we remain with it as the main way for the algorithmic construction of $\mathcal{R}(\mathbf{P})$. However in traditional Descriptive Geometry the planar manner and Monge's method for the determination of the roof $\mathcal{R}(\mathbf{P})$ is preferred.

First, let us assume the counter clockwise orientation (right-handed) of each component C_i of the given generalized polygon \mathbf{P} . Therefore every side c_{ij} of \mathbf{P} can be completed by the perpendicular unit vector \vec{c}_{ij}^\perp which points towards the interior of \mathbf{P} . Then every pair of vectors $(\vec{c}_{1j}, \vec{c}_{1j}^\perp)$ has a right-handed orientation, and every pair $(\vec{c}_{ij}, \vec{c}_{ij}^\perp)$ for $i = 2, 3, \dots, k$ is left-handed (see Fig. 4a).

Let us denote by α_{ij} the half-plane of the hipped roof end with the eaves c_{ij} as the edge line, and let α'_{ij} denote its orthogonal projection. Notice that α'_{ij} is parallel to $\alpha_{i''l}$ if and only if $\alpha'_{i'j} \subset \alpha_{i''l}$ or $\alpha_{i''l} \subset \alpha'_{i'j}$.

Each pair of non-parallel half-planes ($\alpha_{i'j}$, $\alpha_{i''l}$) has a common half-line $b_{i'j;i''l}$, a so-called *b-half-line*. The b-half-line $b_{i'i;i''j}$ ($b'_{i'i;i''j}$) for $i' = i''$ and $|i - j| \in \{1, l_i - 1\}$ will be called a *corner b-half-line* (for a line segment a *corner ridge* of a roof). Each corner ridge $b_{i'i;i''j}$ (and its orthogonal projection $b'_{i'i;i''j}$) of the the roof can briefly be denoted by $b_{i'i}$ ($b'_{i'i}$) if $i + 1 = j$ or by $b_{i''j}$ ($b'_{i''j}$) if $j + 1 = i$. Without loss of generality we can assume that in the symbol $b_{i'i;i''j}$ we mean $i' \leq i''$, $i \leq j$.

Among all corner b-half-lines we distinguish

- (1) the *hips b-half-line* (for a line segment: *roof hips ridge*), if the angle $\angle(\alpha'_{i'i}, \alpha'_{i'j})$ is convex, and
- (2) the *valley b-half-line* (for a line segment: *valley roof ridge*), if the angle $\angle(\alpha'_{i'i}, \alpha'_{i'j})$ is concave.

Each roof has roof hips ridges and may or may not have some valley roof ridges. As it was already said, the set of all such b-half-lines will be denoted by \mathcal{B} .

On the other hand, each pair ($\alpha'_{i'j}$, $\alpha_{i''l}$) of half-planes $\alpha'_{i'j}$, $\alpha_{i''l}$ with non-parallel lines which includes suitable edges $c_{i'j}$, $c_{i''l}$ determines an angle $\angle(\alpha'_{i'j}, \alpha_{i''l})$. Every half-plane $\alpha'_{i'j}$ is uniquely determined by two objects: the line spanned by the line segment c_{ij} and the unit vector \vec{c}_{ij}^\perp . Then the angle $\angle(\alpha'_{i'j}, \alpha_{i''l})$ may be defined by two pairs $(\vec{c}_{i'j}, \vec{c}_{i''l}^\perp)$, $(\vec{c}_{i''l}, \vec{c}_{i'j}^\perp)$.

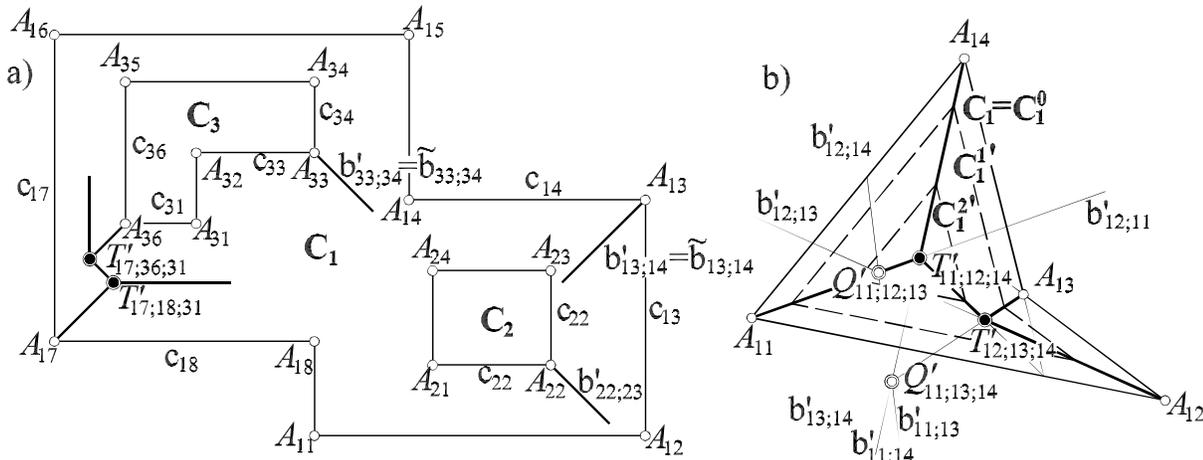


Figure 5: An illustration of the algorithm for the determination of the roof: a) elements of the described algorithm; b) an illustration of the topographic method

Summarizing, for a given generalized simple polygon \mathbf{P} with selected orientations of their components we can uniquely define all possible angles which form all possible pairs of half-lines which contain eaves of the roof $\mathcal{R}(\mathbf{P})$. Let us denote by $\tilde{b}_{i'j;i''l}$ the bisectrix of the angle $\angle(\alpha'_{i'j}, \alpha_{i''l})$ if the lines $c_{i'j}$, $c_{i''l}$ are not parallel, and the line of symmetry of lines determined by line segments $c_{i'j}$, $c_{i''l}$ in the opposite case.

It is well-known from Descriptive Geometry that the bisectrix $\tilde{b}_{i'j;i''l}$ is the orthogonal projection $b'_{i'j;i''l}$ of the b-half-line $b_{i'j;i''l}$ ([5]). Hence $\tilde{b}_{i'j;i''l}$ coincides with $b'_{i'j;i''l}$, i.e., $\tilde{b}_{i'j;i''l} = b'_{i'j;i''l}$ (see Fig. 5a).

For every three $\alpha_{i'i}$, $\alpha_{i''j}$, $\alpha_{i''l}$ pairwise not parallel half-planes the point $Q_{i'i;i''j;i''l}$ can be defined as an element of the set $\alpha_{i'i} \cap \alpha_{i''j} \cap \alpha_{i''l}$ (see Fig. 5b). Every point $Q_{i'i;i''j;i''l}$ is

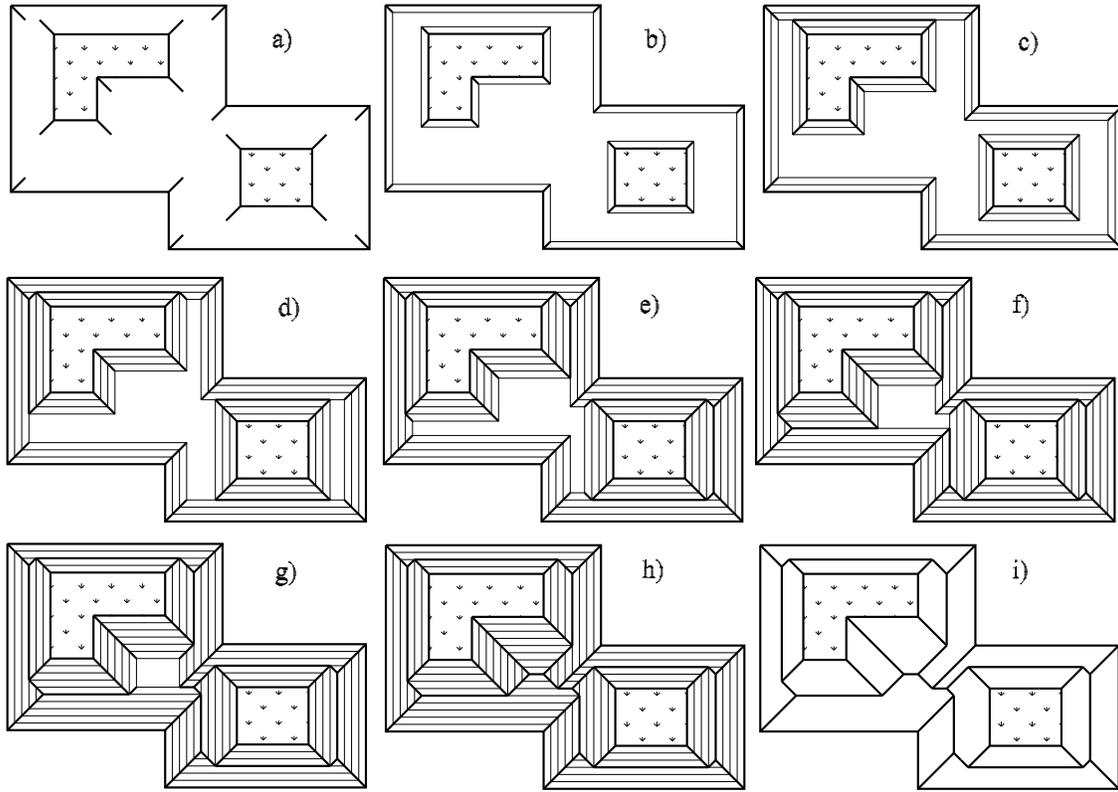


Figure 6: An illustration of the topographic method for the algorithmic determination of the roof defined by the polygon displayed in Figs. 4a, 5a

also a common point of the b-half-lines $b_{i'i''j}$, $b_{i'i''l}$, and $b_{i''j;i''l}$. Every such point may be called a *top point* of the roof. Note that the orthogonal projection $Q'_{i'i''j;i''l}$ of $Q_{i'i''j;i''l}$ is the common point of three appropriate bisectrices $b'_{i'i''j}$, $b'_{i'i''l}$, $b'_{i''j;i''l}$ of the angles $\angle(\alpha'_{i'i}, \alpha'_{i''j})$, $\angle(\alpha'_{i'i}, \alpha'_{i''l})$, $\angle(\alpha'_{i''j}, \alpha'_{i''l})$. Let us denote by \mathcal{Q} (\mathcal{Q}') the set of all points $Q_{i'i''j;i''l}$ ($Q'_{i'i''j;i''l}$) for $i', i'', i''' \in \{1, 2, \dots, k\}$, $i = 1, 2, \dots, l_{i'}$, $j = 1, 2, \dots, l_{i''}$, $l = 1, 2, \dots, l_{i'''}$, $(i', i) \neq (i'', j)$, $(i', i) \neq (i''', l)$, $(i'', j) \neq (i''', l)$. It is easy to see that the cardinality of the set \mathcal{Q} (\mathcal{Q}') is at most equal $(\sum_{i=1}^k l_i)$.

Let us assign to every point $Q_{i'i''j;i''l}$ ($Q'_{i'i''j;i''l}$) of the set \mathcal{Q} (\mathcal{Q}') the height (distance) $h_{i'i''j;i''l}$ ($d_{i'i''j;i''l}$) of this point (determined by the line segments $c_{i'i}$, $c_{i''j}$, $c_{i''l}$) with regard to the plane π . Note that $d_{i'i''j;i''l} = d(Q'_{i'i''j;i''l}, c_{i'i}) = d(Q'_{i'i''j;i''l}, c_{i''j}) = d(Q'_{i'i''j;i''l}, c_{i''l})$. Let $\mathbb{H}(\mathbb{D})$ denote the set of all such numbers $h_{i'i''j;i''l}$ ($d_{i'i''j;i''l}$) for $Q \in \mathcal{Q}$ ($Q' \in \mathcal{Q}'$). We have $\text{card}(\mathbb{H}) = \text{card}(\mathbb{D}) \leq \text{card}(\mathcal{Q}) = \text{card}(\mathcal{Q}')$.

3.3. The algorithm

Let $h_1, h_2, \dots, h_\sigma$ be a monothonic sequence of numbers which belong to \mathbb{H} , and $d_1, d_2, \dots, d_\sigma$ be a monothonic sequence of numbers which belong to \mathbb{D} , i.e., $\text{card}(\mathbb{H}) = \text{card}(\mathbb{D}) = \sigma$.

Let us define the sequence

$$\{\{\mathbf{P}_j^i\}_{j=1,2,\dots,k}\}_{i=0,1,2,\dots} = \{\{\mathbf{P}_j^i(\mathbf{C}_{j1}, \mathbf{C}_{j2}, \dots, \mathbf{C}_{j,l_j})\}_{j=1,2,\dots,k}\}_{i=0,1,2,\dots} \quad (2)$$

of generalized polygons $\{\mathbf{P}_j^i(\mathbf{C}_{j1}, \mathbf{C}_{j2}, \dots, \mathbf{C}_{j,l_j})\}_{j=1,2,\dots,k}$ constructed by induction on i in the following manner.

Let $g_0, g_1, g_2, \dots, g_{\sigma+1}$ be a monothonic sequence of numbers defined as follows. We set $h_0 = 0$, $g_0 = 0$, $g_1 = (h_0 + h_1)/2$. Let us denote by β^1 any plane parallel to π such that $d(\beta^1, \pi) = g_1$. Let us denote by $A_{i'i''j}^1$ the common point of the plane β^1 and the half-line $b_{i'i''j}$ for $i', i'' \in \{1, 2, \dots, k\}$, $i = 1, 2, \dots, l_{i'}$, $j = 1, 2, \dots, l_{i''}$. For the sake of uniformity we adopt a convention $\beta^0 = \pi$. Then $d(\beta^0, \pi) = g_0$ and $A_{i'i}^0 = A_{i'i}$ for $i' = 1, 2, \dots, k$, $i = 1, 2, \dots, l_{i'}$. The set of all points $A_{i'i''j}^1$ stands for the first (exactly, after standardization, the second) element $\{\mathbf{P}_j^1\}_{j=1,2,\dots,l_1}$ of the sequence (2).

The polygon $\{\mathbf{P}_j^1\}_{j=1,2,\dots,l_1}$ induces an unbounded prism $\{\mathbf{P}_j^1\}_{j=1,2,\dots,l_1}^\infty$ which has this polygon as its base. The edges are b-half-lines perpendicular to π with $A_{i'i''j}^1$ as initial points. Let

$$\mathcal{Q}_1 = \left\{ Q \in \mathcal{Q} : d(Q, \pi) = h_1 \wedge Q \in \{\mathbf{P}_j^1\}_{j=1,2,\dots,l_1}^\infty \right\}.$$

For every $Q \in \mathcal{Q}_1$ there exists at least one triad of pairs $(i_1, j_1; i_2, j_2; i_3, j_3)$ such that $d(Q_{i_1, j_1; i_2, j_2; i_3, j_3}, \pi) = h_1$. We denote each such point by $T_{i_1, j_1; i_2, j_2; i_3, j_3}$ and include it in the new set \mathcal{T} . We include every corner roof ridge (bisectrix) which contains an element of the set \mathcal{V} as the first endpoint and an element of \mathcal{T} as its second endpoint to the new set \mathcal{R} . We will also add to \mathcal{R} every such ridge which has the connectable elements of the set \mathcal{T} as endpoints. The latter ridges, if they exist, are parallel to the plane π , because it belongs to the plane β^1 . All hips of b-half-lines (bisectrices) which have vertices of the base of the roof (elements of the set \mathcal{V}) as starting-points are included in the *running set* \mathcal{A} .

Now we may create the sets \mathbb{R} , \mathbb{T} . Each three-set $\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$ of every point $T_{i_1, j_1; i_2, j_2; i_3, j_3}$ is included in \mathbb{T} . If the two-set $\{(i_s, j_s), (i_u, j_u)\}$ defines the corner ridge of the roof, then we include this two-set in \mathbb{R} . We have two cases:

- (1) for every three-set from \mathbb{T} all their subsets (two-sets) belong to \mathbb{R} ;
- (2) there exists a three-set $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ in \mathbb{T} such that one or two subsets of it do not belong to \mathbb{R} .

Without loss of generality we can assume that the condition (2) is satisfied by the two-set $\{(u_1, v_1), (u_2, v_2)\}$ (there are the two-sets $\{(u_1, v_1), (u_3, v_3)\}$, $\{(u_2, v_2), (u_3, v_3)\}$). This two-set (these two-sets) identifies such a b-half-line (identify b-half-lines) on which the ridge of the roof is not constructed yet. This two-set (these two-sets) is included in the set \mathbb{A} and obviously added to \mathcal{A} of the suitably indexed b-half-line (b-half-lines) which has the top point $T_{u_1, v_1; u_2, v_2; u_3, v_3}$ as the origin.

Now we go to the second step of the induction:

Suppose now that the set \mathcal{Q}_m and the remaining sets \mathcal{T} , \mathcal{R} , \mathcal{A} , \mathbb{T} , \mathbb{R} , \mathbb{A} on this level have already been defined. We put $g_{m+1} = (h_m + h_{m+1})/2$, we consider the plane β^{m+1} parallel to π and satisfying the condition $d(\beta^{m+1}, \pi) = g_{m+1}$, and we determine the points $A_{i'i''j}^{m+1}$ as intersections of the plane β^{m+1} with half-lines (bisectrices) $b_{i'i''j}$ for $i', i'' = 1, 2, \dots, k$, $i = 1, 2, \dots, l_{i'}$, $j = 1, 2, \dots, l_{i''}$. Similarly to the first step of induction, we define the polygon $\{\mathbf{P}_j^{m+1}\}_{j=1,2,\dots,l_{m+1}}$ and the unbounded prism $\{\mathbf{P}_j^{m+1}\}_{j=1,2,\dots,l_{m+1}}^\infty$. Let

$$\mathcal{Q}_{m+1} = \left\{ Q \in \mathcal{Q} : d(Q, \pi) = h_{m+1} \wedge Q \in \{\mathbf{P}_j^{m+1}\}_{j=1,2,\dots,l_{m+1}}^\infty \wedge Q \in \mathcal{A} \right\}.$$

For every $Q \in \mathcal{Q}_1$ there exists at least one triad of index pairs $(i_1, j_1; i_2, j_2; i_3, j_3)$ such that $d(Q_{i_1, j_1; i_2, j_2; i_3, j_3}, \pi) = h_{m+1}$. If the three-set $\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$ is incident with any element of the set \mathbb{A} , then add it to the set \mathbb{T} , and we denote the point $Q_{i_1, j_1; i_2, j_2; i_3, j_3}$ by $T_{i_1, j_1; i_2, j_2; i_3, j_3}$ and add it to the set \mathcal{T} . The two-sets which connect $T_{i_1, j_1; i_2, j_2; i_3, j_3}$ with the remaining points of the set \mathcal{T} are added to \mathbb{R} and indexed by the ridges we add to \mathcal{R} .

Similarly to the first step of induction, with regard to the set \mathbb{T} there are two cases:

- (1) for every three-set of \mathbb{T} all subsets (two-sets) belong to \mathbb{R} ,
- (2) there exists a three-set $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ in \mathbb{T} such that one or two subsets of it do not belong to \mathbb{R} .

Without loss of generality we can assume that the condition (2) is satisfied by the two-set $\{(u_1, v_1), (u_2, v_2)\}$ (there are the two-sets $\{(u_1, v_1), (u_3, v_3)\}$, $\{(u_2, v_2), (u_3, v_3)\}$). This two-set (these two-sets) identifies such a b-half-line (identify b-half-lines) on which the ridge of the roof is not constructed yet. We include this two-set (these two-sets) in the set \mathbb{A} and obviously we add to \mathcal{A} the suitably indexed b-half-line (b-half-lines) starting at $T_{u_1, v_1; u_2, v_2; u_3, v_3}$.

We stop the process when the set of numbers $\{h_1, h_2, \dots, h_\sigma\}$ is running out. The sets \mathcal{T} and \mathcal{V} define the skeleton of the roof uniquely determined by the simple generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$.

4. The Graph Theory approach

We can treat every generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ as the plan projection of a certain base of a roof (see Fig. 4a with $k = 3$ and Fig. 7a with $k = 1$). A building with such a roof is $(k - 1)$ -atrial, i.e., it has $k - 1$ court-yards. In the previous section we presented an algorithmic geometric construction of the skeleton of such roof. Now we are going to discuss properties of roof from the view point of Graph Theory.

4.1. Notations

First we introduce the following notations: Let v be a number of vertices V_i and obviously edges e_i , $i = 1, 2, \dots, v$, of the generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ (see Fig. 7b). Let us suppose:

- 1° at every top point of the skeleton there meet exactly three ridges of the roof (one or two corner ridges and one or two ridges of the roof),
- 2° v edges of the base of the roof induce exactly v polygons (hipped roof ends), and
- 3° each hipped roof end contains exactly one eaves.

The above supposition is equivalent to the general position of the vertices of the roof. Then the roof $\mathcal{R}(\mathbf{P})$ induced by $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ we will called *regular*. More precisely, the roof is said to be *regular*, if

- (i) every top point is located exactly on three ridges, and
- (ii) every hipped roof end contains exactly one eaves.

Otherwise a roof is *irregular* (see Figs. 8a', 8a'', 8b', 8c'). The roofs in Figures 8a' and 8c' don't fulfil condition (i), the roofs in Figures 8a'' and 8b'' fail condition (ii). Notice that a roof is *irregular* if and only if there exists at least one top point at which at least two corner ridges are meeting which do not belong to a same triangular hipped roof end.

Furthermore we denote by V_1, V_2, \dots, V_v the corner vertices of the roof, by e_1, e_2, \dots, e_v the edges (eaves) of the roof ($e_i = V_{i-1}V_i$ for $i = 2, 3, \dots, v$ and $e_1 = V_vV_1$), by $r_1, r_2, \dots, r_v, r_{v+1}, r_{v+2}, \dots, r_{v+d}$ the ridges of the roof ($r_{v+i} = T_jT_l$), for $i = 1, 2, \dots, d$ and for appropriate j, l , by r_1, r_2, \dots, r_v the corner roof ridges, by $r_{v+1}, r_{v+2}, \dots, r_{v+d}$ the disappeared ridges), by T_1, T_2, \dots, T_t the top points. Let r denote the number of all ridges of the roof $\mathcal{R}(\mathbf{P})$. Then $r = v + d$.

We will distinguish four sets:

- $\mathbf{V} = \{V_1, V_2, \dots, V_v\}$ is the set of vertices of the base of the roof,

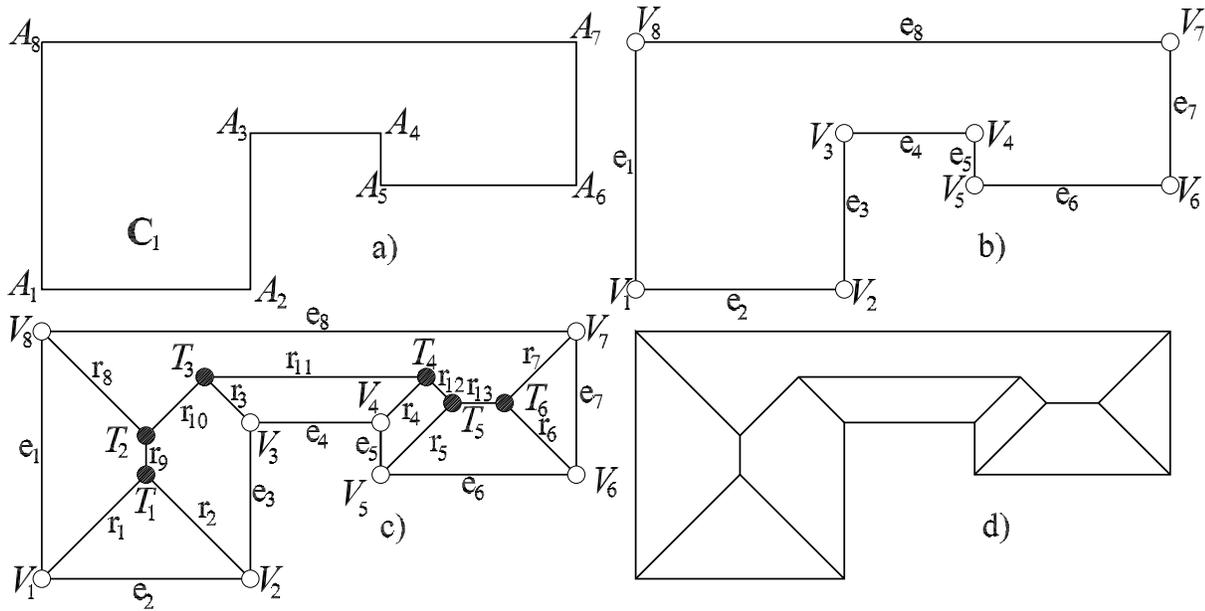


Figure 7: An illustration of the scheme of description of a roof from the view point of Graph Theory: a) the generalized polygon $\mathbf{P}(\mathbf{C}_1)$ — the base of the roof $\mathcal{R}(\mathbf{P})$, b) the basic graph of the contour of this roof, c) the graph of the skeleton of the roof $\mathcal{R}(\mathbf{P})$: $r_i = V_i T_j$ are corner roof ridges, $T_1 T_2 \dots T_6$ is the line of disappearing ridges, $r_9 = T_1 T_2$, $r_{11} = T_3 T_4$, $r_{13} = T_5 T_6$ are roof ridges, $r_{10} = T_2 T_3$, $r_{12} = T_4 T_5$ are disappearing ridges, d) the projection of the skeleton of the roof $\mathcal{R}(\mathbf{P})$

- $E = \{e_1, e_2, \dots, e_v\}$ is the set of edges of the base of the roof,
- $R = \{r_1, r_2, \dots, r_v, r_{v+1}, r_{v+2}, \dots, r_{v+d}\}$ is the set of ridges of the roof,
- $R' = \{r_{v+1}, r_{v+2}, \dots, r_{v+d}\}$ is the set of disappearing ridges,
- $T = \{T_1, T_2, \dots, T_t\}$ the set of top points.

The generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ determines the *basic graph* $BG = (V, E)$ of the contour of the roof. It follows from the geometric properties of a roof (previous section) that the graph $BG = (V, E)$ induces the *graph* $RG = (V \cup T, E \cup R)$ of the projection of the skeleton of the roof uniquely. We will also distinguish the *graph of disappearing ridges* $DRG = (T, R')$.

From the view point of Graph Theory the elements of the set V ($V \cup T$, T) are called *vertices*; the elements of E ($E \cup R$, R') are called *edges*. Consequently the set V ($V \cup T$, T) will be called the *vertex set*, and E ($E \cup R$, R') the *edge set*.

4.2. Euler formula

The vertex set of the graph RG has $v + t$ elements, and the edge set has $v + r$ elements. Every edge joins two vertices of the graph, so there are $2(v + r)$ elements. But any vertex of the graph BG is an endpoint of three edges of the graph (two edges of the base of the roof or three ridges of the roof). To obtain the number of all vertices of the graph we must divide the number $2(v + r)$ by 3. Hence there are $2(v + r)/3$ vertices of the graph BG . On the other hand the number of vertices of the graph is $v + t$. Comparing the last two expressions we obtain the equation $2(v + r)/3 = v + t$, and next the formula

$$v = 2r - 3t. \quad (3)$$

Note that eq. (3) does not depend on the connectivity of the generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$.

In classical Graph Theory Euler's formula for the planar graph embedded in a plane is well known:

$$reg + ver - edg = 2, \quad (4)$$

where reg denotes the number of regions determined by the graph (cf. [1, 2]), ver means the number of vertices, and edg the number of edges. Applying Euler's formula to our considerations we can set $reg = v + k - 1 + 1$ (v hipped roof ends plus $k - 1$ regions of the k -connected generalized polygon plus one region, the exterior of the polygon \mathbf{C}_1), $edg = v + r$, and $ver = v + t$. After substituting this in (4) we obtain

$$v = r - t + 2 - k. \quad (5)$$

By (3) we get

$$t = \frac{r - 2 + k}{2}. \quad (6)$$

From (6) and (5) we finally obtain

$$r = 2v + 3(k - 2), \quad (7)$$

$$t = v + 2(k - 2), \quad (8)$$

$$d = v + 3(k - 2). \quad (9)$$

So we can formulate

Theorem 1 (Euler formula for regular roofs). *If the base of a regular roof is a k -connected generalized v -polygon, then the number r of ridges of this roof is $2v + 3(k - 2)$, the number t of top points is $v + 2(k - 2)$, and the number d of disappearing ridges is $v + 3(k - 2)$.*

Note that every hipped roof end roof is an m -polygon for $3 \leq m \leq v$. Indeed, every hipped roof end must be at least a triangle and at most it may neighbour $v - 1$ other hipped roof ends. Hence, taking into account one edge of the base and adding 1, we can write $3 \leq m \leq v - 1 + 1$. Let m_i denote the number of i -gonal hipped roof ends with $i = 3, 4, \dots, v$. Since the number of hipped roof ends equals v we obtain

$$m_3 + m_4 + \dots + m_v = v. \quad (10)$$

Let us consider now all hipped roof ends (polygons). The number of all line segments which are the edges or ridges of the roof skeleton is $v + 2r$, because every ridge is common for two hipped roof ends and every hipped roof end contains one edge of the v -gon. Therefore we have

$$3m_3 + 4m_4 + \dots + vm_v = v + 2r. \quad (11)$$

We substitute (7) in (11) and obtain

$$3m_3 + 4m_4 + \dots + vm_v = 5v + 6(k - 2). \quad (12)$$

We call (12) and (10) the *equations of a regular roof* generated by any k -connected v -gon. Note that the k -connected regular roofs are geometric configurations of the form $((2v + 2(k - 2))_3, (3v + 3(k - 2))_2)$.

5. Some remarks on irregular roofs

As already said, a roof is *irregular* if and only if there exists at least one top point in which there meet at least two corner ridges which do not belong to the same triangular hipped roof end. We call such top points *singular*. There are two possible cases:

- a) These two corner ridges belong to the same hipped roof end. Then we have a singular top point of the *first kind* or a *break top point* (see Figs. 8a'', 8b'', 8c'').
- b) These two corner ridges do not belong to the same hipped roof end (see Figs. 8a', 8b', 8c''). Then they belong to four different hipped roof ends which have a common top point called singular of the *second kind*.

Usually, such top points are vertices of degree greater than 3 (see Figs. 8a', 8b'). We obtain the singular top points as a limiting position of two (or more) adjacent (i.e., belonging to the same ridge) top points which are coinciding (see Figs. 8a', 8b'). Or one top point would split (Fig. 8a''), if we change the position of the sides of the base. In the first case we obtain a singular top point of the second kind, in the second case the singular top point is of the first kind. In both cases a ridge joining initially two (or more) degenerating top points shrinks into one point. We distinguish even a third *mixed kind of irregularity* of a roof (see Fig. 8c'').

According to the above considerations and earlier notations, we have two useful inequalities for irregular graphs,

$$|\mathbf{T}| < v + 2(k - 2), \quad |\mathbf{R}'| < v + 3(k - 2). \quad (13)$$

Suppose now case a) for a certain roof $\mathcal{R}(\mathbf{P})$ and suppose the break top point belongs to the hipped roof end Δ (see Fig. 9a). Then Δ contains more than two corner ridges r_s, r_i, r_{i+1}, r_l , more than one eaves and at least two top points, the break top point T_n and, e.g., T_j (see Fig. 9a). T_n as the common point of two corner ridges r_i, r_{i+1} does not belong to other sides of the polygon Δ . Moreover, the point T_n is separate from the broken line $T_j T_{j+1} \dots T_m T_{m+1}$. These remarks will be used in the study of the connectivity of the line of disappearing ridges for arbitrary regular roof.

5.1. On the connectivity

In order to examine the connectivity of graphs associated with roofs we prove the following auxiliary lemmas:

Let Δ_i with $i \in \{1, 2, \dots, v\}$ be a sequence of all hipped roof ends of the given roof $\mathcal{R}(\mathbf{P})$ generated by the v -gon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$. Let \mathbf{T} be the set of top points, and \mathbf{R}' the set of disappearing ridges of this roof. Consider a sequence of pairs $(\mathbf{T}|\Delta_i, \mathbf{R}'|\Delta_i)$ where $\mathbf{T}|\Delta_i$ and $\mathbf{R}'|\Delta_i$ are the sets \mathbf{T}, \mathbf{R}' , restricted to Δ_i for $i \in \{1, 2, \dots, v\}$.

Lemma 1. *Let $\mathcal{R}(\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k))$ be a given regular roof with \mathbf{T} as the set of top points and \mathbf{R}' as the set of disappearing ridges. Then for any hipped end $\Delta_i, i \in \{1, 2, \dots, v\}$, of this roof the graph $(\mathbf{T}|\Delta_i, \mathbf{R}'|\Delta_i)$ is connected.*

Proof. Suppose the assertion of this lemma is false. Then there exists a hipped roof end Δ_{i_0} for which the graph $(\mathbf{T}|\Delta_{i_0}, \mathbf{R}'|\Delta_{i_0})$ is not connected. This means that the hipped roof end Δ_{i_0} must contain a break top point (see Fig. 9a), say T_n . Then the hipped roof end Δ contains more than two corner ridges r_s, r_i, r_{i+1}, r_l , and consequently more than one eaves. This is impossible because in a regular roof every the hipped roof end contains exactly one eaves. \square

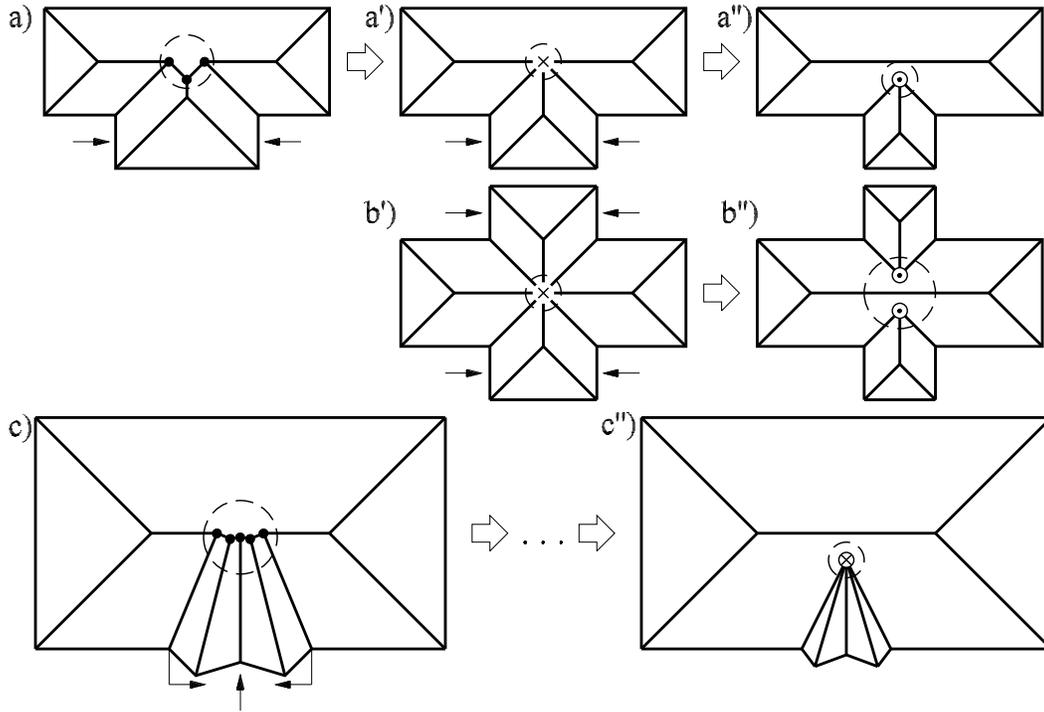


Figure 8: Regular roofs and their irregular mutations:

- a) a regular roof with distinguished regular points (\bullet) which are transformed;
- a') its derivative roof with the singular top point of the first kind (\times), subjected to a further transformation,
- a'') its derivative roof with the singular point of the second kind (\odot);
- b') an irregular roof with one singular top point of the first kind (\times),
- b'') its derivative roof with the singular point of the second kind (\odot);
- c) a regular roof with distinguished regular points (\bullet) which are transformed,
- c'') its derivative roof with singular point of the first and the second (mixed) kind (\otimes) simultaneously

Lemma 2. For any two hipped roof ends Δ, Γ of the roof $\mathcal{R}(\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k))$ there exists a sequence of hipped roof ends $\Lambda_i, i = 1, 2, \dots, n$, such that $\Lambda_i \cap \Lambda_{i+1} \neq \emptyset$, $\Lambda_1 = \Delta$, and $\Lambda_n = \Gamma$.

Proof. We proceed by induction on k :

$k = 1$: We take one of two orientations of the simply connected polygon \mathbf{C}_1 . Starting at the hipped roof end Δ we construct in the specified orientation the sequence of successive hipped roof ends Λ_s which satisfy the condition $\Lambda_s \cap \Lambda_{s+1} = r_j$, where r_j are corner ridges. The construction ends when we meet Γ , i.e., $\Lambda_n = \Gamma$.

$k = 2$: There are two cases:

C21) Δ and Γ contain eaves which belong simultaneously to \mathbf{C}_1 and \mathbf{C}_2 . Here the proof is similar to the case $k = 1$.

C22) Δ contains an eaves belonging to \mathbf{C}_i and Γ contains an eaves belonging to \mathbf{C}_j for $i \neq j$, $i, j \in \{1, 2\}$. Then there exist two hipped roof ends $\Omega_i, i \in \{1, 2\}$, such that Ω_i contains an eaves from \mathbf{C}_i and $\Omega_1 \cap \Omega_2 \neq \emptyset$.

The existence of such hipped roof ends can be concluded as follows: If they do not exist,

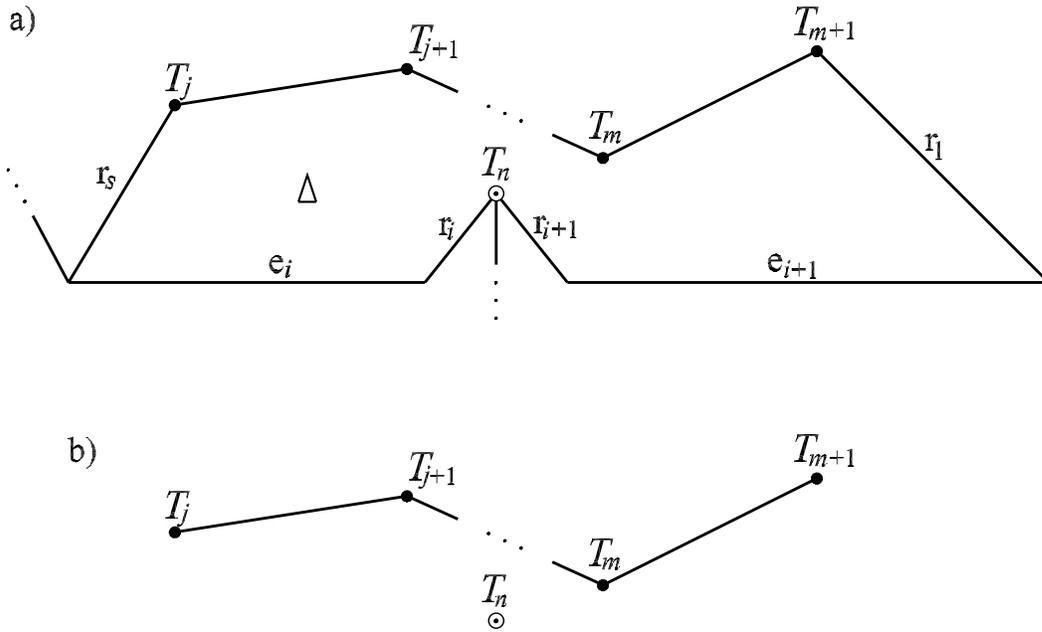


Figure 9: A fragment of an irregular roof with break top points: a) a hipped roof end with a break top point T_n , b) a fragment of a line of disappearing ridges

then all hipped roof ends of \mathbf{C}_1 would form a roof $\mathcal{R}(\mathbf{P})$ generated by the simply connected polygon $\mathbf{P}(\mathbf{C}_1)$, and the bounded region of \mathbf{C}_2 would be covered by the roof $\mathcal{R}(\mathbf{P}(\mathbf{C}_1))$ which is impossible.

Further we assume without loss of generality that Δ contains an eaves which belongs to \mathbf{C}_1 and Γ contains an eaves belonging to \mathbf{C}_2 . The required sequence is constructed in the following way: Starting at Δ we create similarly to the case $k = 1$ in a specified orientation of \mathbf{C}_1 the subsequence $\Lambda_1 := \Delta, \dots, \Lambda_m := \Omega_1$. Continuing, we set $\Lambda_{m+1} := \Omega_2$ and similarly to the case $k = 1$ we create a further part $\dots, \Lambda_{m+2}, \dots, \Lambda_{n-1}$ of the constructed sequence in a certain orientation of \mathbf{C}_2 , up to $\Lambda_n := \Gamma$.

In the second step of induction we assume that Lemma 2 is true for $k = l$. Without loss of generality for hipped roof ends Δ, Γ we consider two cases:

CL1) Δ contains an eaves which belongs to any \mathbf{C}_i with $1 \leq i \leq l$, and Γ contains an eaves which belongs to any \mathbf{C}_j with $1 \leq j \leq l$. This case is easily satisfied because Lemma 2 is true for $k = l$.

CL2) Δ contains an eaves which belongs to any \mathbf{C}_i with $1 \leq i \leq l$ and Γ contains an eaves which belongs to \mathbf{C}_{l+1} . There exist two hipped ends Ω_1, Ω_2 such that Ω_1 contains an eaves which belongs to $\mathbf{C}_{i_0}, 1 \leq i_0 \leq l$; Ω_2 contains an eaves which belongs to \mathbf{C}_{l+1} and $\Omega_1 \cap \Omega_2 \neq \emptyset$.

By inductive assumption for $k = l$ we can construct the subsequence $\Lambda_1 = \Delta, \Lambda_2, \dots, \Lambda_m = \Omega_1, \Lambda_{m+1} = \Omega_2$.

We create the remaining part of the constructed sequence taking successive hipped roof ends in a selected orientation, starting at Ω_2 and finishing at $\Lambda_n := \Gamma$. \square

Now we examine the connectivity of the graph (T, R') , geometrically the line of disappearing ridges. We formulate and prove the following

Proposition 1. *If the roof generated by the k -connected generalized v -polygon is regular, then the graph (T, R') of the line of disappearing ridges is connected.*

Proof. We consider the roof $\mathcal{R}(\mathbf{P})$ generated by the generalized polygon $\mathbf{P}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)$ and with the determined graph (T, R') . In order to prove the connectivity of (T, R') it suffices to construct a path which connects two arbitrary vertices of this graph. Geometrically we must construct a (continuous) broken line included in the line of disappearing ridges which connects any two top points of the roof $\mathcal{R}(\mathbf{P})$.

Consider two vertices T_d, T_g of (T, R') . We assume that they belong to the hipped roof ends Δ, Γ of the roof $\mathcal{R}(\mathbf{P})$, respectively. In the case $\Delta = \Gamma$ due to Lemma 1 we can “run” from T_d to T_g along the sides of the polygon of the hipped roof end $\Delta (= \Gamma)$.

Otherwise according to Lemma 2 there exists a sequence Λ_i such that $\Lambda_1 = \Delta, \Lambda_n = \Gamma, \Lambda_i \cap \Lambda_{i+1} \neq \emptyset$, and any two consecutive hipped roof ends $\Lambda_i, \Lambda_{i+1}, 0 \leq i \leq n$, meet at a ridge. Let r_i denote the common ridge of the hipped roof ends Λ_i, Λ_{i+1} for $i = 1, 2, \dots, v$. It is easy to see that due to Lemma 1 we can pass from the top point T_d to the top point T_1 which belongs to r_1 on the hipped roof end Λ_1 , and which also belongs to Λ_2 (cf. Lemma 2) — we run along the sides of Λ_1 which are disappearing ridges. Next from the top point T_1 we can pass on the hipped roof end Λ_2 to the top point T_2 which belongs to r_2 , and so on. We arrive at T_n , and from T_n to T_g , because $T_n, T_g \in \Gamma$.

We can say that a chimney-sweep who goes along the ridges of a regular roof may pass from an arbitrary top point of this roof to any other top point. \square

The fundamental graph-theoretical properties of roofs as presented above are important for architectural design in order to classify all topological types of roofs (spread over an arbitrary polygon). More useful theorems concerning all possible shapes of roofs will be formulated and proved in a second article.

Acknowledgement

This work has been supported by the Białystok University of Technology through the research grant no. W/IIB/7/03. The author is indebted to Hellmuth STACHEL from Vienna for valuable suggestions and improvements of the editorial layout of the paper.

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Received December 20, 2002; final form June 21, 2004