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On Curvature Properties of Medial Axes in $\mathbb{E}^3$

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Abstract. Here aspects of the geometry of the symmetric axis (= medial axis), which is a subset of Voronoi diagrams, in the Euclidean three-space $\mathbb{E}^3$ are discussed. A relation between the symmetric axis curvature on the one hand and the radius curvature and the boundary curvature on the other is derived.

Key Words: Medial axis in three dimensions, symmetric axis, Voronoi diagrams, curvature of surfaces

MSC 2000: 68U05, 53A05, 51N05

1. Introduction

Shape is a concept of fundamental importance in many disciplines. It remains still difficult to define and even more difficult to measure. H. Blum [3, 4] has introduced a transformation, variously known as the *symmetric axis transform* (SAT), the *medial axis transform*, or the *skeleton*, that induces a unique, coordinate system independent decomposition of a figure into simpler figures. Consequently, the divide-and-conquer strategy can be applied to describe the figure's shape: Divide the figure into several smaller figures, describe each of them, then combine the results to one single description.

In [4] Blum and Nagel propose an elegant method for applying this strategy to describe figures bounded by piecewise smooth, simply closed curves in the plane. Extending their method to 3D figures seems to be very important, e.g., in the field of highway lanes generating and recovery [2]. To do this, it is necessary to generalize the mathematical 2D-tools used by Blum and Nagel to three dimensions. In this paper, we develop the local differential geometry of the symmetric axis in $\mathbb{E}^3$.

We begin by reviewing briefly some important properties of 2D *symmetric axis (= medial axis)* [5, 6]. Let C be a smooth, simply closed curve in $\mathbb{E}^2$ bounding a figure F. The symmetric axis of F is the locus of centers of all maximal discs of F. Equivalently, is C is the outline of F, the symmetric axis $\mathrm{SA}(C)$ is the set of points in F having at least two nearest neighbors on C.

The points of $\mathrm{SA}(C)$ can be classified into three types depending on the position of the point and the number of nearest neighbors on C, which is called *order*. *End points* are of

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order one, *normal points* of order two, and *branch points* of order three or more, corresponding to the maximal discs touching one, two, or more disjoint arcs of C, respectively (see Fig. 1). Additionally, at C we are speaking of *point contact* if each touching maximal disk contacts in a single point only, and otherwise of *finite contact*. We assume that $\mathrm{SA}(C)$ is the union of simple arcs, each a sequence of normal points bounded at each end by a branch or end point, such that different arcs intersect each other only at branch points (Fig. 1).

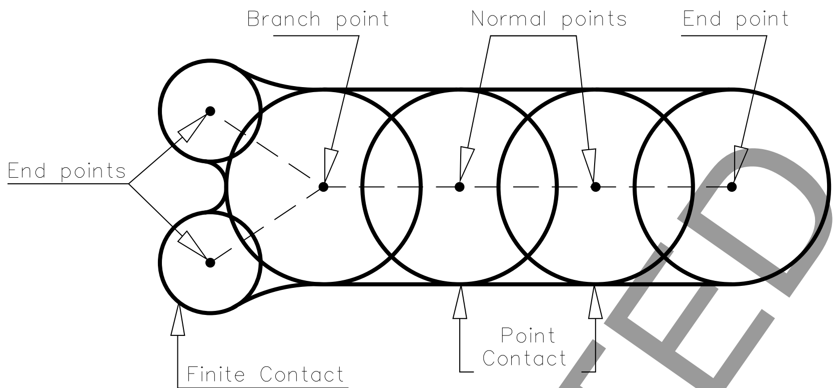

Figure 1: Symmetric axis point types

Let τ be the mapping from C onto $\mathrm{SA}(C)$ that maps a point $P_C \in C$ onto the center of the maximal disc tangent to C at P_C. With each contiguous interval of normal points, which is called *simplified segment*, the inverse relation τ^{-1} associates two disjoint arcs of C. Consequently, C can be decomposed into a collection of pairs of arcs associated with the simplified segments of $\mathrm{SA}(C)$ together with a collection of (possibly degenerate) circular arcs associated with branch and end points.

Choose a direction for traversing a simplified segment and call the two associated arcs of C the *left* and the *right boundary arc*. The angle between the tangent line of C at a point P_C and the tangent line of $\mathrm{SA}(C)$ at $\tau(P_C)$ is called the *object angle*; it is the arcsin of the first derivative of the disc radius at $\tau(P_C)$ with respect to axis arc length (see Fig. 2).

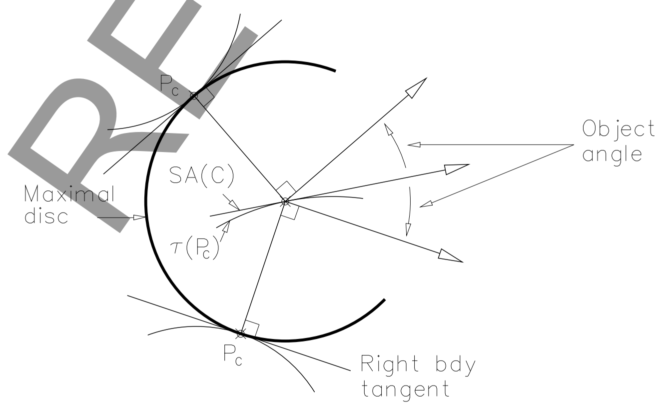

Figure 2: Normal point geometry (point contact)

The algebraic signs of the object angle and its derivative, the *object curvature*, partition the segment into *width shapes* juxtaposed one after the other. The curvature of the simplified segment reflects the degree to which the associated boundary arcs turn into the same direction. The object curvature reflects the symmetry of the associated boundary arcs about the simplified segment. If, for example, the disc radius is held constant while the axis curvature changes, the associated boundary arcs may change from convex to straight to concave in a manner depending on the curvature.

This paper generalizes the explicit functional relationship among the axis curvature, the object curvature, the object angle, and the associated boundary arc curvature to $\mathbb{E}^3$.

The next section defines the SAT in $\mathbb{E}^3$ and gives an intuitive presentation of the major results of the paper. These results are then proved in the following section using results from elementary differential geometry of surfaces. The paper concludes with a brief discussion of the intuitive meanings of radius and axis curvature in the context of shape description.

2. Boundary and symmetric surface curvature relations

In $\mathbb{E}^3$ the *outline* of any spatial figure becomes a smooth, closed surface without self-intersections, and the maximal discs become maximal spheres. In general, the *symmetric axis (= medial axis)* is a surface rather than a curve, though it sometimes degenerates into a space curve or a point. Connected sets of normal points, again called *simplified segments*, are bounded by possibly degenerate space curves of branch or end points. As before, the figure can be decomposed into a collection of paired parts associated with simplified segments, together with pieces of canal surfaces associated with branch and end point curves. A *canal surface* is the envelope of a family of spheres, possibly of varying radius, with centers lying on a space curve. Here, we analyse simplified segments and their associated boundary surfaces.

2.1. Background

First, it is necessary to digress briefly to discuss the curvature of smooth surfaces S in general. Denote the tangent plane of S at P by T_PS. In a small neighborhood of P the curvature of S can be characterized by examining the curvature of curves on S through P. Consider the *normal sections* at P, those curves defined by the intersection of S with planes containing the normal at P. Each normal section is a planar curve, and hence it has a well defined curvature at P measuring its deviation from the tangent line at P. Furthermore, since the tangent line lies in T_PS, the *normal section curvature* also measures the deviation of S from T_PS in the direction of the tangent line. By rotating the defining plane about the normal, we get all normal sections and their curvatures, and hence a complete characterization of the deviation of the surface from its tangent plane (Fig. 3).

To express all normal section curvatures in a finite way, we arbitrarily call one side of the tangent plane *positive* and the other *negative*, and attach a sign to the normal section curvatures according to whether the normal section lies on the positive or negative side of the tangent plane. It can then be shown that while the defining plane is rotating about the normal, either the normal section curvature assumes maximum and minimum values, called *principal curvatures*, in two orthogonal directions, called *principal directions*, or the normal section curvature is constant. Furthermore, each normal section curvature is completely determined by the principal curvatures and the angle between the defining plane and the principal directions.

The product of the principal curvatures is called *Gaussian curvature* K_S of S at P, while their average is its *mean curvature* H_S. The behavior of S at P is characterized by the signs of the Gaussian and mean curvature. For $K_S > 0$, in a local neighborhood of P, all normal sections lie on one side of the tangent plane, the side determined by the sign of the mean curvature. The surface is *cup-shaped* at P. On the other hand, for $K_S < 0$ the normal sections

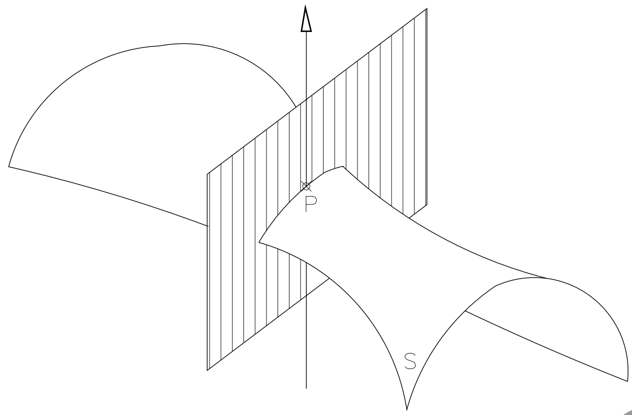

Figure 3: Geometry of surface curvature

about one principal direction lie above the tangent plane and those about the other lie below, giving S a *saddle shape* at P. The remaining case, $K_S = 0$, is a transition between the two cases: in one principal direction the surface has flattened while in the other it may remain curved. When both principal curvatures are zero, S is planar at the point and the principal directions cease to exist.

2.2. Characterization of sphere radius

Let S be a simplified segment surface of any symmetry axis in $\mathbb{E}^3$ and let $P \in S$ be a normal point with point contact, i.e., the maximal sphere centered at P touches the outline in two disjoint points, sometimes called the *associated boundary points*. Furthermore, we assume that S and the radius function r (defined precisely below) are twice continuously differentiable at P.

We now turn to characterize the behavior of the sphere radius. In 2D the disc radius was a function of a single parameter, the arc length of the symmetric axis. Unfortunately, in 3D one parameter is not sufficient. Instead, we examine the first and second derivatives of the radius function along curves in infinitely many directions through the point P.

Pick any direction through P. Then the *first directional derivative* of the radius function at P in the specified direction is the first derivative with respect to the arc length along any curve of S tangent to that direction. It is easy to show that this first directional derivative is independent from the choice of the curve in the specified direction.

Similarly, the *second directional derivative* of the radius function at P in the specified direction can be defined to be the second derivative of the radius function with respect to arc length along the curve. Unfortunately, this is not well defined without constraining the choice of the curve. Since we are interested in the behavior of the radius function, not in the curvature of the curve in S, we require the curve to be "straight" in a small neighborhood of P. More precisely, we require that in an infinitesmal neighborhood about P, the orthogonal projection of the curve onto $T_P S$ is a line in the specified direction. There is a unique curve, called a *geodesic*, that satisfies this condition. Hence, we define the second directional derivative of the radius function with respect to the arc length along the geodesic in that direction.

Below, we prove that, like normal section curvatures, the second directional derivative of the radius function assumes its maximum and minimum values in two orthogonal directions which by analogy will be called *principal curvatures* and *principal directions of the radius* *function*, respectively. Furthermore, the second directional derivative in any direction is completely determined by the principal curvatures and the angle between this direction and a principal direction. We also define the *Gaussian* and the *mean curvature of the radius function* analogously and denote them by K_R and H_R.

We can now state our goal more precisely. We seek a functional relationship among the Gaussian and the mean curvature of S at P, the Gaussian and the mean curvature of the outline at the associated boundary points, and the Gaussian and the mean curvature of the radius function at P.

2.3. Definitions and notation

We begin by imposing a local curvilinear coordinate systems about normal points on a simplified segments S in $\mathbb{E}^3$, thus bringing the techniques of calculus to bear. Except at finite contact normal points, which we ignor hereafter, we assume S to be a C^2 surface. Hence, if U is an open subset of $\mathbb{R}^2$ with coordinates u^1, u^2, we let $\mathbf{s}\colon U \to S$ be a C^2 surface patch on S with linearly independent partial derivatives $\mathbf{s}_i = \partial\mathbf{s}/\partial u^i$ called *coordinate vectors*.

Choose a set of basis vectors in $\mathbb{E}^3$ and let $\mathbf{Y}$ and $\mathbf{Z}$ be two vectors represented in terms of that basis. To distinguish between a vector $\mathbf{X}$ and the n-tuple that represents it with respect to some basis, we denote the n-tuple by X. Then an inner product of $\mathbf{Y}$ and $\mathbf{Z}$ denoted by $\langle \mathbf{Y}, \mathbf{Z} \rangle$, is given by $Y^T G Z$, where G is a 3 by 3 matrix such that $\langle \mathbf{Y}, \mathbf{Z} \rangle = \langle \mathbf{Z}, \mathbf{Y} \rangle$ and $\langle \mathbf{Y}, \mathbf{Y} \rangle > 0$ for all nonzero $\mathbf{Y}$. For the remainder of this paper, we will use the particular inner product defined by $G = I$ (the identity matrix) when the basis vectors are orthonormal. This is nothing else than the dot product $Y^T Z$ in $\mathbb{E}^3$. Though the representation of the inner product $\langle\ ,\ \rangle$ depends on the basis vectors chosen, the inner product itself is basis independent.

It is always possible to choose $\mathbf{s}$ so that the coordinate vectors are orthonormal at the point $P = \mathbf{s}(0,0)$. Thus, without loss of generality, we choose $\mathbf{s}$ so that $\langle \mathbf{s}_i(0,0),\ \mathbf{s}_j(0,0) \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

The tangent plane of S at $\mathbf{s}(u^1, u^2)$ is a two-dimensional subspace of $\mathbb{R}^3$ spanned by the coordinate vectors $\mathbf{s}_1$ and $\mathbf{s}_2$. Consequently, the unit normal $\mathbf{n}_S(u^1, u^2)$ at $\mathbf{s}(u^1, u^2)$ is $(\mathbf{s}_1 \times \mathbf{s}_2)/|\mathbf{s}_1 \times \mathbf{s}_2|$.

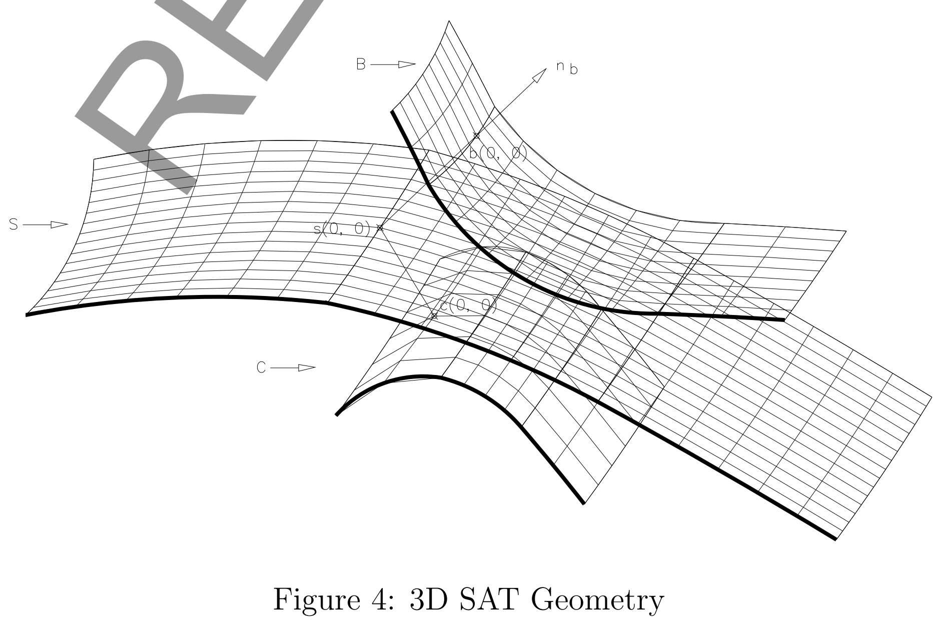

Figure 4: 3D SAT Geometry

Similarly, let B and C be the boundary surfaces associated with S as shown in Fig. 4. Let $\mathbf{b}(u^1,u^2)$ and $\mathbf{c}(u^1,u^2)$ be the points on B and C associated with $\mathbf{s}(u^1,u^2)$, and let $r\colon S \to \mathbb{R}$ map a point on S to the radius of the maximal sphere centered at that point.

The maximal sphere centered at $\mathbf{s}(u^1,u^2)$ is tangent to the boundary surface B at $\mathbf{b}(u^1,u^2)$ with the boundary normal $\mathbf{n}_b(u^1,u^2)$, lying along a diameter of the sphere. The direction of $\mathbf{n}_b$ is supposed to point away from S as shown in Fig. 4. Letting $r(u^1,u^2)$ denote $r(\mathbf{s}(u^1,u^2))$, then

$$\mathbf{b}(u^1,u^2) = \mathbf{s}(u^1,u^2) + r(u^1,u^2)\,\mathbf{n}_b(u^1,u^2). \tag{1}$$

Similarly,

$$\mathbf{c}(u^1,u^2) = \mathbf{s}(u^1,u^2) + r(u^1,u^2)\mathbf{n}_c(u^1,u^2). \tag{2}$$

Let $\alpha(t)\colon\ I \subset \mathbb{R} \ \to\ S$ be the geodesic on S passing through P, where I is some interval of $\mathbb{R}$ containing 0, t is the arc length along the curve, and $\alpha(0) = P$. Let $\mathbf{X} = d\alpha/dt(0)$ be the tangent vector of α at P. Since α is parametrized by its arc length and lies on S, $\mathbf{X}$ is a unit vector in the tangent plane T_PS of S at P.

Definition 1. *The first and second directional derivatives of $r(u^1,u^2)$ at P in the direction of $\mathbf{X}$ are, respectively,*

$$r_{\mathbf{X}} = \frac{dr(\alpha)}{dt}(0) \quad \textit{and} \quad r_{\mathbf{XX}} = \frac{d^2r(\alpha)}{dt^2}(0).$$

We let λ_1,λ_2 for $\lambda_1 \le \lambda_2$ denote the principal curvature of S at P and let $\mathbf{e}_1,\mathbf{e}_2$ be the unit vectors in the corresponding principal directions. Since each principal direction is determined by a line in T_PS, the orientations of $\mathbf{e}_1$ and $\mathbf{e}_2$ are still to choose. As shown below, we can without loss of generality require that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}_s$; the results of this paper are independent of the choice made from the remaining two possibilities. Similarly, let γ_1,γ_2 and $\mathbf{f}_1,\mathbf{f}_2$ denote the principal curvatures and principal directions of the radius function.

2.4. Boundary curvature equations

In two dimensions, the object angle, i.e., the angle between the tangent line of SA at a point P and the tangent line at the associated boundary point is determined by the arcsin of the first derivative of the radius function. A similar relation holds in three dimensions. The following statement will be proved later (page 138):

Lemma 1. *Let $\mathbf{X}$ be a unit vector in T_PS. Then, the directional derivative of the radius function $r(u^1,u^2)$ in the $\mathbf{X}$-direction is $r_{\mathbf{X}} = -\langle \mathbf{n}_b, \mathbf{X}\rangle$.*

That is, in 3D the angle between a symmetric surface tangent vector at a normal point and the normal at the associated boundary point is determined by arccos of the first directional derivative of the radius function in direction of the tangent vector. An analogous result holds for $\mathbf{n}_c$.

The major result of the paper follows:

Theorem 2. *Let*

$$h := \frac{\gamma_1(1-r^2_{\mathbf{f}_2}) + \gamma_2(1-r^2_{\mathbf{f}_1})}{2\,\langle \mathbf{n}_s,\mathbf{n}_b\rangle^2} + \frac{\lambda_1(1-r^2_{\mathbf{e}_2}) + \lambda_2(1-r^2_{\mathbf{e}_1})}{2\,\langle \mathbf{n}_s,\mathbf{n}_b\rangle^2}\,, \tag{3}$$

and

$$k := \lambda_1\lambda_2 + \frac{\gamma_1\gamma_2}{\langle \mathbf{n}_s,\mathbf{n}_b\rangle^2} + \frac{\lambda_1 r_{\mathbf{e}_2\mathbf{e}_2} + \lambda_2 r_{\mathbf{e}_1\mathbf{e}_1}}{\langle \mathbf{n}_s,\mathbf{n}_b\rangle}\,. \tag{4}$$

Then the Gaussian and the mean curvature of the boundary surface B at $\mathbf{b}(0,0)$ *are*

$$H_B = \frac{h - rk}{1 - 2rh + r^2k} \quad \textit{and} \quad K_B = \frac{k}{1 - 2rh + r^2k}\,. \tag{5}$$

These equations express the Gaussian and mean curvatures of the boundary surface B in terms of the radius and the symmetric surface S, together with the angle between the boundary normal $\mathbf{n}_b$ and the symmetric surface normal $\mathbf{n}_s$. Analogous equations for the boundary surface C are obtained when the subscripts b and B are replaced by c and C, respectively.

At a first glance, it appears that the knowledge of the boundary normal is a prerequisite to evaluate h and k, and hence the boundary curvatures. This is not the case. Since $\mathbf{n}_s, \mathbf{e}_1$, and $\mathbf{e}_2$ are orthonormal, $\langle \mathbf{n}_s, \mathbf{n}_b\rangle^2 + \langle \mathbf{n}_b, \mathbf{e}_1\rangle^2 + \langle \mathbf{n}_b, \mathbf{e}_2\rangle^2 = 1$. Hence, up to the sign, $\langle \mathbf{n}_s, \mathbf{n}_b\rangle$ is determined by $r_{\mathbf{e}_1}$ and $r_{\mathbf{e}_2}$.

Specifying the sign of $\langle \mathbf{n}_s, \mathbf{n}_b\rangle$ means choosing one of the boundary surfaces B or C. As symmetry suggests and Lemma 1 reveals, $\mathbf{n}_b$ and $\mathbf{n}_c$ are mirror images of each other with respect to the tangent plane of the symmetric surface. Thus by symmetry, $\langle \mathbf{n}_s, \mathbf{n}_b\rangle = \langle \mathbf{n}_c, -\mathbf{n}_s\rangle$ and hence $\langle \mathbf{n}_s, \mathbf{n}_b\rangle = -\langle \mathbf{n}_c, \mathbf{n}_s\rangle$. Consequently, if we replace $\langle \mathbf{n}_s, \mathbf{n}_b\rangle$ by $\pm\langle \mathbf{n}_s, \mathbf{n}_b\rangle$ in (3) and (4), the curvature relations hold for both boundary surfaces.

To discuss the geometric significance of h and k, consider the surface B' defined by

$$\mathbf{b}'(u^1, u^2) = \mathbf{s}(u^1, u^2) + r'(u^1, u^2)\mathbf{n}_b(u^1, u^2),$$

where $r'(u^1, u^2) = r(u^1, u^2) - r(0,0)$. B' passes through the point $P = \mathbf{s}(0,0)$ and at each (u^1, u^2) B and B' share the unit normal vector. B' and B are called *parallel surfaces* (see Fig. 5). Since the derivatives of r' and r are identical, we can evaluate (5) at $(0,0)$. Substituting r' for r gives $k = K_B$ and $h = H_B$. Thus, the terms h and k in (5) are the mean and Gaussian curvature, respectively, of the surface parallel to B passing through P. Therefore, (5) expresses the dependence of the boundary curvature from the distance to the symmetric surface. Blum and Nagel [3, 4] use a similar relationship in the two-dimensional case to derive the boundary curvature from the parallel curve curvature. Analogous results hold for the surface parallel to C through P when the sign of $\langle \mathbf{n}_s, \mathbf{n}_b\rangle$ is changed.

Both, the symmetric surface curvature and the radius function are involved in the curvature of the boundary surfaces. Examining each alone reveals different aspects of the boundary surface. Intuitively, symmetric surface curvature reflects the overall "curvatur trend" of the pairs of associated parts, i.e., the degree to which the boundary surfaces are turning in the same direction. The radius curvature, on the other hand, reflects the symmetry of the boundary surfaces about the symmetric surface, i.e., the degree to which both boundary surfaces are curved in opposite directions.

To see this, note that in (3) the symmetric surface curvatures λ_1 and λ_2 contribute with equal magnitude but opposite sign to the mean curvature of the two boundary surface, while the radius curvatures γ_1 and γ_2 contribute equally to each. Since the boundary surface normals are directed away from the symmetric surface, boundary surface mean curvatures of opposite sign imply curvature in the same direction. Furthermore, it can be shown that the signs of the Gaussian and mean curvatures of each boundary surface are equal to the signs of the curvatures of the corresponding parallel surface. Hence, our intuitive notions of the meanings of the symmetric surface curvature and the radius curvature are confirmed.

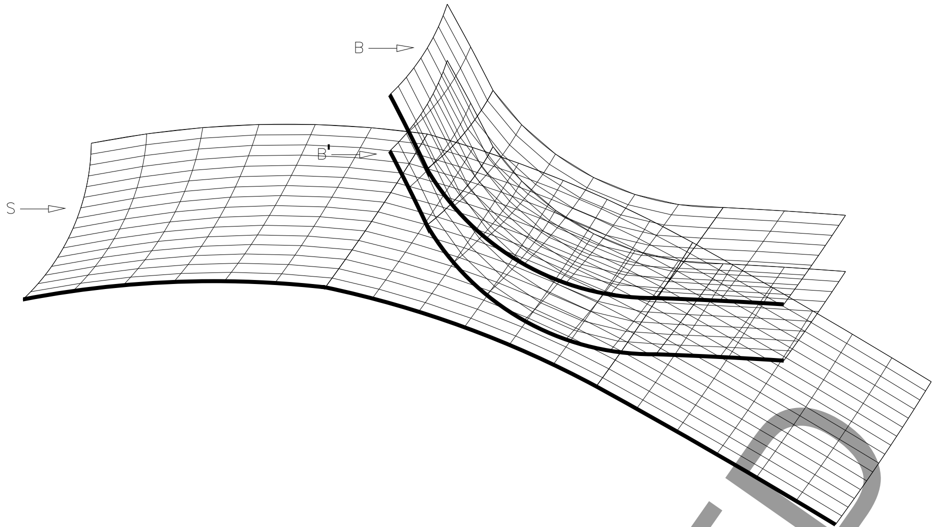

Figure 5: Surface parallel to the boundary surface

3. Proof of curvature relations

In this section we prove the results presented in Section 2.4, using results from elementary differential geometry of surfaces.

3.1. Quadratic curvature forms

First, we show that the second directional derivative of the radius function is a quadratic form. Hence, by properties of quadratic forms the principal curvatures and principal directions exist and behave as claimed in Section 2.2.

Lemma 3. *$r_{\mathbf{XX}}$ is a quadratic form over the unit vectors $\mathbf{X}$ in T_PS.*

Proof: Let $\alpha(t) = \mathbf{s}(\alpha^1(t), \alpha^2(t))$ be given. Then, since T_PS is a vector space spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$, there are scalars X^i such that $\mathbf{X} = \sum_{i=1}^2 X^i\mathbf{s}_i$. Using the chain rule, $d\alpha/dt = \sum_{i=1}^2(d\alpha^i/dt)\mathbf{s}_i$, so $(d\alpha^i/dt)(0) = X^i$. Applying the chain rule again,

$$r_{\mathbf{X}} = \frac{dr(\alpha)}{dt}(t) = \sum_{i=1}^{2} \frac{\partial r(\mathbf{s})}{\partial u^i}\,\frac{d\alpha^i}{dt}\,.$$

Differentiating and substituting X^i for $d\alpha^i/dt$,

$$r_{\mathbf{XX}} = \frac{d^2r(\alpha)}{dt^2}(t) = \sum_{i=1}^{2} \frac{\partial r(\mathbf{s})}{\partial u^i}\,\frac{d^2\alpha^i}{dt^2} + \sum_{i=1}^{2}\sum_{j=1}^{2} X^iX^j\,\frac{\partial^2 r(\mathbf{s})}{\partial u^i\partial u^j}\,.$$

The geodesic α is characterized by the differential equations

$$\frac{d^2\alpha^k}{dt^2} = -\sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij}^k\frac{d\alpha^i}{dt}\,\frac{d\alpha^j}{dt}\,, \quad k = 1, 2,$$

where the Γ_{ij}^k measure the tangential components of the second partial derivatives $\mathbf{s}_{ij}$. Combining the last two equations, denoting $\partial r(\mathbf{s})/\partial u^i$ by r_i and $\partial^2 r(\mathbf{s})/\partial u^i \partial u^j$ by r_{ij}, and rearranging terms, we see that since $r_{ij} = r_{ji}$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$, $r_{\mathbf{XX}}$ is a quadratic form in $\mathbf{X}$,

$$r_{\mathbf{XX}} = Q(X) = X^T Q X \tag{6}$$

with

$$Q = [q_{ij}] = \left[r_{ij} - \sum_{k=1}^{2} r_k \Gamma_{ij}^k \right]. \qquad \square \tag{7}$$

Since Q represents the quadratic form $Q(\mathbf{X})$ with respect to an orthonormal basis of $T_P S$, over all unit vectors $\mathbf{X}$ in $T_P S$, $Q(\mathbf{X})$ assumes its minimum value at the eigenvector of Q corresponding to the smallest eigenvalue γ_1 and its maximum value at the eigenvector corresponding to the largest eigenvalue γ_2. Furthermore, the values assumed are γ_1 and γ_2, respectively, and the eigenvectors are orthogonal if the eigenvalues are distinct. By solving the characteristic equation of Q, it is easy to see that $\gamma_1\gamma_2 = \det Q$ and $\gamma_1 + \gamma_2 = \operatorname{tr} Q$.

Similarly, the *second fundamental form* $II(\mathbf{X})$ of S is a quadratic form over unit vectors in $T_P S$ that gives the curvature of the normal section in the direction $\mathbf{X}$. Letting $L_s = [L_{sij}]$ be the matrix defining the second fundamental form with respect to the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ of $T_P S$, we have $II(\mathbf{X}) = X^T L_s X$.

Thus, two quadratic forms are defined at each point P of S. One, the second fundamental form, gives the curvature of normal sections through P in any direction, while the other gives the second derivative of the radius along the geodesic in the same direction. Since the normal to a geodesic is everywhere normal to the surface on which it lies, the geodesic and the normal section share a common normal vector. By construction, they have the same tangent vector and hence, the same curvature. Therefore, one quadratic form measures the curvature of S along the geodesic and the other measures the second derivative of the radius function along the same geodesic.

3.2. Matrix formation

In this section, we derive a relation between the matrices Q and L_s that determine the radius curvature and the symmetric surface curvature, respectively, to the matrix defining the second fundamental form and hence the curvature of each boundary surface.

The partial derivatives of (1) are

$$\mathbf{b}_i = \mathbf{s}_i + r_i\, \mathbf{n}_b + r\, \mathbf{n}_{bi}\,. \tag{8}$$

We can solve them for r_i by taking the inner product with $\mathbf{n}_b$. Since $\mathbf{n}_b$ is a vector of constant norm, it is perpendicular to its derivative $\mathbf{n}_{bi}$. Thus, since $\mathbf{b}_i$ is perpendicular to $\mathbf{n}_b$ by definition,

$$r_i = -\langle \mathbf{s}_i, \mathbf{n}_b \rangle. \tag{9}$$

We take partial derivatives again and get

$$r_{ij} = -\langle \mathbf{s}_{ij}, \mathbf{n}_b \rangle - \langle \mathbf{s}_i, \mathbf{n}_{bj} \rangle.$$

Using the *Gauss's equation* $\mathbf{s}_{ij} = L_{s\,ij}\mathbf{n}_s + \sum_{k=1}^{2} \Gamma_{ij}^k \mathbf{s}_k$ and the definition of the coefficients of the second fundamental form, $L_{s\,ij} = \langle \mathbf{s}_{ij}, \mathbf{n}_s \rangle$, we obtain

$$r_{ij} = -L_{s\,ij} \langle \mathbf{n}_s, \mathbf{n}_b \rangle - \sum_{k=1}^{2} \Gamma_{ij}^k \langle \mathbf{s}_k, \mathbf{n}_b \rangle - \langle \mathbf{s}_i, \mathbf{n}_{bj} \rangle. \tag{10}$$

Analogus results for boundary surface C follow from (2), though for brevity we defer further consideration of C until the end of this section.

The matrices $G_b = [G_{b\,ij}] = [\langle \mathbf{b}_i, \mathbf{b}_j \rangle]$ and $L_b = [L_{b\,ij}]$ represent the first and second fundamental form of B at $\mathbf{b}(u^1, u^2)$ with respect to $\{\mathbf{b}_1(u^1, u^2), \mathbf{b}_2(u^1, u^2)\}$. Since $\mathbf{n}_b$ is a vector of constant norm, the $\mathbf{n}_{bj}$ can be expressed as a linear combination of the $\mathbf{b}_i$ by *Weingarten's equations*

$$\mathbf{n}_{bj} = -\sum_{i=1}^{2} W^i_{bj} \mathbf{b}_i\,, \quad \text{where} \quad W_b = \left[W^i_{bj}\right] = G_b^{-1} L_b\,. \tag{11}$$

Letting $A = [\langle \mathbf{s}_i, \mathbf{b}_j \rangle]$ and combining Weingarten's equations with (7), (9), and (10), we obtain

$$AW_b = [r_{ij}] + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s - \sum_{k=1}^{2} r_k \left[\Gamma^k_{ij}\right] = Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s\,. \tag{12}$$

Equation (12) relates the boundary curvature expressed by W_b to the radius curvature expressed by Q, and to the symmetric surface curvature expressed by L_s. We seek the boundary curvatures in terms of the radius and the symmetric surface. Our approach is to solve the matrix equation (12) for the two invariants, the determinant and trace. We then solve the resulting two equations simultaneously.

3.3. Determinant equations

Substitute Weingarten's eqs. (11) into (8) and solving for the $\mathbf{s}_j$ gives

$$\mathbf{s}_1 = (1 + rW^1_{b1})\mathbf{b}_1 + rW^2_{b1}\mathbf{b}_2 - r_1\mathbf{n}_b\,, \tag{13}$$

$$\mathbf{s}_2 = rW^1_{b2}\mathbf{b}_1 + (1 + rW^2_{b2})\mathbf{b}_2 - r_2\mathbf{n}_b\,. \tag{14}$$

Recalling that $A = [\langle \mathbf{s}_i, \mathbf{b}_j \rangle]$ and defining

$$T = \begin{bmatrix} 1 + rW^1_{b1} & rW^2_{b1} \\ rW^1_{b2} & 1 + rW^2_{b2} \end{bmatrix},$$

we use (13) and (14) to obtain $A = TG_b$ and consequently, since $W_b = G_b^{-1}L_b$, that $AW_b = TL_b$. Substituting into (12) gives

$$TL_b = Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s. \tag{15}$$

To evaluate the determinant of the left side of (15), we use Lemma 1 (which we now prove) and the following Lemma 4.

Proof of Lemma 1: Let X^1 and X^2 be the components of $\mathbf{X}$ in the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$, i.e., $\mathbf{X} = \sum_{i=1}^{2} X^i \mathbf{s}_i$. So, $\langle \mathbf{n}_b, \mathbf{X} \rangle = \sum_{i=1}^{2} X^i \langle \mathbf{n}_b, \mathbf{s}_i \rangle$ which by (9) is $-\sum_{i=1}^{2} X^i r_i$. Thus by the proof of Lemma 3, $r_{\mathbf{X}} = -\langle \mathbf{n}_b, \mathbf{X} \rangle$. □

Lemma 4. $g_b = \det G_b$ *implies* $g_b \det^2 T = \langle \mathbf{n}_s, \mathbf{n}_b \rangle^2$.

Proof. Recall that $[\langle \mathbf{s}_i, \mathbf{s}_j \rangle] = I$, where I is the two-by-two unit matrix. Then, using (13) and (14), by straightforward algebra, it is not difficult to show that $TG_bT^T = I - \begin{bmatrix} r_1^2 & r_1r_2 \\ r_1r_2 & r_2^2 \end{bmatrix}$ implies

$$(TG_bT^T)^{-1} = \frac{1}{g_b \det^2 T}(I - R) \quad \text{where} \quad R = \begin{bmatrix} r_2^2 & -r_1r_2 \\ -r_1r_2 & r_1^2 \end{bmatrix}. \tag{16}$$

We take the determinant of both sides in (16) and apply Lemma 1,

$$g_b \det{}^2 T = 1 - r_1^2 - r_2^2 = 1 - \langle \mathbf{n}_b, \mathbf{s}_1\rangle^2 - \langle \mathbf{n}_b, \mathbf{s}_2\rangle^2 = \langle \mathbf{n}_s, \mathbf{n}_b\rangle^2,$$

where the last step follows because $\mathbf{n}_s$, $\mathbf{s}_1$, and $\mathbf{s}_2$ are orthonormal. □

Thus the determinant of the left side of (15) is

$$\det(TL_b) = \det(TG_b)\det(G_b^{-1}L_b) = \frac{\langle \mathbf{n}_s, \mathbf{n}_b\rangle^2 \det(G_b^{-1}L_b)}{\det T} = \frac{\langle \mathbf{n}_s, \mathbf{n}_b\rangle^2 K_B}{\det T}, \tag{17}$$

where $K_b = \det W_b$ is the Gaussian curvature of B.

We now evaluate the determinant of the right side of (15). Recalling that the determinant is invariant under change of basis, we change from the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ of T_PS to that defined by the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of L_s corresponding to the eigenvalues λ_1 and λ_2, respectively. Since eigenvectors are determined only up to a nonzero multiplicative constant and since $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the tangent plane T_PS and are orthogonal to each other, we can, without loss of generality, choose $\mathbf{e}_i$ to be unit vectors so that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}_s$. Similarly, let $\mathbf{f}_1$ and $\mathbf{f}_2$ be unit vectors of Q corresponding to the eigenvalues γ_1, γ_2 so that $\mathbf{f}_1 \times \mathbf{f}_2 = \mathbf{n}_s$. In terms of their respective eigenvectors bases, the transformations represented by L_s and Q in terms of the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ are represented by $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, i.e., $L_s \approx \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $Q \approx \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, where $\approx$ denotes matrix similarity.

Representing both transformations in terms of the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ requires examinating the relationship between $\mathbf{e}_i$ and $\mathbf{f}_i$. Let θ be the counterclockwise angle from $\mathbf{e}_1$ to $\mathbf{f}_1$. Then, with respect to the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$, $\mathbf{e}_i = \Theta\mathbf{f}_i$, where $\Theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. As shown in Fig. 6, θ is determined only up to a multiple of π; thus, Θ is determined only up to sign.

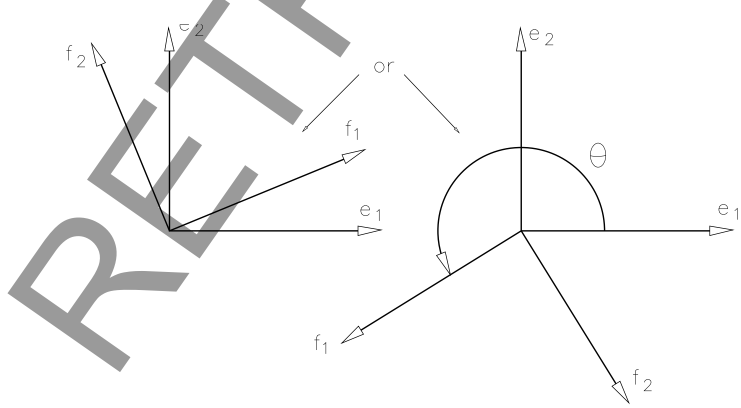

Figure 6: Relation between principal directions

Changing from the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$,

$$\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \approx \pm\Theta^{-1}\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}(\pm\Theta) = \Theta^T\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}\Theta.$$

Therefore, $Q + \langle \mathbf{n}_s, \mathbf{n}_b\rangle L_s$ is similar to

$$\Theta^T\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}\Theta + \langle \mathbf{n}_s, \mathbf{n}_b\rangle\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

which is easily seen to have a determinant of

$$\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 \lambda_1 \lambda_2 + \gamma_1 \gamma_2 + \langle \mathbf{n}_s, \mathbf{n}_b \rangle (\lambda_1 \gamma_1 + \lambda_2 \gamma_2 - (\gamma_1 - \gamma_2)(\lambda_1 - \lambda_2) \cos^2 \theta). \tag{18}$$

Note that (18) is independent of θ if either $\gamma_1 = \gamma_2$ or $\lambda_1 = \lambda_2$. Consequently, when one pair of eigenvalues fails to be distinct, the principal directions can be chosen arbitrarily.

Combining (17) and (18) and rearranging terms,

$$\frac{K_B}{\det T} = \lambda_1 \lambda_2 + \frac{\gamma_1 \gamma_2}{\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2} + \frac{\lambda_1(\gamma_1 \sin^2 \theta + \gamma_2 \cos^2 \theta) + \lambda_2(\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta)}{\langle \mathbf{n}_s, \mathbf{n}_b \rangle}. \tag{19}$$

Recall that $\mathbf{e}_1 = \mathbf{f}_1 \cos\theta - \mathbf{f}_2 \sin\theta$ and $\mathbf{e}_2 = \mathbf{f}_1 \sin\theta + \mathbf{f}_2 \cos\theta$. Equation (19) can be simplified by observing that $Q(\mathbf{e}_1) = [\cos\theta \;\; -\sin\theta] \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} [\cos\theta \;\; -\sin\theta]^T = \gamma_1 \cos^2\theta + \gamma_2 \sin^2\theta$ and $Q(\mathbf{e}_2) = \gamma_1 \sin^2\theta + \gamma_2 \cos^2\theta$. Hence, by (6) and (19) we obtain

$$\frac{K_B}{\det T} = \lambda_1 \lambda_2 + \frac{\gamma_1 \gamma_2}{\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2} + \frac{\lambda_1 r_{\mathbf{e}_2 \mathbf{e}_2} + \lambda_2 r_{\mathbf{e}_1 \mathbf{e}_1}}{\langle \mathbf{n}_s, \mathbf{n}_b \rangle}. \tag{20}$$

3.3.1. Trace equations

The second equation relating the boundary curvature to radius and symmetric surface curvature results from taking the trace of (12). It follows from (12) and (16) that

$$\left(g_b \det^2 T\right) (T^T)^{-1} W_b = (Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s) - R(Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s).$$

Hence, since $\operatorname{tr} W_b = 2H_B$, $\det W_b = K_B$, $\operatorname{tr} Q = 2H_R$, and $\operatorname{tr} L_s = 2H_S$, taking the trace of both sides gives

$$(2 g_b \det T)(r K_B + H_B) = 2(H_R + \langle \mathbf{n}_s, \mathbf{n}_b \rangle H_S) - \operatorname{tr}(RQ) - \langle \mathbf{n}_s, \mathbf{n}_b \rangle \operatorname{tr}(R L_s). \tag{21}$$

Two observations enable us to evaluate $\operatorname{tr}(RL_s)$ and, by analogous reasoning, $\operatorname{tr}(RQ)$. First, simple algebra reveals that $\operatorname{tr}(RL_s)$ is nothing more than the second fundamental form of S, evaluated at $[r_2 \; -r_1]$, i.e., $[r_2 \; -r_1] \, L_s \, [r_2 \; -r_1]^T$. Second, with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ the second fundamental form is represented by the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Hence, letting $[a^1 \; a^2]$ represent, with respect to $\{\mathbf{e}_1, \mathbf{e}_2\}$ the vector represented by $[r_2 \; -r_1]$ in the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$, $\operatorname{tr}(RL_s) = [a^1 \; a^2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [a^1 \; a^2]^T$.

Let V be the matrix of transition from the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, i.e. the matrix such that $[r_2 \; -r_1]^T = V [a^1 \; a^2]^T$. Since the columns of V as the coordinates of the $\mathbf{e}_i$ are orthonormal, $V^T V = I$. Thus, $\det V = \pm 1$ which is nonzero. Therefore, we can solve for $[a^1 \; a^2]$ obtaining $\pm [r_1 V_{12} + r_2 V_{22} \;\; -r_1 V_{11} - r_2 V_{21}]$. Since by the definition of V, $\mathbf{e}_i = \sum_{j=1}^2 V_{ji} \mathbf{s}_j$, by using (9) we see that $[a^1 \; a^2] = \pm [-\langle \mathbf{n}_b, \mathbf{e}_2 \rangle \; \langle \mathbf{n}_b, \mathbf{e}_1 \rangle]$ and hence, that $\operatorname{tr}(RL_s) = (\lambda_1 \langle \mathbf{n}_b, \mathbf{e}_2 \rangle^2 + \lambda_2 \langle \mathbf{n}_b, \mathbf{e}_1 \rangle^2)$. Analogously, $\operatorname{tr}(RQ) = (\gamma_1 \langle \mathbf{n}_b, \mathbf{f}_2 \rangle^2 + \gamma_2 \langle \mathbf{n}_b, \mathbf{f}_1 \rangle^2)$. Finally, combining these results with (21), Lemma 4, and the definition of mean curvature as the average of principal curvatures, we obtain

$$\begin{aligned} 2\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 \frac{r K_B + H_B}{\det T} &= \gamma_1 (1 - \langle \mathbf{n}_b, \mathbf{f}_2 \rangle^2) + \gamma_2 (1 - \langle \mathbf{n}_b, \mathbf{f}_1 \rangle^2) \\ &+ \langle \mathbf{n}_s, \mathbf{n}_b \rangle (\lambda_1 (1 - \langle \mathbf{n}_b, \mathbf{e}_2 \rangle^2) + \lambda_2 (1 - \langle \mathbf{n}_b, \mathbf{e}_1 \rangle^2)), \end{aligned} \tag{22}$$

which, using Lemma 1, can be written as

$$\frac{r K_B + H_B}{\det T} = \frac{\gamma_1 (1 - r_{\mathbf{f}_2}^2) + \gamma_2 (1 - r_{\mathbf{f}_1}^2)}{2 \langle \mathbf{n}_s, \mathbf{n}_b \rangle^2} + \frac{\lambda_1 (1 - r_{\mathbf{e}_2}^2) + \lambda_2 (1 - r_{\mathbf{e}_1}^2)}{2 \langle \mathbf{n}_s, \mathbf{n}_b \rangle}. \tag{23}$$

3.4. Solution

The right sides of Eqs. (23) and (20) are the right sides of (3) and (4), respectively. Recall that $K_B = \det W_b$ and $H_B = \frac{1}{2}\operatorname{tr} W_b$. Then, by straightforward algebra, $\det T = 1 + r^2 K_B + 2rH_B$. Substituting this into (20) and (23), we obtain a linear system of two equations in the two unknowns H_B and K_B, with solutions (5). This proves Theorem 2.

4. Summary and conclusions

Blum's symmetric axis transform defines a unique decomposition of a figure into disjoint pairs of pieces, each with its own surface (axis) of symmetry and associated boundary surfaces. In previous sections of this paper, we have defined measures of the radius function and have shown how these measures and the symmetric surface curvature are related to the boundary surface curvatures. In particular, we have shown that Gaussian and mean curvatures of the boundary surfaces are determined by nine measures, each with a geometric interpretation:

(1) the symmetric surface curvature as determined by two principal curvatures and a principal direction;

(2) the radius curvature as determined by two principal curvatures and a principal direction;

(3) directional derivatives of the radius function as determined by the angles between either boundary normal and the two symmetric surface principal directions, called *width angles*; and

(4) the radius function itself.

Other, equivalent sets of measures are easily found. It can also be shown that these measures, and the curvature relationship derived from them, subsume the two-dimensional measures and curvature relationship given by Blum and Nagel [3, 4].

It appears possible to use the measures defined here to further partition a simplified segment into a set of canonical pairs of pieces, yielding a symbolic description, much as Blum and Nagel have done in two dimensions. Before such a symbolic description can be obtained, however, further work needs to be done in two directions. First, though the assumptions we have made about the topology of the symmetric surface seem to be true for smooth outlines in general, a definitive study of the symmetric surface topology is lacking. We have also completely ignored the analysis of finite contact normal points, branch points, and end points. Second, a suitable algorithm for computing symmetric surfaces of three-dimensional figures is required in order to evaluate which of several possible schemes is suitable for specific three-dimensional shape description problems. This is already implemented in Pascal programming as in [1]. Indeed, the purpose of this paper is not to propose specific features for three-dimensional shape description, but rather to provide mathematical tools for further studies.

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