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A Note on Bang's Theorem on Equifacial Tetrahedra

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Abstract. In this note, we give an analytic proof based on Pythagoras' theorem of a theorem of Bang stating that if the faces of a tetrahedron have equal areas then they are congruent. We also place Bang's theorem in the more general context that deals with the existence and uniqueness of a tetrahedron $PABC$ having a given base ABC and having lateral faces of given areas. Our approach shows also how to construct counter-examples to Bang's statement in higher dimensions.

Key Words: isosceles tetrahedron, equifacial tetrahedron, Bang's Theorem, regular simplex, barycentric coordinates, trilinear coordinates

MSC 2000: 51M20, 52B11

1. Introduction

Bang's theorem states that if the faces of a tetrahedron have equal areas then they are congruent. Proofs of this elegant theorem are given in [1], [3], [4] and [6, Chapter 9, pp. 90–97]. These proofs, however, are too geometric (or geo-trigonometric) to shed any light on the situation in higher dimensions. In this paper, we adopt an analytic approach that places Bang's theorem in a more general context and indicates how to treat the problem in any dimension. As a result, we obtain a new proof of Bang's theorem and counter-examples to Bang's statement in higher dimensions. Other related results are obtained. We note that the failure of Bang's theorem in dimensions higher than three follows also from [8] where more general results are established. Related issues are investigated in [7].

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2. An analytic proof of Bang's theorem

We start with a theorem that implies a detailed version of Bang's theorem.

Theorem 1. *Let ABC be a triangle of side-lengths a, b, c with $0 < a \leq b \leq c$, and let its angles be denoted by A, B, C. Suppose that there exists a non-planar tetrahedron $PABC$ whose four faces have equal areas. Then*

1. *ABC is acute-angled.*
2. *The tetrahedron $PABC$ is unique up to a symmetry about the plane of ABC.*
3. *Letting*
$$Q = \sqrt{1 - \frac{a^2}{c^2}} + \sqrt{1 - \frac{b^2}{c^2}} \quad \text{and} \quad q = \tan A \tan B,$$
the projection P' of P on the plane of ABC lies inside, outside or on the boundary of the triangle ABC according as $Q < 1$, $Q > 1$ or $Q = 1$. These conditions are equivalent to $q > 2$, $q < 2$ or $q = 2$, in this order.

Conversely, if ABC is acute-angled, then there exists a non-planar tetrahedron $PABC$ whose four faces are congruent (and hence have equal areas).

Proof. Suppose that there exists a non-planar tetrahedron $PABC$ whose four faces have equal areas. Let Δ be the area of the triangle ABC. For any point P, we let P' be the projection of P on the plane of ABC. By symmetry, we let P range in only one of the two closed half-spaces into which the plane of ABC divides the space. Thus a point P is uniquely determined by P' and by the distance PP'.

If h is the distance from the point P to the plane of ABC (i.e., to P') and if α, β and γ are the areas of the triangles PBC, PCA and PAB (respectively), then the heights PA', PB' and PC' of these triangles are $2\alpha/a$, $2\beta/b$ and $2\gamma/c$ (respectively). Since BC is perpendicular to both PA' and PP', it follows that it is perpendicular to $P'A'$. Thus the actual trilinear coordinates of P' with respect to ABC are $P'A'$, $P'B'$ and $P'C'$. By Pythagoras' theorem (applied to the triangles $PP'A'$, $PP'B'$ and $PP'C'$), these trilinear coordinates are given by

$$\pm\sqrt{\left(\frac{2\alpha}{a}\right)^2 - h^2}, \quad \pm\sqrt{\left(\frac{2\beta}{b}\right)^2 - h^2}, \quad \pm\sqrt{\left(\frac{2\gamma}{c}\right)^2 - h^2}.$$

Therefore the actual barycentric coordinates of P' are

$$\pm\frac{1}{2}\sqrt{4\alpha^2 - a^2h^2}, \quad \pm\frac{1}{2}\sqrt{4\beta^2 - b^2h^2}, \quad \pm\frac{1}{2}\sqrt{4\gamma^2 - c^2h^2}.$$

Hence the existence of a non-planar tetrahedron $PABC$ having faces of equal areas is equivalent to the existence of a positive solution h to one of the eight equations

$$2\Delta = \pm\sqrt{4\Delta^2 - a^2h^2} \pm \sqrt{4\Delta^2 - b^2h^2} \pm \sqrt{4\Delta^2 - c^2h^2}. \tag{1}$$

The uniqueness of such a $PABC$ is equivalent to the condition that there do not exist two distinct values of h such that each is a solution to one of these eight equations. Setting

$$u = \left(\frac{a}{2\Delta}\right)^2, \quad v = \left(\frac{b}{2\Delta}\right)^2, \quad w = \left(\frac{c}{2\Delta}\right)^2 \quad \text{and} \quad T = h^2, \tag{2}$$

(1) takes the form

$$1 = \pm\sqrt{1 - uT} \pm \sqrt{1 - vT} \pm \sqrt{1 - wT}. \tag{3}$$

Since $0 < a \le b \le c$, it follows that $0 < u \le v \le w$ and that T ranges over the interval $(0, 1/w]$. It also follows that each of the six quantities

$$-\sqrt{1-uT} \pm \sqrt{1-vT} \pm \sqrt{1-wT}, \quad \pm\sqrt{1-uT} - \sqrt{1-vT} \pm \sqrt{1-wT}$$

is less than 1. Actually, the largest of these six quantities is $\sqrt{1-uT} - \sqrt{1-vT} + \sqrt{1-wT}$ and this is clearly less than or equal to $\sqrt{1-uT}$. Thus of the eight equations in (3), the only relevant ones are

$$f(T) \;:=\; \sqrt{1-uT} + \sqrt{1-vT} + \sqrt{1-wT} = 1, \tag{4}$$

$$g(T) \;:=\; \sqrt{1-uT} + \sqrt{1-vT} - \sqrt{1-wT} = 1. \tag{5}$$

Note that if $u + v \le w$, then $(1-uT)(1-vT) > (1-wT)$ and

$$\begin{aligned} & (\sqrt{1-uT} + \sqrt{1-vT})^2 - (1 + \sqrt{1-wT})^2 \\ = \;& (w-u-v)T + 2(\sqrt{1-uT}\sqrt{1-vT} - \sqrt{1-wT}) > 0. \end{aligned}$$

Therefore if $u+v \le w$, then $\sqrt{1-uT} + \sqrt{1-vT} > 1 + \sqrt{1-wT}$ and neither of the equations (4) and (5) has a solution. Thus we restrict our attention to the case when $u + v > w$. By (2), this is equivalent to the condition that $a^2 + b^2 > c^2$ which in turn is the requirement that ABC is acute-angled (since c is the largest side). This proves part 1.

Let

$$Q = \sqrt{1 - u/w} + \sqrt{1 - v/w}.$$

We first note that if $Q = 1$, then $T = 1/w$ is a solution to both (4) and (5) and this solution gives a tetrahedron with P' on the boundary of the triangle ABC. So we restrict our attention to solutions of (4) and (5) in $(0, 1/w)$ and we show that (4) (respectively (5)) has a solution if and only if $Q < 1$ (respectively $Q > 1$) and that the solution is unique. Clearly, a solution $T \in (0, 1/w)$ of (4) (respectively of (5)) corresponds to a tetrahedron $PABC$ with P' lying inside (respectively outside) ABC.

Since $f(T)$ decreases with T (on $[0, 1/w]$) and since $f(0) = 3 > 1$ and $f(1/w) = Q$, it follows that (4) has a (necessarily unique) solution in $(0, 1/w)$ if and only if $Q < 1$.

We next consider (5). If $v = w$, then $g(T) = \sqrt{1-uT}$ and (5) has no solutions in $(0, 1/w]$. Otherwise,

$$\begin{aligned} & g'(T) = 0 \\ \iff \quad & \frac{-u}{2\sqrt{1-uT}} + \frac{-v}{2\sqrt{1-vT}} + \frac{w}{2\sqrt{1-wT}} = 0 \\ \iff \quad & \frac{u\sqrt{1-wT}}{\sqrt{1-uT}} + \frac{v\sqrt{1-wT}}{\sqrt{1-vT}} = w. \end{aligned}$$

Setting

$$H(T) = \frac{u\sqrt{1-wT}}{\sqrt{1-uT}} + \frac{v\sqrt{1-wT}}{\sqrt{1-vT}},$$

we easily see that

$$H'(T) = \frac{u(u-w)}{2\sqrt{1-wT}\sqrt{(1-uT)^3}} + \frac{v(v-w)}{2\sqrt{1-wT}\sqrt{(1-vT)^3}} < 0.$$

Also, $H(0) = u + v > w > 0 = H(1/w)$. It follows that there is a unique T in $(0, 1/w)$ for which $H(T) = w$, i.e., for which $g'(T) = 0$. Since $g'(0) = (-u - v + w)/2 < 0$ and $g(0) = 1$, it follows that there exists $T \in (0, 1/w)$ for which $g(T) = 1$ if and only if $g(1/w) > 1$, i.e., if and only if $Q > 1$. Here too, we record that when this happens then the solution is unique.

To complete the proof of parts 2 and 3, it remains to show that the conditions $Q < 1$ and $\tan A \tan B > 2$ are equivalent. This is seen as follows:

$$\begin{aligned}
& \sqrt{1 - u/w} + \sqrt{1 - v/w} < 1 \\
\iff\ & \sqrt{w - u} + \sqrt{w - v} < \sqrt{w} \\
\iff\ & 2\sqrt{w - u}\sqrt{w - v} < u + v - w \\
\iff\ & 4(w - u)(w - v) < (u + v - w)^2 \\
\iff\ & 3w^2 - 2uw - 2vw + 2uv < u^2 + v^2 \\
\iff\ & 2(c^4 - a^4 - b^4 + 2a^2b^2) < 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\
\iff\ & 2(c^2 - a^2 + b^2)(c^2 + a^2 - b^2) < 16(\Delta)^2 = (4\Delta)(4\Delta) \\
& \text{(by Heron's Formula [2, page 12])} \\
\iff\ & 2(2bc \cos A)(2ac \cos B) < (2bc \sin A)(2ac \sin B) \\
\iff\ & \tan A \tan B > 2.
\end{aligned}$$

To prove the converse, let ABC be a given acute-angled triangle. We complete ABC to a parallelogram $ABCD$ and fixing ABC, we swing ACD against the edge AC. As we do so, the angle between BC and CD decreases from $\angle BCA + \angle ACD = \angle C + \angle A > \angle B$ to $\angle BCA - \angle ACD = \angle C - \angle A < \angle B$. If we stop the swinging when BC and CD make an angle equal to $\angle B$, then it is easy to check that we get the desired tetrahedron. □

Bang's theorem now follows as a trivial corollary. We mention that tetrahedra with congruent faces are referred to in the literature as equifacial or isosceles. Some of their properties are investigated in [5].

Corrolary 2 (Bang's Theorem). *Let $PABC$ be a non-planar tetrahedron. If the faces have equal areas, then they are congruent.*

Before turning to higher dimensions, let us note that Bang's theorem can be viewed as an answer to a very special case of the following question: For a given triangle (of area δ say) and positive numbers α, β and γ, what are the conditions that guarantee the existence and uniqueness of a non-planar tetrahedron $PABC$ such that the areas of the faces PBC, PCA and PAB are α, β and γ? Bang's theorem guarantees existence and uniqueness in the case when ABC is acute-angled and when $\alpha = \beta = \gamma = \delta$. The following theorem shows that the restriction $\alpha = \beta = \gamma$ is not sufficient to guarantee uniqueness.

Theorem 3. *Let α, β, γ, and δ be positive numbers such that*

$$\alpha + \beta - \gamma > \delta, \quad \alpha - \beta + \gamma > \delta, \quad -\alpha + \beta + \gamma > \delta.$$

Then there exist a triangle ABC of area δ and four tetrahedra P_jABC, $1 \le j \le 4$, such that, for each j, the areas of the faces P_jBC, P_jCA and P_jAB are α, β and γ, respectively, and such that the projections of the P_j's on the plane of ABC are the incenter and the three excenters of the triangle ABC.

Proof. Since $\delta > 0$, it follows that α, β and γ qualify as the side-lengths of a triangle. Thus there is a triangle ABC with area δ and with

$$BC : CA : AB = \alpha : \beta : \gamma. \tag{6}$$

Let I be the incenter of ABC and draw from I a half-line perpendicular to the plane of ABC. Then for every point P on L, the heights from P of the triangles PBC, PCA and PAB are equal. Thus it follows from (6) that

$$\text{area}(PBC) : \text{area}(PCA) : \text{area}(PAB) = \alpha : \beta : \gamma. \tag{7}$$

Let h be the distance IP. As P moves along L away from I, h increases from 0 to ∞ and the quantity

$$F(h) := \text{area}(PBC) + \text{area}(PCA) + \text{area}(PAB) \tag{8}$$

moves from δ to ∞. Since

$$\delta < \alpha + \beta + \gamma, \tag{9}$$

it follows that there is $h > 0$ such that

$$F(h) = \alpha + \beta + \gamma. \tag{10}$$

By (7), the corresponding P satisfies the conditions

$$\text{area}(PBC) = \alpha, \quad \text{area}(PCA) = \beta, \quad \text{area}(PAB) = \gamma,$$

as desired.

To construct the other three tetrahedra, we replace I by any of the three excenters and repeat the same procedure. Equations (8), (9) and (10) will be modified accordingly. For example, for the excenter corresponding to C, these equations are replaced by

$$\begin{aligned} F(h) &= \text{area}(PBC) + \text{area}(PCA) - \text{area}(PAB) \\ \delta &< \alpha + \beta - \gamma \\ F(h) &= \alpha + \beta - \gamma, \end{aligned}$$

respectively. This completes the proof. □

3. Higher dimensions

We now turn to higher dimensions. Let $ABCD$ be a given tetrahedron with volume Δ and with face-areas a, b, c and d. The existence of a four-dimensional simplex $PABCD$ whose tetrahedral faces have equal volumes is equivalent to the existence of a solution to one of the 16 equations

$$1 = \pm\sqrt{1 - uT} \pm \sqrt{1 - vT} \pm \sqrt{1 - wT} \pm \sqrt{1 - sT} \tag{11}$$

where

$$u = \left(\frac{a}{3\Delta}\right)^2, \quad v = \left(\frac{b}{3\Delta}\right)^2, \quad w = \left(\frac{c}{3\Delta}\right)^2, \quad s = \left(\frac{d}{3\Delta}\right)^2, \tag{12}$$

and where T is the square of the height from P to the hyperplane of $ABCD$. The conditions for the existence of such a solution are worth exploring. As for uniqueness, it is not sufficient to show that none of the 16 equations in (11) can have more than one solution. It may happen that for a given tetrahedron two or more of these equations have solutions that do not coincide. This gives a clue to how to construct an example of a simplex whose tetrahedral faces have equal volumes without being congruent. The following theorem provides many such examples.

Theorem 4. *Let $ABCD$ be a tetrahedron with volume Δ and with face-areas a, b, c and d, $0 < a \leq b \leq c \leq d$. Suppose that*

$$a^2 + b^2 + c^2 + d^2 > d^2 \quad \text{and} \quad \sqrt{1 - \frac{a^2}{d^2}} + \sqrt{1 - \frac{b^2}{d^2}} + \sqrt{1 - \frac{c^2}{d^2}} \leq 1. \tag{13}$$

Then there exist at least two points P and R not in the hyperplane of $ABCD$ such that the tetrahedral faces of the simplexes $PABCD$ and $RABCD$ have equal volumes and such that the projections of P and R on the hyperplane of $ABCD$ lie, respectively, inside and outside the tetrahedron $ABCD$. In particular, there exists a four-dimensional simplex whose five tetrahedral faces have equal volumes but are not congruent (i.e. isometric).

Proof. Let u, v, w and s be as in (12). Then it follows from (13) that

$$u + v + w > s \quad \text{and} \quad \sqrt{1 - \frac{u}{s}} + \sqrt{1 - \frac{v}{s}} + \sqrt{1 - \frac{w}{s}} \leq 1.$$

Also let

$$\begin{aligned} F(T) &:= \sqrt{1 - uT} + \sqrt{1 - vT} + \sqrt{1 - wT} + \sqrt{1 - sT} \\ G(T) &:= \sqrt{1 - uT} + \sqrt{1 - vT} + \sqrt{1 - wT} - \sqrt{1 - sT} \end{aligned}$$

It is clear that F decreases on $[0, 1/s]$ and that

$$F(0) = 4 > 1 \quad \text{and} \quad F(1/s) = \sqrt{1 - \frac{u}{s}} + \sqrt{1 - \frac{v}{s}} + \sqrt{1 - \frac{w}{s}} \leq 1.$$

Therefore, there exists $T \in (0, 1/s]$ for which $F(T) = 1$.

Calculations similar to those we have performed on g in the proof of Theorem 1 show that $G'(0) < 0$ and that $G'(T) = 0$ at exactly one point in $(0, 1/s)$. Since $G(0) = 2$ and $G(1/s) \leq 1$, it follows that there is one $T \in (0, 1/s)$ at which $G(t) = 1$. This finishes the proof of the first statement.

It remains to construct a simplex $PABCD$ for which Bang's statement does not hold. Start with a regular tetrahedron $ABCD$ of volume 1 and let P' be the point whose barycentric coordinates are $(1/2, 1/2, 1/2, -1/2)$. Let L be the half-line from P' perpendicular to the hyperplane of $ABCD$, let P move on L and let h be the distance from P' to P. Since the tetrahedra $P'BCD$, $P'ACD$, $P'ABD$ and $P'ABC$ have equal volumes, it follows that their heights from P' are equal. Pythagoras' theorem then implies that the heights from P of the tetrahedra $PBCD$, $PACD$, $PABD$ and $PABC$ are equal and hence their volumes are equal. As h increases from 0 to ∞, the volume of $PABC$ increases from $1/2$ to ∞. Thus for some $h > 0$, the tetrahedra will each have volume 1 and the tetrahedral faces of $PABCD$ will have equal volumes. The tetrahedral faces of $PABCD$ cannot be congruent because that would imply that $PABCD$ is the regular simplex [2, pages 396-414], and that the barycentric coordinates of P' would be $(1/4, 1/4, 1/4, 1/4)$. Finally, our $PABCD$ is non-degenerate because it corresponds to $h > 0$. This completes the proof. $\square$

The conditions imposed on $ABCD$ in the previous theorem are not severely restrictive. In fact, if $ABCD$ is regular (or has equal faces), then

$$u+v+w=3s \quad \text{and} \quad \sqrt{1-\frac{u}{s}}+\sqrt{1-\frac{v}{s}}+\sqrt{1-\frac{u}{s}}=0$$

and our conditions are lavishly satisfied. Thus in constructing a simplex for which Bang's theorem does not hold, we have a rich variety of choices for the base tetrahedron $ABCD$.

We close by posing the question regarding what tetrahedra qualify as facets of four-dimensional simplices with congruent facets. In view of Theorem 1, such tetrahedra would be, in a sense, three-dimensional analogues of acute-angled triangles.

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