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Relativistic Homology as a Way of Tying or Untying Singular Points

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Abstract. This is an introduction into the author's concept of a 'relativistic geometry'. The paper focuses on the generation and resolution of singular points of algebraic curves by relativistic homologies and inversions.

Key Words: relativistic homology, relativistic inversion, generation and resolution of singular points

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1. Introductory considerations

1.1. On arbitrary space antipodal point

All spheres, from those infinitesimally small to the largest spherical 'planes', are *harmonically equivalent*, i.e., they can be bijectively and conformally inverted into each other. Therefore one can study everything that takes place in a 'plane' also on an inversely corresponding sphere, and vice versa. In such a 'plane', the most distant point from an observer is his antipodal point A, and all the circles and all the spheres in the universe which pass through this unique geometrical point A are the observer's 'straight lines' and 'planes'.

Wherever we imagine that our space antipodal point A is positioned, it will always be at a finite, however great distance r, so that its inversely corresponding point $\overline{A}$ can never be at the geometrical centre I of the sphere s_I of inversion, for the reciprocal distance $\overline{r} = 1/r$ must also be finite, however tiny (Fig. 1).

It seems that there is no limit as to how closely $\overline{A}$ can approach the centre I, but $\overline{A}$ can never attain it. First, geometrically, no sphere orthogonal to the sphere s_I can ever pass through its centre I, and, second, according to the fundamental principle *Ex nihilo nihil*, the minimal sphere ($\overline{r} = I\overline{A}$) must exist so that it could be inverted into the corresponding maximal sphere ($r = IA$). This is in accordance with the ancient Pythagorean system (*"The universe is spherical in shape and finite in size. Outside it is infinite void which enables the universe to breathe, as it were"*), as well as with the modern theory of Big Bang (from

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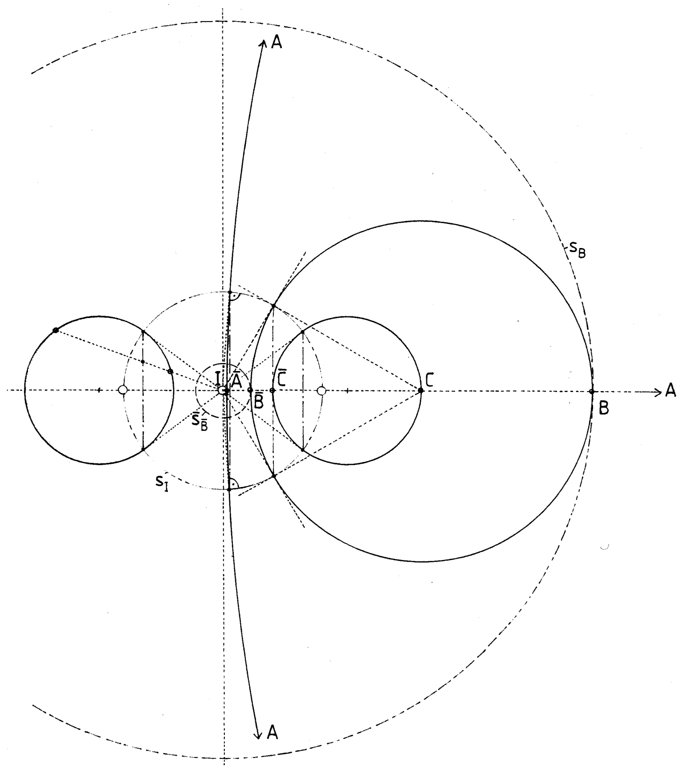

Figure 1: Arbitrary direction and arbitrary distance of the observer's antipodal point in space

the singular *'primal egg'*) and FRIEDMAN's model of a closed and pulsating universe. The 'breathing' of the universe is harmonical, for any two state-spheres are mutually symmetrical with respect to an intermediate state-sphere.

From Fig. 1 it is now clear that there are non-enumerable pairs of diametrically opposite positions of the point $\overline{A}$ on the minimal sphere, and because of this point-like sphere, the contour circles of all the greatest self-inverted spheres ($\overline{A} \leftrightarrow A$) seemingly coincide with the 'rectilinear' diametrical rays "through" the pole I.

1.2. On the parallelism of 'straight lines' and 'planes'

The observer's most distant 'straight lines' in a 'plane' (= sphere) in all geodesical directions are antipodal infinitesimal circles through A, which fill up an infinitesimal circle (or rather a calotte) with A as its centre. Parallel 'planes' touch each other in this calotte, i.e., they have all the antipodal 'straight lines' in common. In fact, observing micro geometrically, we could say that through a given point S (Fig. 2) there can be imagined two opposite parallels with the given 'line' (or 'plane') o: the westward parallel w and the eastward parallel e; they touch o at the common segments W-A and E-A respectively, and eventually w and e are coincident at the single 'straight line' s through S, parallel to the 'line' (or 'plane') o, having a common segment between the geometrical points W and E. The reader can observe in Fig. 2 two similar procedures for the genesis of the common segment of two parallel 'lines', and of the common antipodal 'lines' inside the common circular point of two parallel 'planes'.

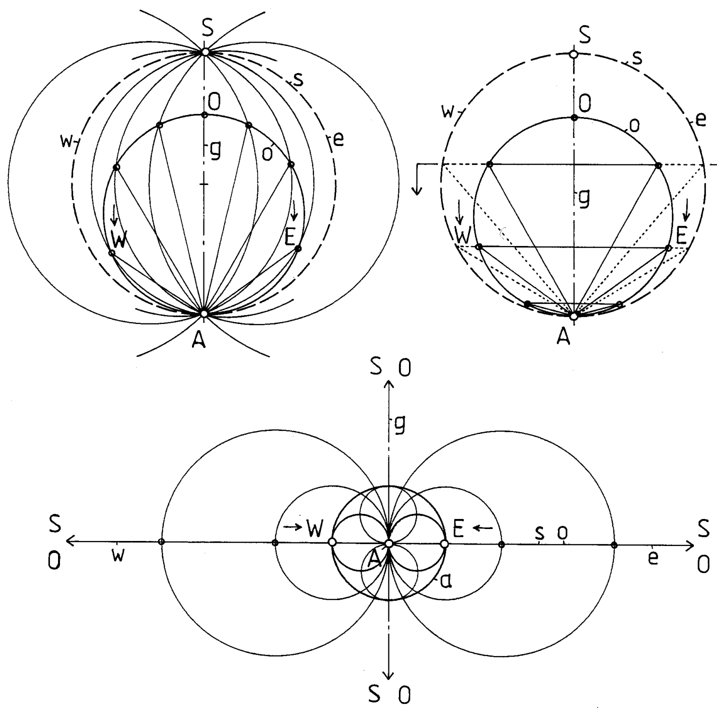

Figure 2: Genesis of parallel 'lines' or 'planes' ($s \parallel o$) and of the common antipodal 'lines' of parallel 'planes'

2. Homology in relativistic geometry

In relativistic geometry a *homology* (perspective collineation) in space is a central projection (center S) from one plane into another (planes 1 and 2 in Fig. 3). In classical projective geometry, the vanishing lines u_2 and v_1 are projected into the infinitely distant lines u_1 (in 1) and v_2 (in 2), respectively. In relativistic geometry, however, the vanishing 'lines' u_2 and v_1 really "vanish" as they are blown down into the infinitesimal antipodal 'straight lines' u_1 and v_2. The procedure becomes more obvious if the 'planes' and the 'rectilinear' projecting rays are inverted into the corresponding spheres and circular projecting rays (through S and A).

The parallel 'planes' (1 and $\|1$, 2 and $\|2$) are transformed into the spheres touching at A. Since the relativistic parallelism is invariant under inversion while the absolute equidistance is not, all the mutually parallel 'lines' (s, $\|s$, u_2, v_1, u_1, v_2) are inverted into the circles (seen edgewise) which touch each other at the antipodal point A. Now one can clearly see that the vanishing 'line' v_1 in 1 is projected (by its circular rays through S in opposite directions) onto the corresponding pair of antipodal 'straight lines' v_2 in 2 and vice versa (as well as u_2 in 2 onto u_1 in 1). Actually, the pencil of 'planes' through s, the fixed axis of the homology, is inverted into the pencil of spheres through the fixed circle s ($dia = OA$). So the point A is the common antipodal point for every observer staying diametrically opposite to A on his particular sphere from the pencil (e.g., O_1 and O_2).

The relativistic procedure shows us that such a homology represents a ready-made mechanism for tying (or untying) singular points of curves by tightening (or untightening) the circular loop v_1 (or v_2) and u_2 (or u_1). In Fig. 3 (top on 'planes' and bottom on spheres) the circle TH (seen edgewise in 1) is projected from S onto the hyperbola TH in 2. Since v_1 meets the circle TH at two real points M, N, these points, by tightening the loop v_1 in 1 into v_2 in 2, form the antipodal crunode of the hyperbola, while at the same time the circle's

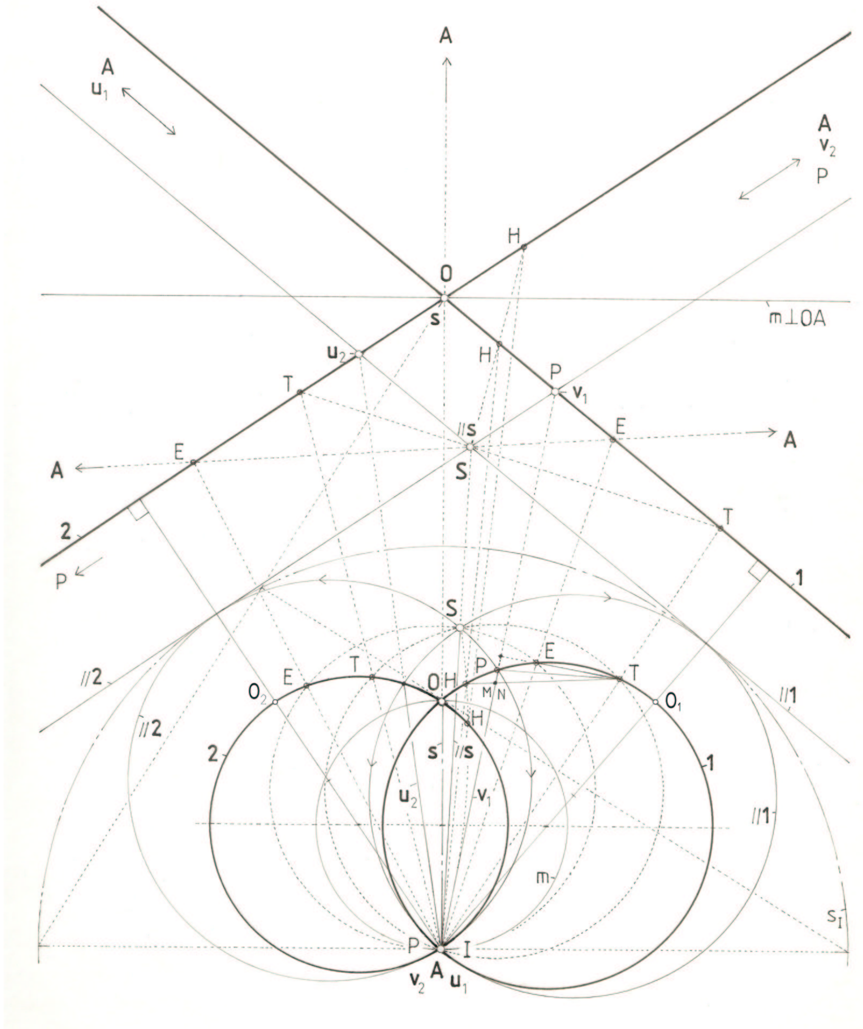

Figure 3: Relativistic homology on ‘planes’ and spheres

antipodal circular acnode u_1 in 1 is untied into two conjugate imaginary intersecting points between u_2 and the ‘spherical hyperbola’ in 2 with the apices T, H and the crunode v_2 at A.

In the same way the circle TE is transformed into the ellipse TE with its antipodal acnode, and the circle TP into the parabola TP with its antipodal cusp.

Before dealing with a few more complex examples let us present some essentials of relativistic geometry and especially about its inversion and stereographical projection.

2.1. Relativistic versus classical interpretation of inversion and stereographical projection

Let us begin with the question why our new geometry is *relativistic* when everybody knows that different observers from different positions see one and the same object differently. However, this everyday experience is a trivial non-mathematical ‘relativity’, where (to a lay person) *“everything is relative”* and no observer can know the views of the others. The relativity principle, on the contrary, means that each observer can know in advance how any other observer will see the same object because between the equivalent positions of two observers, as well as between their respective views, there is a definite mathematical relation (hence *relativity*).

In geometry, this relation is a harmonical and conformal transformation — the *symmetry* with respect to a circle or sphere, in general, and to a 'straight line' or 'plane' in the limiting classical cases. It means that under such a symmetry, i.e., under a 'plane' or space inversion or stereographical projection, the shape of a curve or surface changes the lengths of its 'linear' or curved segments only, whereas all the rest — harmonic relations, angles, and numbers of symmetry axes and centres, singular points, apices, foci — remains invariant.

In classical projective geometry the 4^{th} order cruciform c_2 (dashed in Fig. 4) with its central acnode at the pole O is transformed by an inversion (as a quadratic transformation with respect to the imaginary coordinate triangle O, I, J with sides $o_\infty = IJ$, $i = OI$ and $j = OJ$) into an expected 8^{th} order curve with 4-fold points at O, I, J which splits into the 6^{th} order rose $\overline{c}_2$ and the double line o_∞. As the 2-fold o_∞ passes through the 4-fold absolute points I, J two times, the rose passes also two times through I, J, meaning that it is bicircular. Then, reversely, the 6^{th} order rose with its 4-fold point O and 2-fold points I, J, is transformed into the total 12^{th} order curve with three 6-fold points O, I, J, which splits into the 4^{th} order cruciform c_2, the 4-fold ideal line o_∞, and the two 2-fold imaginary lines i and j (note $4+4+2+2 = 12$). As the 4-fold o_∞ passes through I and J four times, and i passes two times through I and O, and j two times through J and O, the resulting multiplicity of the singular points is: 0 for I $(6-4-2=0)$, 0 for J $(6-4-2=0)$ and 2 for O $(6-2-2=2)$, meaning that the cruciform passes only through O as its acnode. Everything is alright, but rather complicated.

In contrast to this classical procedure, the relativistic inversion is quite simple because it is not a quadratic transformation but a pure harmonic symmetry with respect to a fixed circle; it is a complete *bijection* because the pole does not correspond to the line at infinity but to the unique observer's antipodal point. In the same way, the relativistic stereographic projection is a space inversion of a sphere into a 'plane' with respect to a fixed sphere with its centre at the pole, i.e., there is a complete harmonic symmetry between the 'flat' geometry in a 'plane' and the corresponding curved geometry on a sphere. It is intriguing to note that even the classical inversion, in contrast to other quadratic transformations, is quite limited; it works only between particular curves whose classical orders ranges from n to $2n$. That is why a circle, e.g., cannot be inverted into a 4^{th} order curve but only into a circle or a line.

In Euclidean geometry, the stereographic projection (= inversion) is not bijective for the pole cannot be projected from the sphere onto the plane since the projecting rays, being equidistant to the plane, cannot have any contact with the plane. That is why Viviani's curve (as the intersection of a sphere with radius R and a rotational cylinder with radius $r = R/2$ touching internally) can be stereographically projected, either from its crunode into a 2^{nd} order orthogonal hyperbola $(4-2=2)$, or from a regular point into a 3^{rd} order curve, e.g., a strophoid $(4-1=3)$, or from a pole outside the curve into a 4^{th} order curve (e.g., Bernoulli's lemniscate). In projective geometry we have a similar effect: the bijection cannot be achieved since the pole is projected ('contractually') into the plane's line at infinity.

In relativistic geometry, however, the order of the curve does not change under stereographic projection (= inversion = symmetry) because the pole, as the observer's antipodal point on the sphere, is projected into the observer's antipodal point in the 'plane', and vice versa.

Now it is evident why — because of the fact that relativistic stereographic projection, inversion and conformal symmetry are synonyms — all spatial and planar curves which are symmetrical to Viviani's curve and then consequently with each other, form a family of different but harmonically and conformally equivalent curves of the same order. Naturally, none of them is more elementary than the others, but yet the orthogonal hyperbola distinguishes itself as a unique shape, while the equivalent shapes of 3rd and 4th order curves are non-enumerable.

On the way from projective to relativistic geometry the first step had been done by Möbius in the plane, and by Riemann in the space. Möbius added to the Euclidean plane a unique point at infinity, thus forming the Möbius plane or Euclidean circular plane, on which the inversion became bijective. Riemann mapped such a plane onto a sphere creating his representation of the complex plane by stereographic projection, where the pole corresponds to the plane's point at infinity. However, they could not go further since by replacing the line at infinity by a point at infinity they stepped outside the existing projective geometry and its algebraic interpretation. But above all, they could not free themselves from the reigning conception of infinity, which otherwise was necessary for the final recognition of the classical open-ended plane as a sphere of unlimited size.

Because of the facts, first, that equivalent curves of different classical order (n to $2n$) are mutually entirely symmetrical, and second, that all algebraic and transcendent curves and surfaces are closed, it is evident that today's infinitistic algebraic geometry cannot work in the relativistic geometry. To say it briefly, the discovery of the relativistic nature of 'straight line' and 'plane' leads us finally to the 'end of infinity'.

Since Euclidean geometry was algebraized 2000 years later by Descartes and Fermat, and projective geometry 200 years later by Möbius and Plücker, it is to expect that the algebraization of relativistic geometry — regarding the acceleration of modern science — should be done within 20 years; one decade has already elapsed.

2.2. Tying of the four-petal rose into a six-petal lemniscate

The cruciform c_2 (Fig. 4) with a central acnode and an antipodal 4-fold node (constructed as the intersection of two hyperbolic cylinders with an apex-apex touch; see Fig. 198 in [6]) is inverted into the four-petal rose $\overline{c}_2$ with an antipodal acnode and a central 4-fold node. Both curves are projected from 2 to 1 (note: a *solid* curve is always mapped into a *dashed* curve, and vice versa). The points $2, 6$ and the central acnode of c_2 on u_2 are tied into an antipodal 4-fold point of c_1 on the antipodal u_1, while at the same time the antipodal 4-fold node of c_2 on v_2 is untied into two crunodes $1, 5$ and $3, 7$ of c_1 on v_1. Since u_2 intersects the rose $\overline{c}_2$ at six points ($2, 6$ and $1, 3, 5, 7$), the rose is projected into the six-branch hyperbola $\overline{c}_1$ with a 6-fold antipodal node on u_1. Simultaneously, the rose's antipodal acnode on v_2 is untied into a pair of conjugate imaginary points of $\overline{c}_1$ on v_1.

We will be able to see the structure of the singular antipodal points of c_1 and $\overline{c}_1$ directly after inverting them into c_1' and $\overline{c}_1'$ (Fig. 5). Thus the 6-fold node of the six-petal lemniscate $\overline{c}_1'$ (as well as the equivalent antipodal node of $\overline{c}_1$) is composed of two crosses ($2'$ and $6'$) and four touches ($1'$, $3'$, $5'$ and $7'$) with the common horizontal symmetry axis.

In classical geometry the cruciform c_2 and the rose $\overline{c}_2$ are of the 4th and 6th order, respectively, but in relativistic geometry they both, as mutually symmetrical curves, are of the 8th order with mutually symmetrical singular points (at the pole and at the antipodal point). So the classical 4th order cruciform with its antipodal 4-fold point and the classical 6th order rose with its antipodal 2-fold point are unique shapes among the multitude of various classical 7th and 8th order shapes, which all together form the relativistic family of harmonically equivalent curves of 8th order. Thus, the number of unique shapes is equal to the number of unique singular points. Because of that in the mentioned family there is no curve of classical 5th order as these curves have no 3-fold point.

Because of the fact that the antipodal 'straight line' v_2 meets the classical 4th order

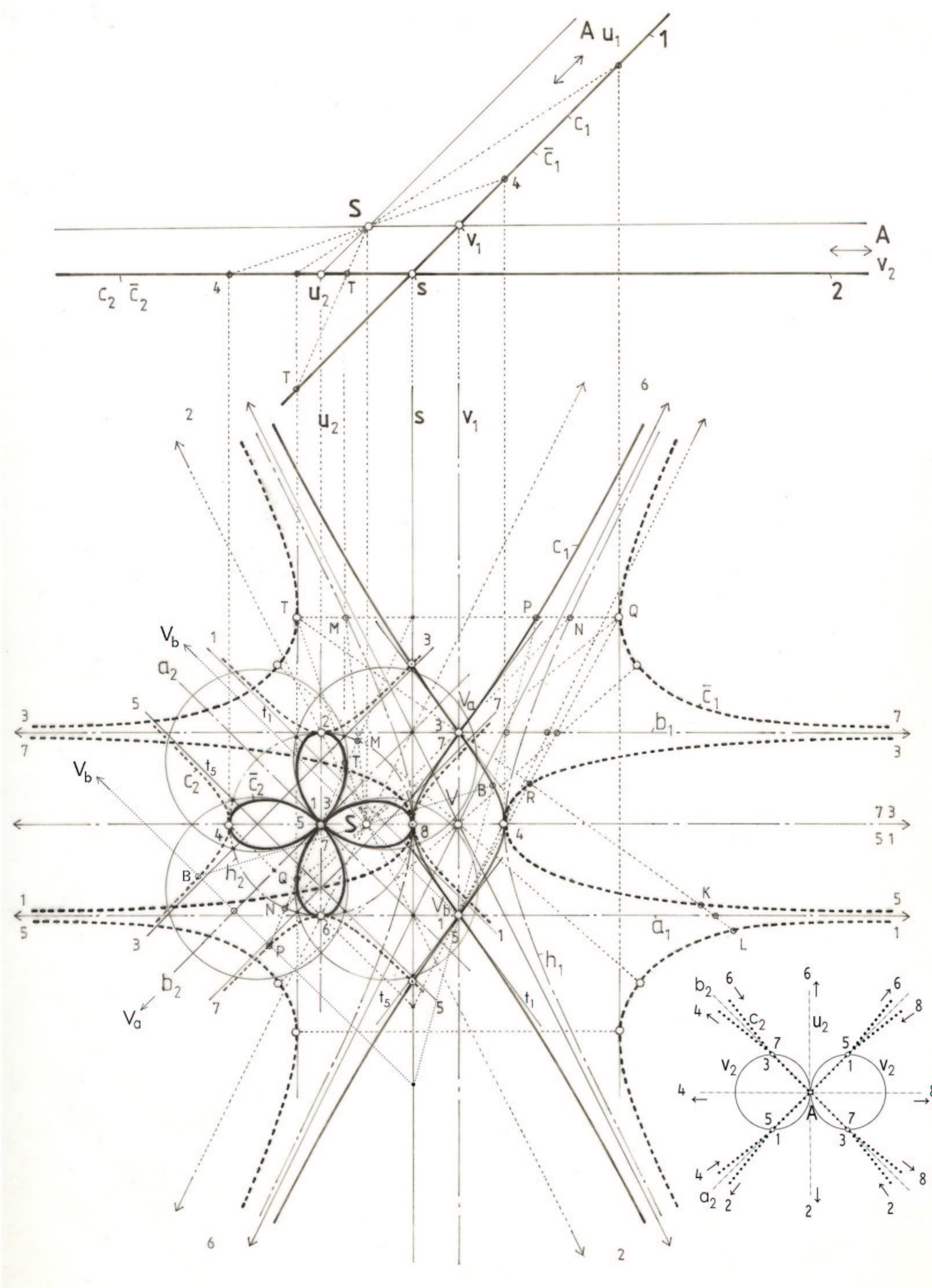

Figure 4: Projection of the rose $\overline{c}_2$ into $\overline{c}_1$ with a 6-fold antipodal point
and the cruciform c_2 into c_1 with a 4-fold antipodal point

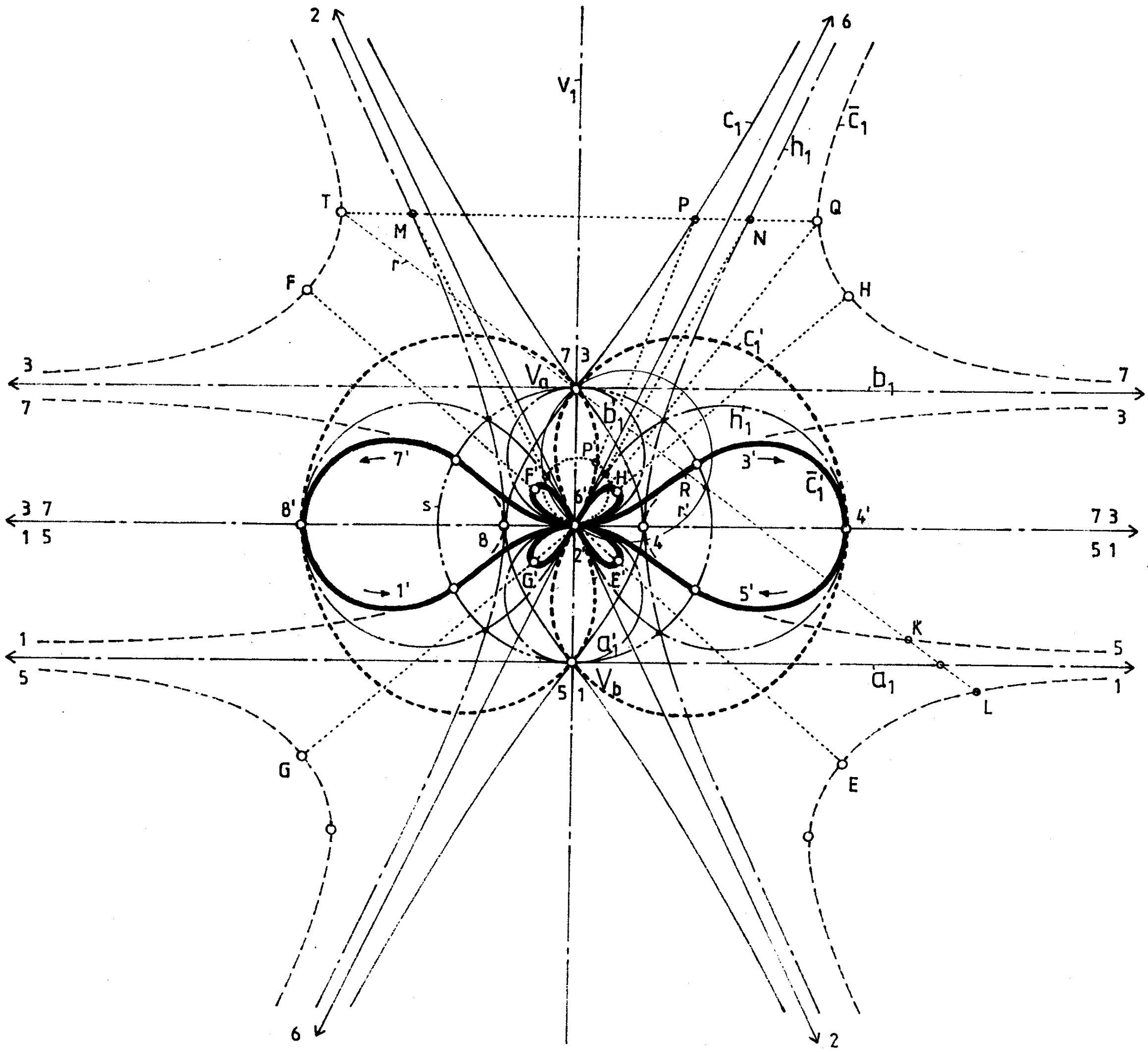

Figure 5: Inversion of the curves c_1 and $\bar{c}_1$ with respect to the circle s into the curves c_1' and $\bar{c}_1'$

cruciform c_2 at four real points $(4 + 4 = 8)$, the classical 6^{th} order rose $\bar{c}_2$ at two conjugate imaginary points $(6 + 2 = 8)$, every classical 7^{th} order curve at one point $(7 + 1 = 8)$, and every 8^{th} order curve at no point (neither real nor imaginary), the classical Bézout theorem is not valid in relativistic geometry.

The micro geometrical structure of the cruciform's antipodal 4-fold node at the infinitesimal environment of the antipodal point A is shown in the corner of Fig. 4. The touching points $1, 5$ and $3, 7$ lie on the lines a_2, b_2, c_2 and v_2, so that neither the axis 4-8 nor the vanishing 'line' u_2, nor its parallel s, pass through these points. (This must be so, for a singular point is not the Gordian knot; it is tied in such a way that it could be untied reversely). The fixed axis s and its parallels u_2 and v_2 pass through A touching each other in the overlapped infinitesimal segments (presumably of different lengths) which, under reverse projection from S, remain on the fixed axis s and on its parallels u_1 and v_1.

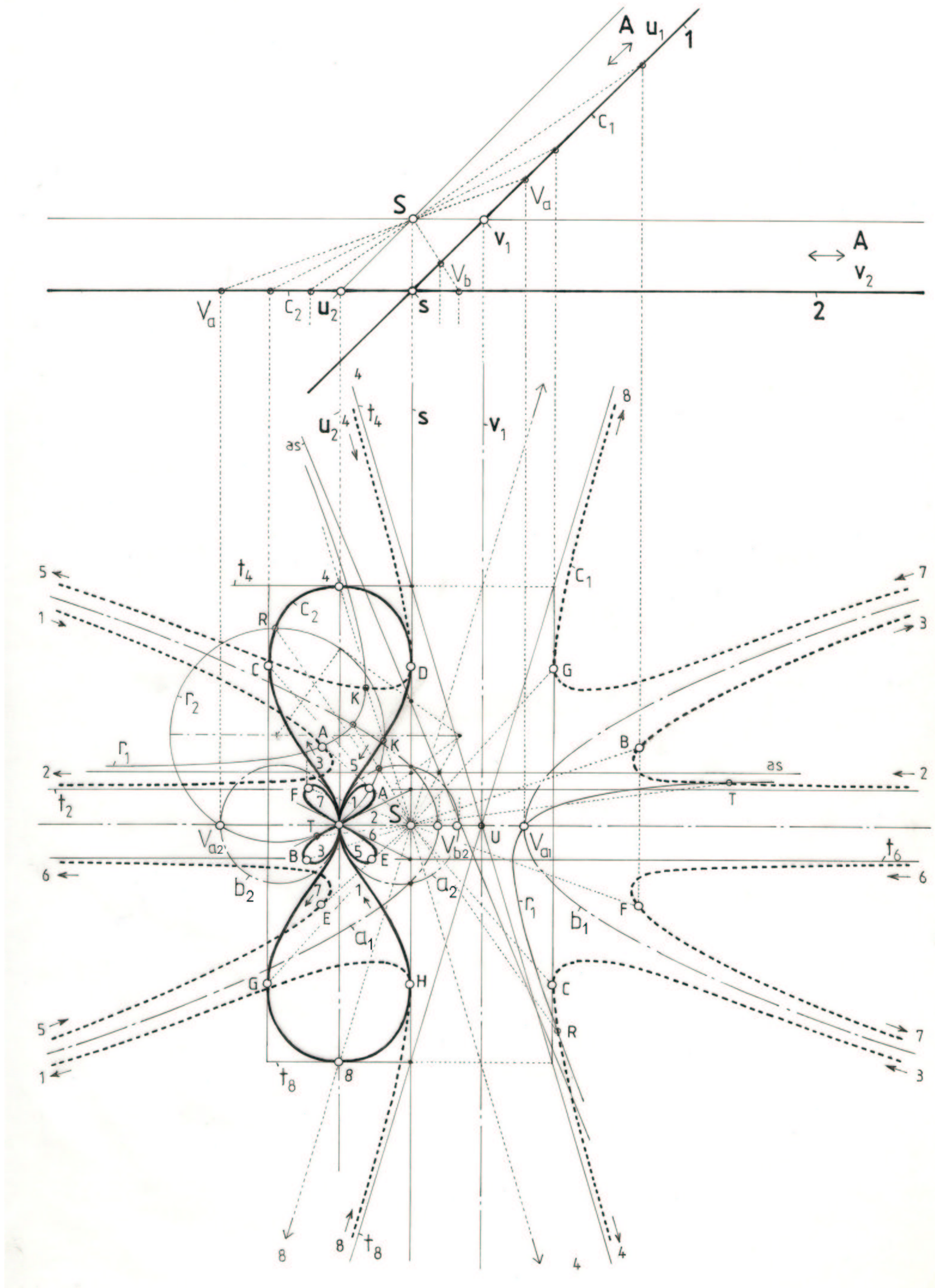

Figure 6: Projection of the 12th order lemniscate into a curve with an antipodal 12-fold point

2.3. Tying of the six-petal lemniscate into an eight-petal curve

After putting the axis of the lemniscate c_2 on the vanishing 'line' u_2 (Fig. 6) and projecting it into the 'plane' 1, we obtain a curve c_1 with four hyperbolic branches $(2, 6$ and $4, 8)$ and eight parabolic branches $(1, 1;\ 5, 5$ and $3, 3;\ 7, 7)$, which all together form the curve's antipodal 12-fold point composed of one crunode (4 with 8), one touching (2 with 6) and four cusps (1-1, 5-5, 3-3, 7-7). It is because of the fact that the vanishing 'line' u_2 intersects the 12$^{\text{th}}$ order lemniscate c_2 four times $(2, 6, 4, 8)$ and touches it four times (1-1, 5-5, 3-3, 7-7). To recognize this, one must note how the lemniscate c_2 consecutively passes through its singular point with respect to u_2. This can be better seen on the inverted curve $\bar{c}_1$ (Fig. 5) which passes through its antipodal singular point, with respect to the common horizontal axis, in the following order: $1, 1\,(\text{touch})$ - E – $2\,(\text{cross})$ - F – $3, 3\,(\text{touch})$ - 4 – $5, 5\,(\text{touch})$ - G – $6\,(\text{cross})$ H – $7, 7\,(\text{touch})$ - $8 \to 1$. In the same way, after inverting the obtained curve c_1 into the eight-petal rose $\bar{c}_1$ (Fig. 7), we can recognize the order of its passings through its central 12-fold point only with the help of c_1: $1, 1\,(\text{cusp})$ - A – $2\,(\text{pass})$ - B – $3, 3\,(\text{cusp})$ - C – $4\,(\text{pass})$ - D – $5, 5\,(\text{cusp})$ - E – $6\,(\text{pass})$ - F – $7, 7\,(\text{cusp})$ - G – $8\,(\text{pass})$ - $H \to 1$.

2.4. Transformations of symmetries

The rose $\bar{c}_2$ (Fig. 4) has five symmetries, one central and four axial. Since the central rays with the centre on u_2 are projected into parallel rays orthogonal to v_1, the central symmetry of c_2 (and $\bar{c}_2$) is projected into the axial v_1-symmetry of c_1 (and $\bar{c}_1$), the axial u_2-symmetry into the central V-symmetry, and the two axial a_2- and b_2-symmetries into the two central-axial (V_a, a_1)- and (V_b, b_1)-symmetry, respectively. For example: There are harmonic quadruples $(V_b b_2, BP) = (V_b b_1, BP) = -1$, $(V_a a_1, KL) = -1$, and $(V_a a_1, RT) = -1$ for two pairs of points of $\bar{c}_1$.

As the rose $\bar{c}_2$ and the cruciform c_2 (Fig. 4) are mutually symmetrical with respect to the circle h_2, so are $\bar{c}_1$ and c_1 symmetrical with respect to the hyperbola h_1 (as the projection of the circle), and therefore the final curves $\bar{c}_1'$ and $\bar{c}_1$ (Fig. 5) are symmetrical with respect to the lemniscate h_1' (= inverted hyperbola h_1); e.g., $(PQ, MN) = -1$ is inverted into $(P'Q', M'N') = -1$ on the circular ray through P' and the pole.

In the same way, the five symmetries of the six-petal lemniscate c_2 (Fig. 6) are transformed as follows: the axial u_2-symmetry into the central U-symmetry, the central one into the axial v_1-symmetry, the central-axial (V_{a2}, a_2)- and (V_{b2}, b_2)-symmetries into the central-axial (V_{a1}, a_1)- and (V_{b1}, b_1)-symmetries of c_1. The 'rectilinear' rays r of the V_a-symmetry (Fig. 5) are inverted into the circular rays r' through V_a and through the pole. The pencil of circular rays, e.g., the circle r_2 in Fig. 6, is projected into the pencil of conics (circle $r_2 \to$ hyperbola r_1). Then the harmonic quadruples $(V_{a2}a_2, AK) = -1$ and $(V_{a2}a_2, RT) = -1$ on r_2 are projected onto $(V_{a1}a_1, AK) = -1$ and $(V_{a1}a_1, RT) = -1$ on r_1. Finally, the hyperbolic ray r_1 (Fig. 7) is inverted into the harmonically equivalent 4$^{\text{th}}$ order ray $\bar{r}_1$ with a crunode at the pole, as well as the parabolic axes a and b of c_1 are inverted into their 4$^{\text{th}}$ order harmonic equivalents $\bar{a}$ and $\bar{b}$ with a cusp at the pole. Consequently, the projective transformations leave the same harmonic relations between four segments on the 4$^{\text{th}}$ order ray $\bar{r}_1$ of the central-axial symmetry of the 24$^{\text{th}}$ order eight-petal rose $\bar{c}_1$ with respect to its center $\overline{V}_a$ and its 4$^{\text{th}}$ order axis $\bar{a}$: $(\overline{V}_a\bar{a}, AK) = -1$ and $(\overline{V}_a\bar{a}, RT) = -1$.

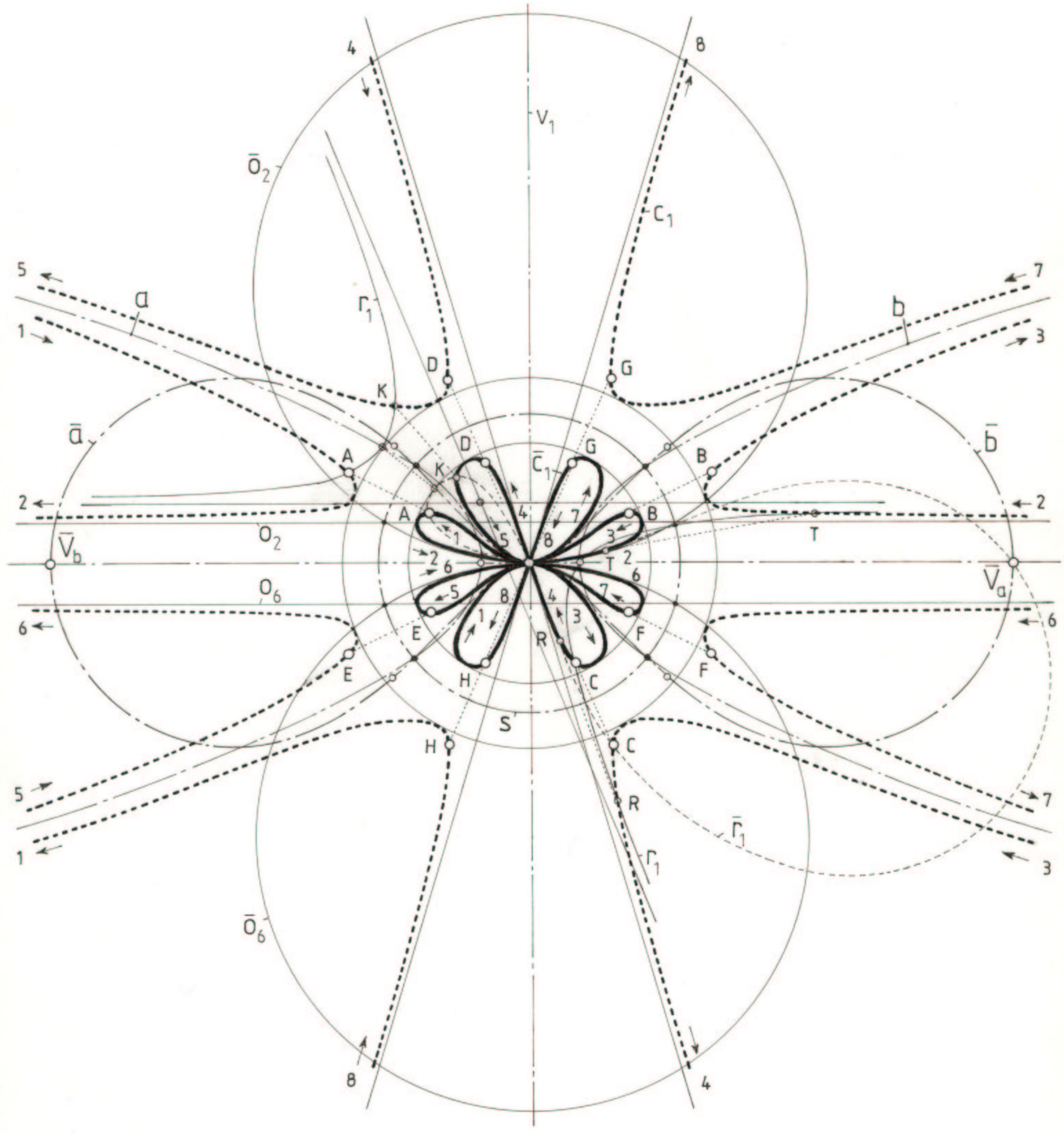

Figure 7: Five symmetries of the two harmonically equivalent curves

3. Conclusion

In relativistic geometry the classical roles of homology and inversion are, in some sense, exchanged: now the relativistic homology does the task of transformation, i.e., it resolves and generates the respective antipodal points of homological curves c_1 and c_2 by blowing up or down the two vanishing 'lines' u_1 or v_1, whereas the relativistic inversion, as a pure symmetry, only does the mere task of transportation of a curve's singular point to or from the observer's antipodal point. Accordingly, the number of points (real + conjugate imaginary), where u_2 (v_1) meets c_2 (c_1), is equal to the multiplicity of the antipodal point of c_1 (c_2) on u_1 (v_2). Therefore, by a homology any curve's order can at most be doubled or halved, (e.g., $c_2 \leftrightarrow c_1$ in Fig. 6), meaning that it can remain unchanged too (e.g., ellipse $\leftrightarrow$ hyperbola). Briefly, the *total order* of a curve is now its classical order + the multiplicity of its antipodal point.

Under a chain of *homology-inversion-homology* transformations the types of curves' symmetries (central, axial, central-axial), symmetry axes and rays ('line', circle, conic, conic equivalent, ...) are obviously changing, but the harmonic cross-ratio of the four segments on a symmetry ray remains invariant.

In general, countless families of harmonically equivalent curves can now be connected by relativistic homologies in an unlimited network. The generation and resolution of singular points of curves can be achieved otherwise by quadratic transformations: analytically by so called σ-processes [1], or constructively by a sequence of restituted MacLaurin's transformations [2, 3].

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