

Some Examples of Using *Mathematica* and *webMathematica* in Teaching Geometry

Sonja Gorjanc

*Faculty of Civil Engineering, University of Zagreb
Kačićeva 26, 10000 Zagreb, Croatia
email: sgorjanc@grad.hr*

Abstract. The paper outlines some ideas on teaching geometry at technical faculties as well as various possibilities and advantages of the application of *Mathematica* in teaching. A few examples demonstrate the use of *Mathematica* visualizations in teaching geometry at the Faculty of Civil Engineering in Zagreb. These examples are extracts from lectures, internet pages and students' projects on algebraic ruled surfaces and on Gaussian and mean curvature.

Key Words: ruled surfaces, hyperbolic paraboloid, Gaussian curvature, mean curvature, visualization of surfaces

MSC 2000: 51N05, 53A05

1. Introduction

At the technical faculties in Croatia geometry is primarily taught within the Descriptive geometry course in the first year. The way of teaching is very traditional. Computers are rarely used. There are many reasons for such situation which are probably similar to those in some other transitional countries: financial conditions, the policy of the university, teaching staff etc..

The author thinks that the most significant positive effect of the traditional way of teaching is that the high level of geometric theory is preserved. On the other hand, there are negative effects, such as students' declining interest and a lack of motivation and an increasing gap between the subjects of geometry and mathematics. Majority of students do not realize that the contents dealt within geometrical and mathematical subjects are often the same but treated in different ways — constructive and analytical.

2. *Mathematica* and *webMathematica* in teaching geometry

In the author's opinion, the main objective of our teaching is to develop cognitive and perceptive abilities and the students' knowledge of the three-dimensional Euclidean space, as well

as to enhance their knowledge of constructive methods necessary for graphic communication between future engineers. In the computerization of teaching, the development of constructive methods and skills is by no doubt connected with CAD programs. For this purpose *Mathematica* cannot help. However, it is very convenient for teaching the geometry of the Euclidean space because of its great graphics possibilities. Moreover, it is an ideal program for connecting the content of geometrical and mathematical subjects.

Mathematica is a program which is used as a numerical and symbolical calculator, computer language, a system for the visualization of functions and data, a platform for designing packages for specific applications and creating interactive documents with text, animations and sound, etc.. *Mathematica* cannot be directly activated by HTML, but *webMathematica* is a program which bridges web server and *Mathematica* and enables interactive calculations and visualizations on web pages. The web pages on the server have to be written in extended HTML which is defined by the rules of *webMathematica*. Such complex systems with very sophisticated graphics can be applied creatively in teaching geometry. They are especially suitable for developing educational material. For example, for designing

- presentation notebooks and web applications with animations and sound,
- web applications for interactive learning on the internet,
- educational notebooks used in computer classrooms.

3. Examples

Since the major advantage of the use of *Mathematica* and *webMathematica* in teaching geometry is that they allow the design of sophisticated animations as well as 3D pictures moved by mouse (*LiveGraphics3D*), a great number of geometric properties can be clearly visualized. Because of the nature of the print medium, the examples listed here in this paper have rather poor quality compared to the original.

3.1. Lecture ‘hyperbolic paraboloid’

Because of their geometric properties ruled surfaces are suitable for roofing; therefore they are especially interesting for the students of civil engineering. The second degree surfaces (the one-sheet hyperboloid and the hyperbolic paraboloid) are the simplest. They have two systems of rulings, i.e., through each point of the surface two rulings are passing — one from each system. The rulings of one system are pairwise skew and each of them cuts each rulings of the other system. Any three rulings of one system can be chosen as directing curves [1, 7, 6].

3.1.1. The generations of the hyperbolic paraboloid

The hyperbolic paraboloid \mathcal{HP} is a ruled quadric with the simplest parametrization

$$(u, v) \rightarrow (u, v, uv). \quad (1)$$

It is a conoidal surface with two lines at infinity. Each system of rulings has one ruling at infinity. The rulings of \mathcal{HP} can be constructed as shown in Figs. 1 and 2.

The hyperbolic paraboloid is also a surface of translation. It can be obtained by moving the parabola of one main section along the parabola of the other main section (Fig. 3). The corresponding parametrization of \mathcal{HP} is still very simple:

$$(u, v) \rightarrow (u, v, u^2 - v^2). \quad (2)$$

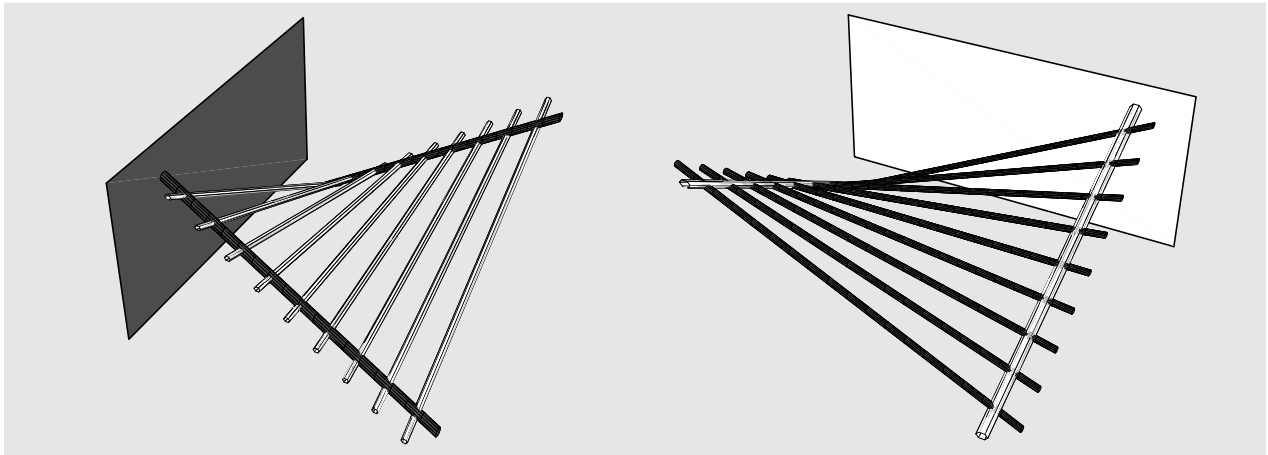


Figure 1: Each system of rulings of \mathcal{HP} can be constructed as a system of lines which cut two skew directing lines and which are parallel to a directing plane — a plane which contains the ruling at infinity. (The directing lines are not parallel to the directing planes.)

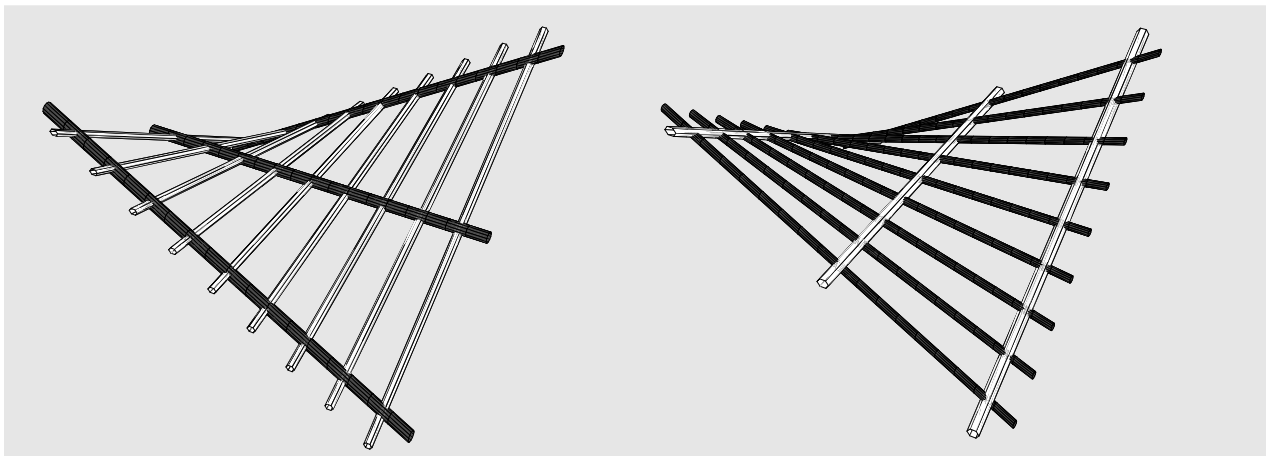


Figure 2: The directing curves are three skew lines parallel to the same plane. Any three rulings of one system can be chosen as directing curves.

3.1.2. Planar sections of the hyperbolic paraboloid

The tangent plane cuts \mathcal{HP} in a pair of rulings (Fig. 4a). Since the plane at infinity contains two rulings, it is the tangent plane at the point of intersection between the directing planes and the plane at infinity. Since \mathcal{HP} has two real lines at infinity, the conic intersection of that surface and a plane possesses two or one real point at infinity. Therefore there are no ellipses on \mathcal{HP} . A plane cuts \mathcal{HP} in a parabola if it is parallel to the intersection of the directing planes (Fig. 4b). In all other cases, excluding tangent planes, the planar sections of \mathcal{HP} are hyperbolas (Fig. 4c).

3.1.3. Mean and Gaussian curvature

For future structural engineers it is important to have a knowledge of the Gaussian and mean curvature. For example: Tensile fabric structures (e.g., a membrane roof) in a uniform state of tensile prestress behaves like a soap film stretched over a wire which is bent in a shape of

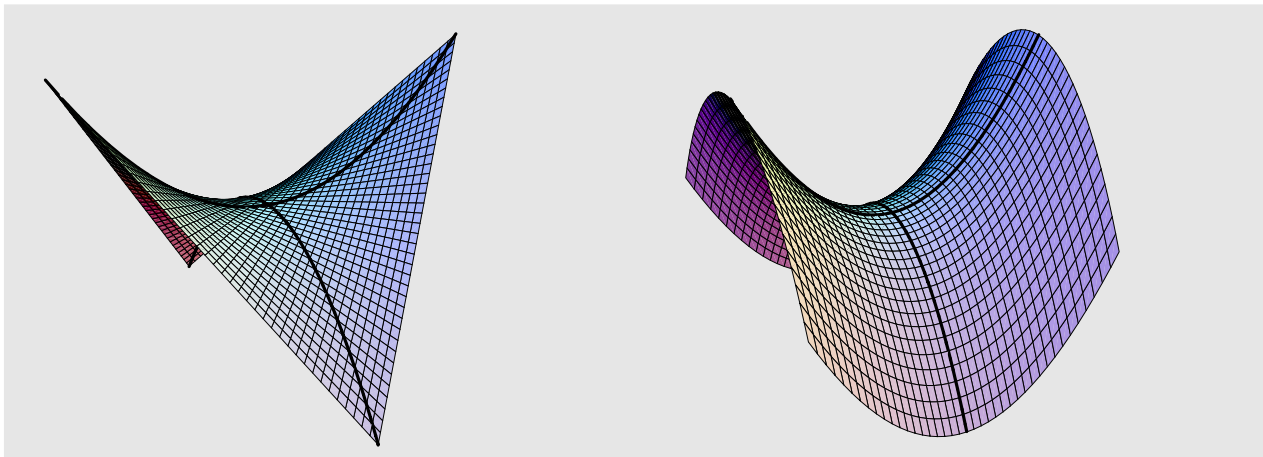


Figure 3: Two parabolas which generate \mathcal{HP} as a surface of translation. *Mathematica* graphics are obtained by the use of the parametrizations (1) and (2).

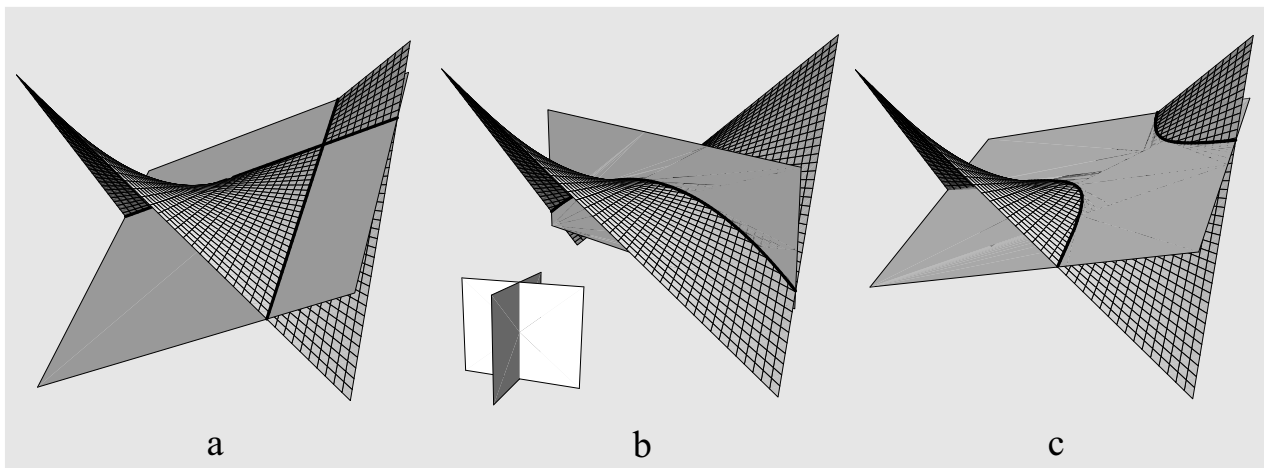


Figure 4: Planar sections of a hyperbolic paraboloid

a closed space curve. The soap film assumes a form which has minimal area relative to all other surfaces stretched over the same wire; this surface is therefore called *minimal* surface. It can be shown that the *mean curvature vanishes* at each point of that surface.

\mathcal{HP} is convenient for some visualizations connected with Gaussian and mean curvature (Fig. 5): At any regular point of a surface the Gaussian curvature is the product and the mean curvature is half of the sum of the principal curvatures. For computing we use the first and the second fundamental form. In the book [3, pp. 394] we find the *Mathematica* definitions of the functions *mcurvature* and *gcurvature* which compute mean and Gaussian curvatures in each regular point of a surface given by parametric equations. These functions enable us to plot the graphs of the Gaussian and mean curvatures of regular surfaces and to paint surfaces with colours which depend on these curvatures. For the following visualizations we used the periodical *Mathematica* colour function `Hue` (period 1) with values on a colour-spectrum. It is obvious that in the black-white print of this paper, which for example does not differentiate orange from blue, the distinction of visual data given by the function `Hue` is decreased. However, even under such conditions the function `Hue` can be applied to show some properties of Gaussian and mean curvatures (Fig. 6).

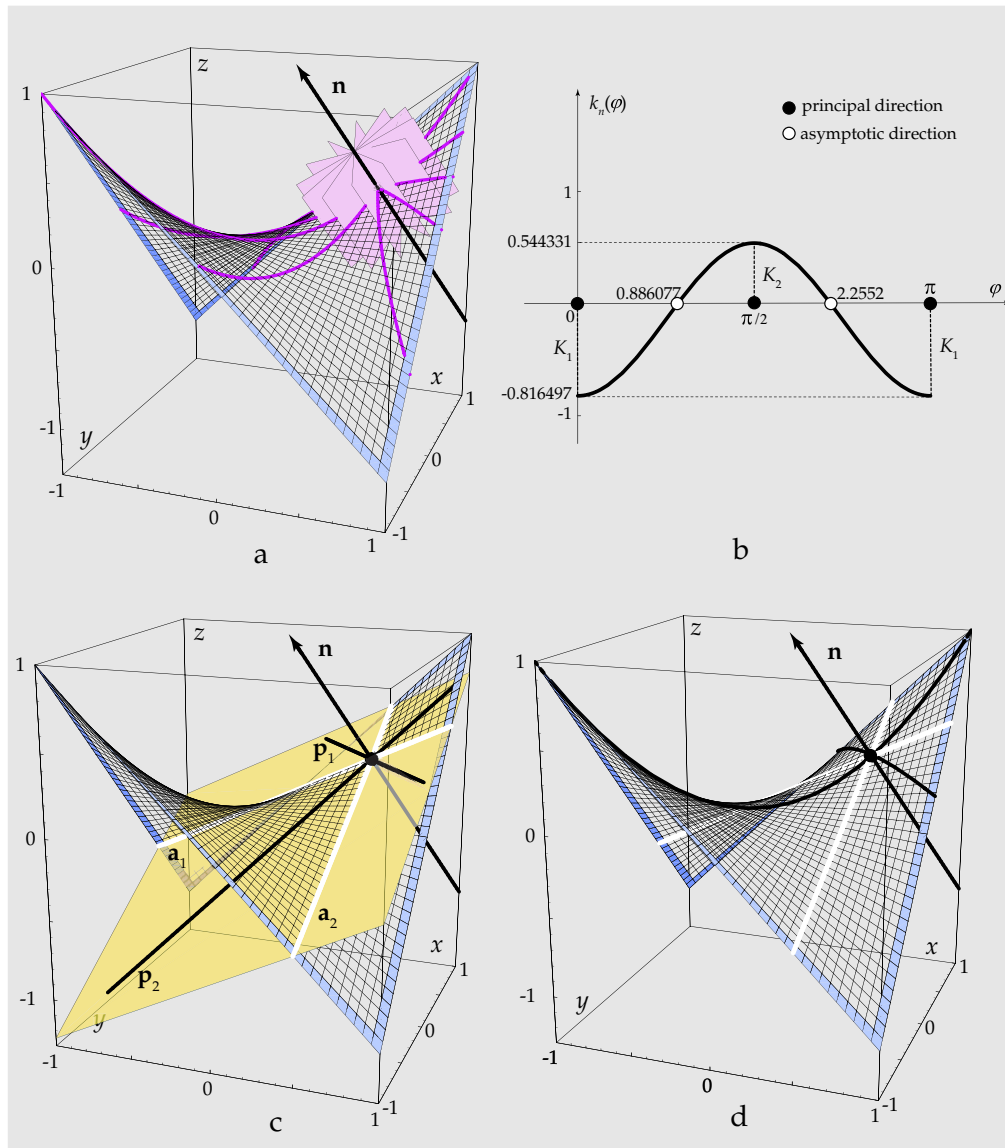


Figure 5: The pictures show a part of the hyperbolic paraboloid defined by the parametrization (1) over the rectangle $[-1, 1] \times [-1, 1]$.

a. Here six planes through a normal line at the point $(0.5, 0.5, 0.25)$ are displayed together with their intersections with the surface.

b. This picture shows the graph of the function of normal curvature at that point with principal and asymptotic directions. Asymptotic directions are defined by the two rulings passing through the given point. For the parametrization (1) the rulings are the u - and v -lines.

c. The principal directions are defined by lines of symmetry of the rulings.

d. In this picture the normal sections tangent to the principal directions and to the asymptotic directions are displayed.

3.2. Internet pages

Internet pages without animations can be simply designed by saving *Mathematica* notebooks as HTML. If a notebook contains animations it would not be correctly saved in such a way;

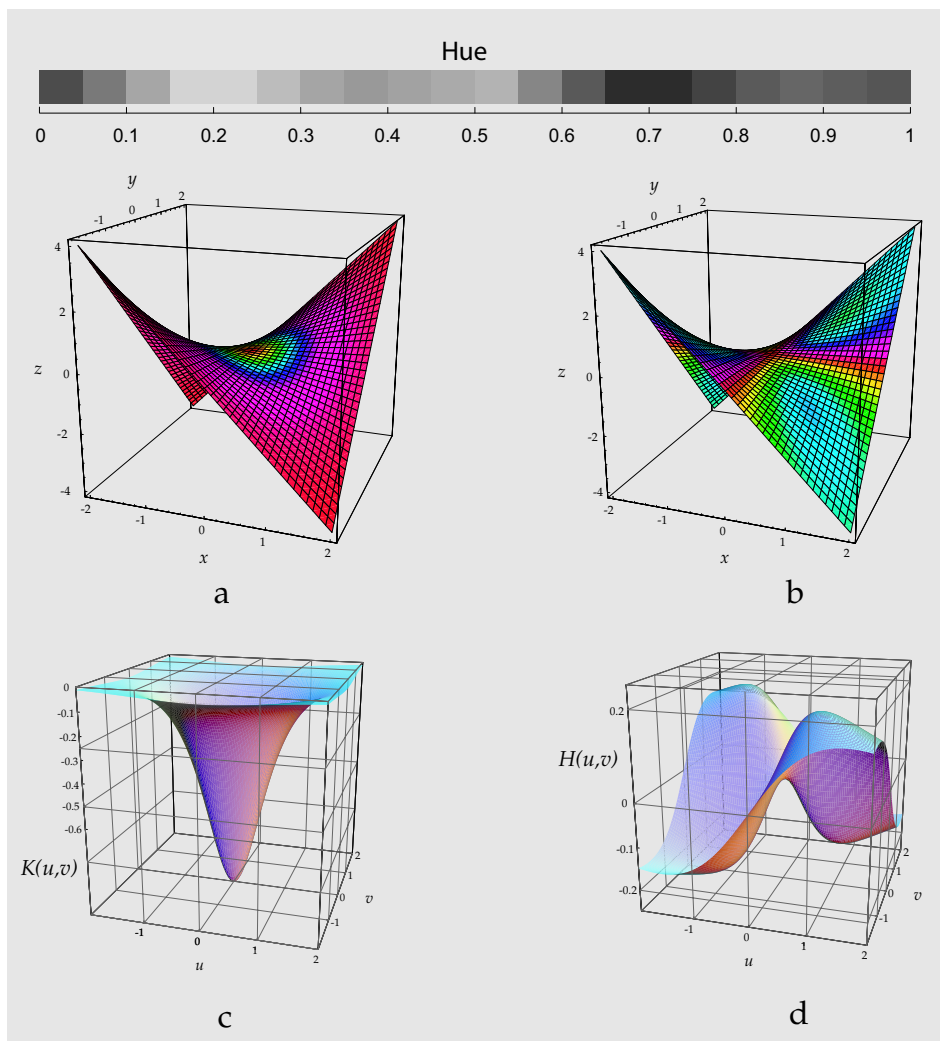


Figure 6: For the hyperbolic paraboloid defined by the parametrization (1) the figure shows:
 a. The surface coloured by $\text{Hue}(gcurvature)$.
 b. The surface coloured by $\text{Hue}(3mcurvature)$.
 c. The graph of the function $gcurvature$ over the rectangle $[-2, 2] \times [-2, 2]$.
 d. The graph of the function $mcurvature$ over the rectangle $[-2, 2] \times [-2, 2]$.

animations have to be exported from *Mathematica* and then manually included into HTML files. It is clear that graphics and animations exported from *Mathematica* can also be included into any file written in HTML. At the address www.grad.hr/nastava/geometrija/ilustracije one can find web pages with *Mathematica* graphics and animations which visualize generations and properties of conics, intersections of planes and cones or cylinders, intersections of cones and cylinders, ruled algebraic surfaces [4, 5], normal, Gaussian and mean curvatures etc.. Five years ago the author started to design these pages for students. The major part (especially files connected with curvatures) was created within an IT project¹ (http://www.grad.hr/itproject_math). Within this project we also installed *webMathematica* on the web server at the Faculty of Civil Engineering in Zagreb and designed some interactive web

¹IT project “*Selected Chapters of Geometry and Mathematics Treated by Means of Mathematica for Future Structural Engineers*” was supported by the Ministry of Science and Technology of the Republic of Croatia through 2002/03.

pages powered by this program.

In Figs. 7 and 8 we show the print-screen of the web page with visualizations of Gaussian and mean curvatures. A user writes his inputs in white rectangles and starts interactive communication by pressing the command button **Visualize ►**.

The Visualizations of Gaussian and Mean Curvatures POWERED BY webMATHEMATICA

For the following visualizations we used the periodical *Mathematica* color function **Hue** (period 1). It is defined in the following way:

In the book [A. Gray: Modern Differential Geometry of Curves and Surfaces with Mathematica](#) (p. 394) you can find the definitions of the *Mathematica* functions **gcurvature** and **mcurvature** that we used in this file.

Define the parametrization $f : U \rightarrow \mathbb{R}^3$ of a surface Φ .

Write the parametric equations of a surface.

$x(u,v) =$
 $y(u,v) =$
 $z(u,v) =$

Define the rectangle $[u_0, u_1] \times [v_0, v_1]$ that is a subset of $U!$

$u_0 =$ $u_1 =$
 $v_0 =$ $v_1 =$

Define the steps of the variables u and $v!$

$\Delta u =$ $\Delta v =$

(How to format your inputs...)

Visualize ►

regular surface with a parametrization $f : \mathcal{U} \rightarrow \mathbb{R}^3$ if:

- $\mathcal{U} \subset \mathbb{R}^2$, \mathcal{U} is open and connected,
- $\forall (u,v) \in \mathcal{U}$, $f(u,v) = (x(u,v), y(u,v), z(u,v))$,
where $x, y, z : \mathcal{U} \rightarrow \mathbb{R}$ are differentiable functions,
- $\forall (u,v) \in \mathcal{U}$, a matrix $\mathcal{J}(f)(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) & \frac{\partial z}{\partial u}(u,v) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v) & \frac{\partial z}{\partial v}(u,v) \end{pmatrix}$ has rank 2,
- $f(\mathcal{U}) = \Phi$.

The Gaussian (total) curvature K and mean curvature H of Φ are the functions $K, H : \Phi \rightarrow \mathbb{R}$ defined in the following way:

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{LG - 2MF + NE}{2(EG - F^2)},$$

where E, F, G and L, M, N are the coefficients of the first and the second fundamental form of a function f , respectively.

[See the lectures for Mathematics IV – Differential Geometry \(in Croatian\).](#)
[See the visualizations connected with the curvatures of a regular surface.](#)

Figure 7: Web page with visualizations of Gaussian and mean curvatures

3.3. Students' projects

Since some of our best students were very interested in working with the program *Mathematica*, the author encouraged them to explore issues beyond the regular curriculum. The results of that collaboration are two students' papers which won the Rector's Prizes. The papers can be downloaded from the address <http://www.grad.hr/sgorjanc/teaching.html>.

3.3.1. Covering with parabolic conoid

In 2001 the second-year students Sanja FILIPAN and Hrvoje KVASNIČKA wrote the paper "*Covering with a Parabolic Conoid*". Selected parts of it, extended with visualizations of Gaussian and mean curvatures, can be found in [2]. Students treated the class of 3rd degree parabolic conoids with directing curves p , l and d^∞ , which are linked with a Cartesian coordinate system as in Fig. 9; they computed their equations and defined the functions for

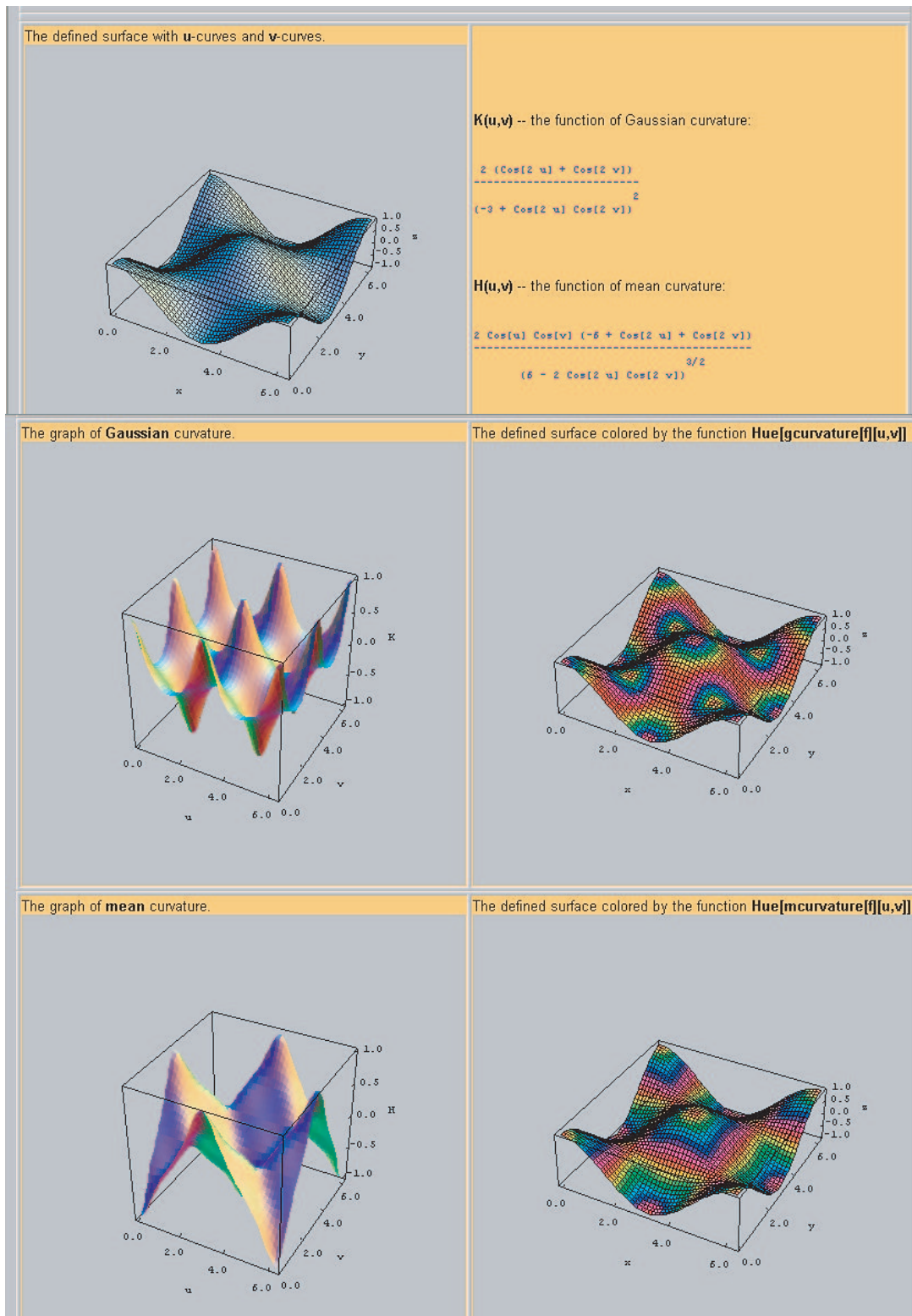


Figure 8: The results are *LiveGraphics3D* on the computer.

drawing the parabolic conoids using *Mathematica* for each choice of the parameters a, b and c . Parabolic conoids can be used for covering in numerous ways. Various parts of conoids can be composed in simple series. Some ideas for the covering a rectangular area are displayed in Fig. 9.

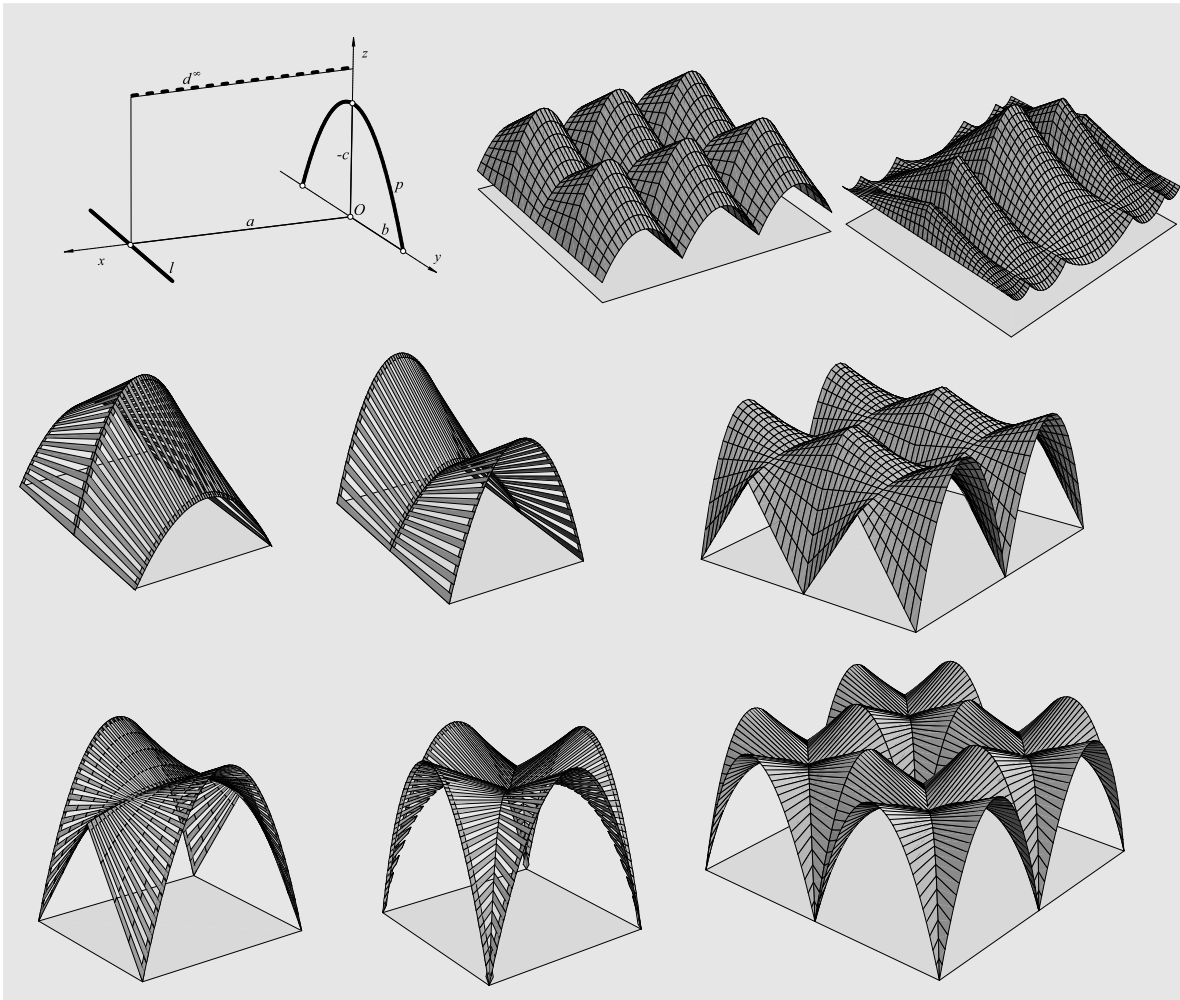


Figure 9: Covering with a parabolic conoid

3.3.2. Gaussian and mean curvatures of surfaces — visualizations with the program *Mathematica*

In 2004 the third and fourth-year students Sanja HAK and Mario UROŠ wrote the paper “*Gaussian and Mean Curvatures of Surfaces — Visualizations with the Program Mathematica*”. Thanks to their interest and talent, it was a pleasure to work with Sanja and Mario and the achieved results were far beyond their expectations. I enclose the summary of their paper together with several pictures (Figs. 10, 11, 12):

“In the first chapter elementary characteristics of tension structures are mentioned, as well as some specific problems during their design. A survey of photographs for this type of structures is given on the CD, which is enclosed in this paper. In the second and third chapter, using methods of analytic and differential geometry, with aid of the program Mathematica, the following terms are worked out: the parametrization of surfaces, the tangent plane and normal vector, the first and second fundamental form, principal and asymptotic directions, as well as the functions of Gaussian and mean curvatures with particular accent on ruled surfaces. In the fourth chapter, the visualizations of Gaussian and mean curvatures are given for eleven examples of surfaces. There is also an overview of the Mathematica notebook, which was used to obtain these visualizations. The program and input data for drawing are accessible on the

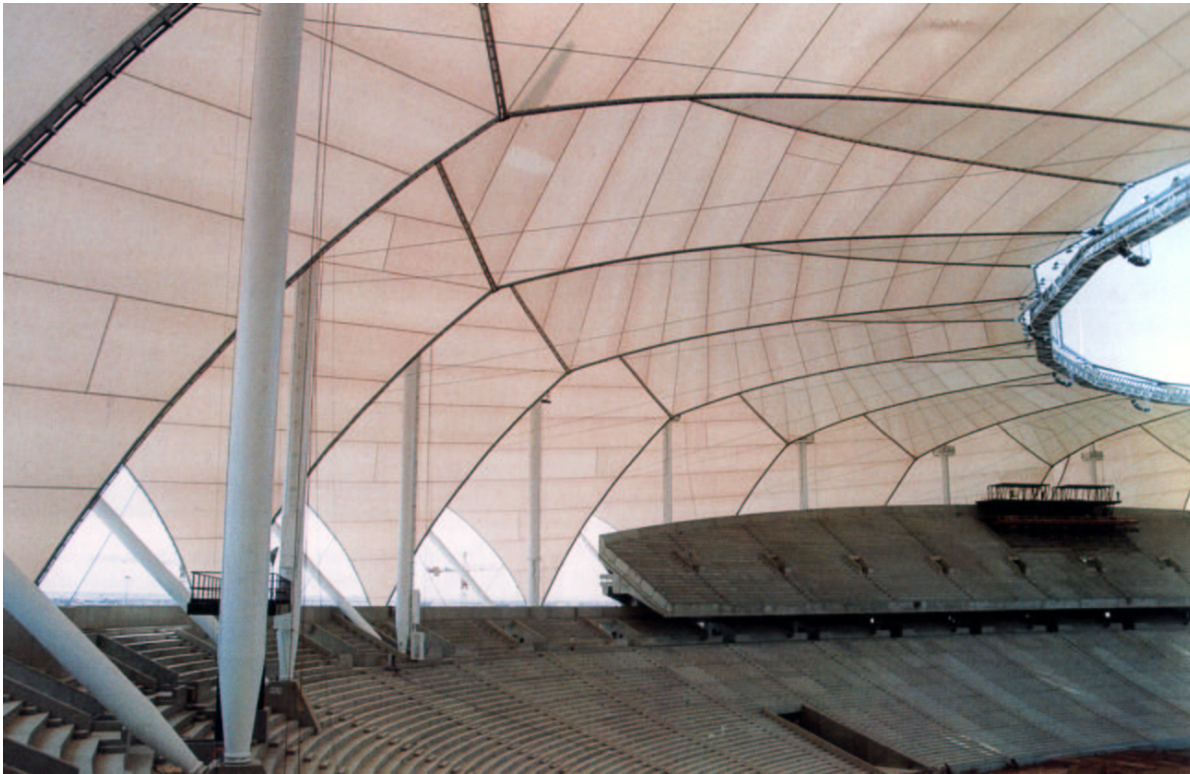


Figure 10: Riyadh International Stadium, Riyadh, Saudi Arabia, 1985

enclosed CD. The fifth chapter contains definitions, as well as eleven examples of minimal surfaces. These examples are worked out through parametrization, calculation of principal curvatures and graphics in Mathematica. The file used for this treatment is also given on the enclosed CD. The sixth chapter contains a short description of numerical and experimental form finding of minimal nets."

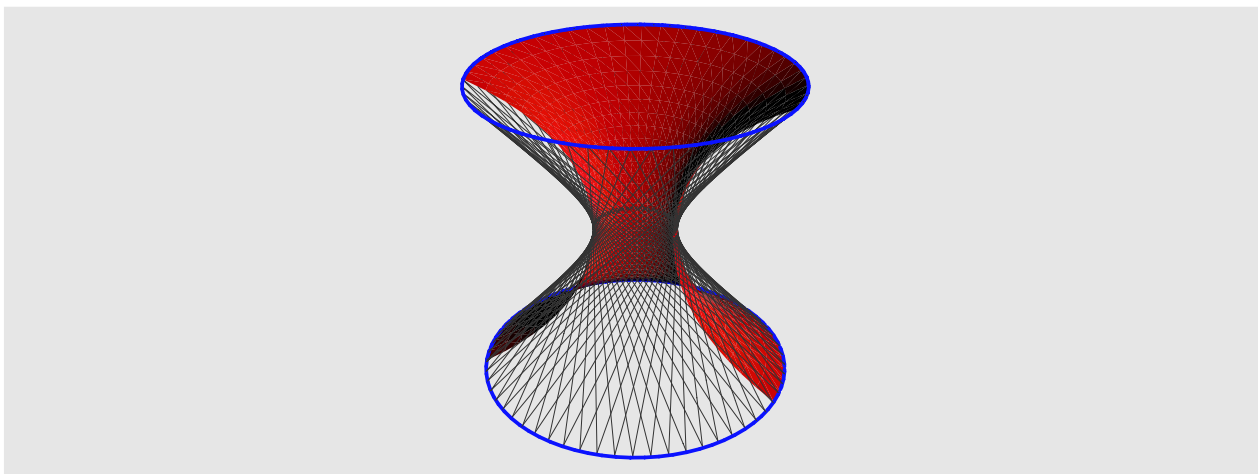


Figure 11: Minimal surface (the catenoid) and minimal net (the one-sheet hyperboloid) bounded by the same parallel circles.

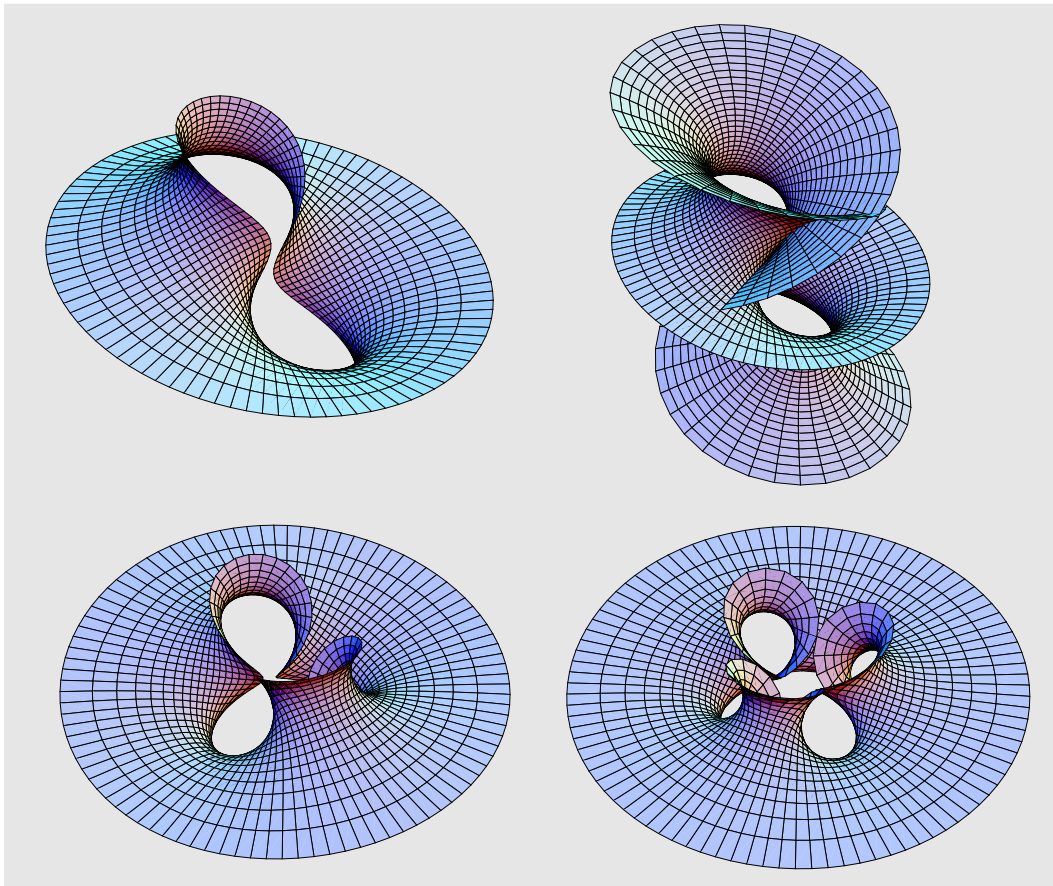


Figure 12: Richmond's minimal surfaces

References

- [1] H. BRAUNER, W. KICKINGER: *Geometrija u graditeljstvu*. Školska knjiga, Zagreb 1980.
- [2] S. FILIPAN, S. GORJANC, H. KVASNIČKA: *Natkrivanje paraboličkim konoidom*. KoG **5**, 57–64 (2000).
- [3] A. GRAY: *Modern Differential Geometry of Curves and Surfaces with Mathematica*. CRC Press, Boca Raton 1998.
- [4] S. GORJANC: *The Generation of Ruled Quartics in Mathematica*. Proc. Special SEFI European Seminar on Geometry in Engineering Education, Bratislava-Smolenice/Slovakia, 16–31 (1997).
- [5] S. GORJANC: *The Generation of Ruled Cubics by using Mathematica*. Proc. 8th ICGG, Austin/Texas, 41–48 (1998).
- [6] B. KUČINIĆ, O. KRISTOFOROVIĆ, I. SALER: *Oble forme u graditeljstvu*. Građevinar, Zagreb 1992.
- [7] V. NIČE: *Deskriptivna geometrija II*. Školska knjiga, Zagreb 1980.

Received August 1, 2002; final form December 31, 2004