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On the Existence of Shapes of Roofs

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Abstract. Roofs considered in this article are defined as special geometric polyhedral surfaces, on the basis of two assumption: (1) all eaves of a roof form a planar (simply connected or k-connected) polygon called the base of this roof, (2) every hipped roof end forms the same angle slope with the (horizontal) plane which contains the base of this roof. Basing on the Euler formula for regular roofs formulated and proved in a previous paper we formulate properties which enable a study of the shapes of roofs. We indicate how these properties may be used to analyse the shapes of roofs. We suggest a method of classification and a detailed classification of shapes of roofs for polygons with a small number of edges.

Key Words: geometry of roofs, generalized polygon, k-connected generalized polygon, regular graphs, Euler formula for regular roofs, classification of shapes of roofs

MSC 2000: 51N05, 52B05, 05C90, 68U05

1. Introduction

This article is the second part of a work which deals with geometrical properties of roofs of buildings. In [7] we formulated and proved an Euler formula for regular roofs and basic properties of roofs considered as a special class of polyhedral surfaces from the view point of Graph Theory. In this paper we develop the geometrical characterization of roofs. We formulate and prove the main theorem (Theorem 2) and useful detailed properties obtained from it. These properties are the key to derive a classification of the shapes of roofs. Hence we obtain a method to study the topological characterization of roofs. Next, we demonstrate how these properties may be used to analyse and classify the shapes of roofs.

In this article we use the notation and terminology from [7]. First, let us recall the main results obtained in [7]: (main) Theorem 1 (= Euler formula for regular roofs), Proposition 1 and the equations of a regular roof generated by a k-connected generalized v-gon. These are

Theorem 1 (Euler formula for regular roofs)**.** *If the base of a regular roof is a k-connected generalized v-polygon, then the number of ridges of this roof is $r = 2v + 3(k-2)$, the number of top points is $t = v + 2(k-2)$, and the number of disappearing ridges is $d = v + 3(k-2)$.*

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Proposition 1. *If a roof generated by a k-connected generalized v-polygon is regular, then the graph* (T,R') *of the line of disappearing ridges induced by this roof is connected.*

And we recall the equations of roofs which belong to the class of regular roofs:

$$\sum_{i=3}^{v} m_i = v, \tag{1}$$

$$\sum_{i=3}^{v} i\, m_i = 5v + 6(k-2), \tag{2}$$

where m_i denotes the number of i-gonal hipped roof ends for $i = 3, 4, \ldots, v$.

2. Algebraic and geometric examination of the roof equations

First, we make two obvious statements. Two eaves of a roof are said to be *adjacent* (they adjoin, border each other) if they have a common corner vertex of the base of this roof. Similarly, two hipped roof ends are said to be *adjacent* (they adjoin, border each other) if they have a common roof hips ridge (corner ridge).

Statement 1. *If two triangles as hipped roof ends adjoin in a regular, simply connected roof, then the base of this roof must be a triangle.*

Proof. Let e_1, e_2 be two adjacent edges of the roof $\mathrm{R}(\mathrm{e}_1,\mathrm{e}_2,\ldots,\mathrm{r}_1,\mathrm{r}_2,\ldots)$. Let us consider two adjoining triangles $(\mathrm{e}_1,\mathrm{r}_1,\mathrm{r}_2)$, $(\mathrm{e}_2,\mathrm{r}_1,\mathrm{r}_3)$ as the hipped roof ends of a given roof with a common ridge r_1. Then the edges r_1, r_2, r_3 meet in a common point $T = (\mathrm{r}_1,\mathrm{r}_2,\mathrm{r}_3)$ which must be a unique top point of the roof. Indeed, from the top point T starts no ridge r different from r_1, r_2, r_3. Then the edges r_2, r_3 must be two corner (roof hips) ridges. Hence the segments r_1, r_2, r_3 are all ridges of the roof. The endpoints of ridges $\mathrm{r}_2,\mathrm{r}_3$ different from T form a third edge e_3 of the roof. Therefore the roof $\mathrm{R}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{r}_1,\mathrm{r}_2,\mathrm{r}_3)$ is a unique roof satisfying our assumptions. Then we obtain the complete roof having the triangle $(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3)$ as the base. □

Statement 2. *If in any regular, simply connected roof there exists a quadrangle as hipped roof end which adjoin with two triangles as hipped roof ends, then the base of this roof must be a quadrangle.*

Proof. Let us consider a quadrangle $(\mathrm{e}_1,\mathrm{r}_1,\mathrm{r}_2,\mathrm{r}_3)$, and two triangles $(\mathrm{e}_2,\mathrm{r}_1,\mathrm{r}_5)$, $(\mathrm{e}_3,\mathrm{r}_3,\mathrm{r}_4)$ adjoin with it as three hipped roof ends of a certain regular simply connected roof $\mathrm{R}(\mathrm{e}_1,\mathrm{e}_2,\ldots,\mathrm{r}_1,\mathrm{r}_2,\ldots)$. Then r_1, r_3 are the common corner (roof hips) ridges of suitably two pairs hipped roof ends $((\mathrm{e}_1,\mathrm{r}_1,\mathrm{r}_2,\mathrm{r}_3),(\mathrm{e}_2,\mathrm{r}_1,\mathrm{r}_5))$, $((\mathrm{e}_1,\mathrm{r}_1,\mathrm{r}_2,\mathrm{r}_3),(\mathrm{e}_3,\mathrm{r}_3,\mathrm{r}_4))$. Since the considered roof is regular the ridge r_2 is not any corner roof hips ridge. It is the consequence of the fact that the corner ridges r_1, r_3 belong to one quadrangle. So the points $T_1 = (\mathrm{r}_1,\mathrm{r}_2,\mathrm{r}_5)$, $T_2 = (\mathrm{r}_2,\mathrm{r}_3,\mathrm{r}_4)$ are unique top points of the regular roof $\mathrm{R}(\mathrm{e}_1,\mathrm{e}_2,\ldots,\mathrm{r}_1,\mathrm{r}_2,\ldots)$ which satisfies the above assumptions. Then the line segments r_5, r_2, r_4 must be the ridges of a certain quadrangle which has the ridges $\mathrm{r}_4,\mathrm{r}_5$ as corner roof hips ridges, the ridge r_2 as disappearing ridge, and a fourth ridge denoted by e_4. Then we obtain the complete roof $\mathrm{R}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4,\mathrm{r}_1,\mathrm{r}_2,\ldots,\mathrm{e}_5)$ having a quadrangle $(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4)$ as the base. □

From Theorem 1, Proposition 1 and due to classical Graph Theory we obtain the following corollaries:

Corollary 1. *The graph* $(\mathrm{T}, \mathrm{R}')$ *of the line of disappearing ridges of a regular simply connected roof is connected.*

Proof. In Proposition 1 we set $k = 1$. □

Corollary 2. *The graph of disappearing ridges* $(\mathrm{T}, \mathrm{R}')$ *of an arbitrary regular simply connected roof is a tree and has at least two leaves.*

Proof. Due to Theorem 1 for $k = 1$ the graph $(\mathrm{T}, \mathrm{R}')$ has n $(= v - 2)$ vertices and $n - 1$ $(= v - 3)$ edges, and by Corollary 1 it is connected. So it is a tree (cf. [1], [2]). Then the graph $(\mathrm{T}, \mathrm{R}')$ has at least two leaves. □

Corollary 3. *Every regular simply connected roof contains at least two triangular hipped roof ends.*

Proof. Due to Corollary 2 the graph of ridges of an arbitrary roof has at least two leaves. From each of them three ridges are starting: one is a disappearing ridge and the two remaining are corner ridges, which with any edge of the base of the roof form a triangle. Then we have at least two triangular hipped roof ends. □

A number of triangular hipped roof ends may be at least 2 and due to Statement 1 at most $\left[\frac{v}{2}\right]$ (the symbol [.] denotes the function *entier*). For example, the "spiral" roof in Fig. 1 has only two triangular hipped roof ends, the roof generated by a base with rectangular half-transepts or cutting off corners has a lot of triangular hipped roof ends (cf. Fig. 1).

Now we formulate the theorem which exemplifies the number of sides of the hipped roof ends.

Theorem 2. *If a regular simply connected roof generated by a* v*-gon has* s_i $(v-n-i+1)$*-gonal hipped roof ends for* $i = 1, 2, \ldots, p$ *with* $n \geq 0$, $n + p + 2 \leq v$, $1 \leq \sum_{i=1}^{p} s_i \leq v$, *then under the assumption that certain* i*-gonal hipped roof ends may have a common disappearing ridge*

$$v \leq n+3+\frac{n + \sum_{j=3}^{p}\sum_{i=2}^{j-1} s_j s_i + (s_1 - 1)\sum_{i=2}^{p} s_i + \sum_{i=2}^{p}\frac{s_i+1}{2}s_i + \frac{s_1(s_1-1)}{2} + \sum_{i=1}^{p} s_i(i-1)}{\sum_{i=1}^{p} s_i - 1} \quad (3)$$

and, under the assumption that every two i*-gonal hipped roof ends do not share any disappearing ridge for* $i = 1, 2, \ldots, p$

$$v \leq n + 3 + \frac{n + \sum_{i=1}^{p} s_i(i-1)}{\sum_{i=1}^{p} s_i - 1}\,. \quad (4)$$

Proof. Case (i): Let us assume, that any i-gonal hipped roof end may have a common disappearing ridge for $i = 1, 2, \ldots, v$. Then we give the following reasoning:

The first $(v-n)$-gonal hipped roof end induces $v - n - 3$ disappearing ridges, the second $(v-n)$-gonal hipped roof end induces $v-n-4$ successive disappearing ridges, and so on: the s_1-th hipped roof end induces successive $v - n - 1 - 3 - (s_1 - 1)$ disappearing ridges. Next, the first $(v - n - 1)$-gonal hipped roof end induces successive $v - n - 1 - (p - 1) - 3 - s_1 - 1$ disappearing ridges, and so on, the last $(v - n - (p - 1))$-gonal hipped roof end induces $v - n - (p-1) - 3 - (s_1 - 1) - s_2 - \ldots - s_{p-1} - s_p$ disappearing ridges. The sum of all such

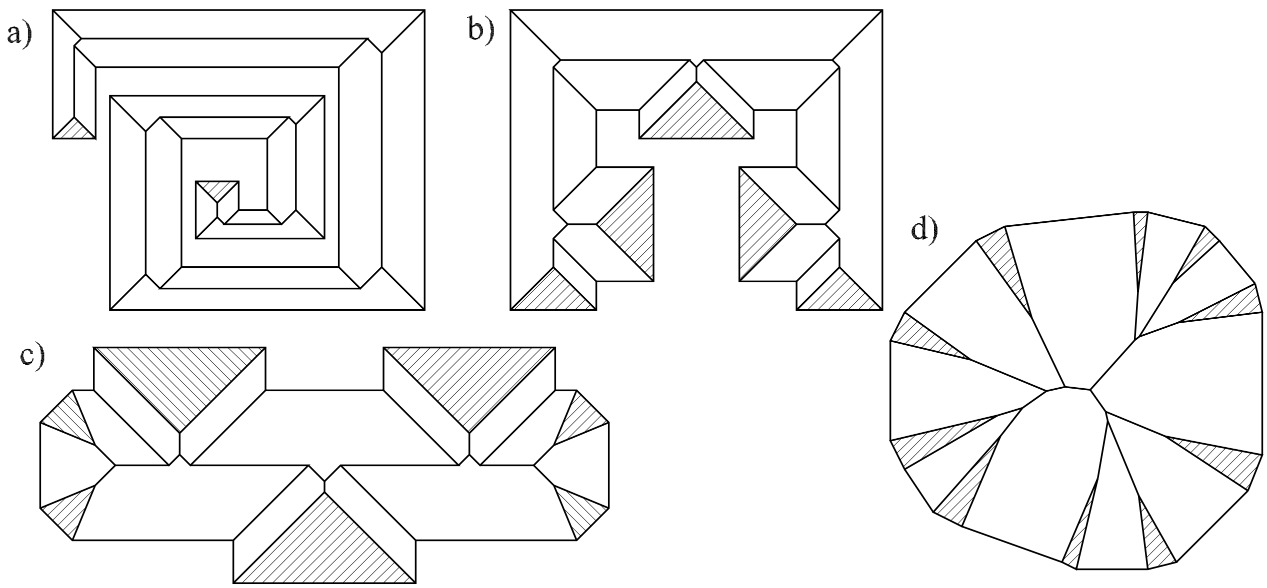

Figure 1: Four regular roofs generated by a 20-gon as basis:
a) a "spiral" roof with the minimal number of two triangular hipped roof ends,
b) a roof on a base with rectangular half-transepts with five triangular hipped roof ends,
c) a roof on a base with three rectangular half-transepts and four cutting off corners with seven triangular hipped roof ends,
d) a nonsymmetric roof with the maximal number of ten triangular hipped roof ends (ten cuttings of corners)

ridges cannot exceed the number $v-3$. Therefore we can write the following inequality

$$\begin{aligned}
&(v-n-3)+(v-n-3-1)+\ \dots\ +(v-n-3-(s_1-1))\\
&+(v-n-1-3-(s_1-1)-1)+(v-n-1-3-(s_1-1)-2)\\
&+\ \dots\ +(v-n-1-3-(s_1-1)-s_2)\\
&+(v-n-1-3-(s_1-1)-s_2-1)+(v-n-1-3-(s_1-1)-s_2-2)\\
&+\ \dots\ +(v-n-1-3-(s_1-1)-s_2-s_3)\\
&+\ \dots\ +\\
&+(v-n-1-3-(s_1-1)-s_2-\ \dots\ -s_{p-1}-1)\\
&+(v-n-1-3-(s_1-1)-s_2-\ \dots\ -s_{p-1}-2)\\
&+\ \dots\ +(v-n-1-3-(s_1-1)-s_2-\ \dots\ -s_{p-1}-s_p)\ \le\ v-3
\end{aligned}\tag{5}$$

Note that the variable v appears $\sum_{i=1}^{p} s_i$-times with "plus" and once on the left side with "minus", the number 3 appears $\sum_{i=1}^{p} s_i$-times, n appears $\sum_{i=1}^{p} s_i$-times (to obtain the expression $n(\sum_{i=1}^{p} s_i - 1)$ we must add n and simultaneously subtract n). Arranging the above expressions we obtain

$$v\Big(\sum_{i=1}^{p} s_i - 1\Big) - 3\Big(\sum_{i=1}^{p} s_i - 1\Big) - n - n\Big(\sum_{i=1}^{p} s_i - 1\Big),\tag{6}$$

next, we arrange other expressions (in the first row we have sum $1+2+\ \dots\ +s_1-1$, in the second row we have sum $1+2+\ \dots\ +s_2$ and s_1-1 occurs s_2-times, in the third row we have sum $1+2+\ \dots\ +s_3$ and s_1-1 occurs s_3-times, and s_2 occurs s_3-times, and so on) and we obtain

$$\begin{aligned}
&-\frac{s_1(s_1-1)}{2}\\
&-s_2(s_1-1)-\frac{s_2+1}{2}s_2\\
&-s_3(s_1-1)-s_3s_2-\frac{s_3+1}{2}s_3\\
&-s_4(s_1-1)-s_4s_2-s_4s_3-\frac{s_4+1}{2}s_4\\
&\dots\dots\dots\dots\\
&-s_{p-1}(s_1-1)-s_{p-1}s_2-s_{p-1}s_3-\ \dots\ -s_{p-1}s_{p-2}-\frac{s_{p-1}+1}{2}s_{p-1}\\
&-s_p(s_1-1)-s_ps_2-s_ps_3-\ \dots\ -s_ps_{p-1}-\frac{s_p+1}{2}s_p.
\end{aligned}\tag{7}$$

Finally, we arrange the remaining expressions of the left hand side of the inequality in the following manner:

$$-0s_1-1s_2-3s_3-4s_4-\ \dots\ -(p-1)s_p\ \leq\ 0.\tag{8}$$

We rewrite the rows (7) in the following form:

$$\begin{array}{ccc}
-0 & -0 & -\dfrac{s_1(s_1-1)}{2}\\
-0 & -s_2(s_1-1) & -\dfrac{s_2+1}{2}s_2\\
-\sum\limits_{i=2}^{2} s_3s_i & -s_3(s_1-1) & -\dfrac{s_3+1}{2}s_3\\
-\sum\limits_{i=2}^{3} s_4s_i & -s_4(s_1-1) & -\dfrac{s_4+1}{2}s_4\\
\dots & \dots & \dots\\
-\sum\limits_{i=2}^{p-2} s_{p-1}s_i & -s_{p-1}(s_1-1) & -\dfrac{s_{p-1}+1}{2}s_{p-1}\\
-\sum\limits_{i=2}^{p-1} s_ps_i & -s_p(s_1-1) & -\dfrac{s_p+1}{2}s_p\,.
\end{array}$$

We rearrange (9) and obtain

$$-\sum_{j=3}^{p}\sum_{i=2}^{j-1}s_js_i-(s_1-1)\sum_{i=2}^{p}s_i-\sum_{i=2}^{p}\frac{s_i+1}{2}s_i-\frac{s_1-1}{2}s_1\,.\tag{10}$$

Finally, we get the following form of the inequality ((6), (7), (8))

$$\begin{aligned}
&v\Big(\sum_{i=1}^{p}s_i-1\Big)-3\Big(\sum_{i=1}^{p}s_i-1\Big)-n-n\Big(\sum_{i=1}^{p}s_i-1\Big)-\sum_{j=3}^{p}\sum_{i=2}^{j-1}s_js_i-\\
&\qquad(s_1-1)\sum_{i=1}^{p}s_1-\sum_{i=2}^{p}\frac{s_i+1}{2}s_i-\frac{s_1-1}{2}s_1-\sum_{i=1}^{p}(i-1)\ \leq\ 0\,.
\end{aligned}\tag{11}$$

Hence we obtain the inequality (3).

Case (ii): Let us assume now, that every two i-gonal hipped roof ends do not share any disappearing ridge for all $i = 1, 2, \dots, v$. Then each $(v-n)$-gon induces $v-n-3$ disappearing ridges. This implies the inequality

$$(v-n-3)s_1 - (v-n-1-3)s_2 - (v-n-2-3)s_3 - \ \dots \ - (v-n-(p-1)-3)s_p \ \leq \ v-3. \tag{12}$$

By arranging appropriate expressions we obtain

$$v\Big(\sum_{i=1}^{p} s_i - 1\Big) - 3\Big(\sum_{i=1}^{p} s_i - 1\Big) - n - n\Big(\sum_{i=1}^{p} s_i - 1\Big) - \sum_{i=1}^{p} s_i(i-1) \ \leq \ 0\,. \tag{13}$$

Hence we easily obtain inequality (4). □

As an immediate consequence of Theorem 2 we have the following corollaries:

Corollary 4. *If a regular simply connected roof generated by a v-gon has s $(v-n)$-gonal hipped roof ends with $n+3 \leq v$ and $1 \leq s \leq v$, then under the assumption that these hipped roof ends may have a common disappearing ridge, the inequality*

$$v \ \leq \ n + 3 + \frac{s}{2} + \frac{n}{s-1} \tag{14}$$

holds, and under the assumption that every two of these hipped roof ends do not share any disappearing ridge, the following inequality holds:

$$v \ \leq \ n + 3 + \frac{n}{s-1}\,. \tag{15}$$

Proof. It suffices to set $p = 1$ and $s_1 = s$ in Eq. (3) of Theorem 2. □

Corollary 5. *If a regular simply connected generated by a v-gon roof has two v-gonal hipped roof ends, then $v \leq 4$.*

Proof. We set $p = 1$, $s_1 = 2$ and $n = 0$ in Eq. (3) of Theorem 2. □

Corollary 6. *If a regular simply connected generated by a v-gon roof has three v-gonal hipped roof ends, then $v \leq 3$.*

Proof. If a roof has three v-gonal hipped roof ends, then it has two v-gonal hipped roof ends and due to Corollary 5 v is equal 3 or 4. Due to Corollary 3 a regular simply connected roof must contain at least two triangular hipped roof ends. Therefore the considered roof has no three quadrangles. Consequently such a roof must contain three triangles as hipped roof ends, hence at least two triangles must adjoin. When according to Statement 1 the considered roof must be generated by a triangle. □

It is convenient to formulate

Theorem 3. *If in a regular simply connected roof generated by a v-gon there are s m_i-gonal hipped roof ends ($3 < m_i$ for $i = 1, 2, \dots, s$) such that no two of them share a disappearing ridge, then*

$$\sum_{i=1}^{s} m_i - 3s + 3 \ \leq \ v\,. \tag{16}$$

Proof. Suppose that a roof has s m_i-gonal hipped roof ends such that no two share a disappearing ridge. Then every such polygon induces $m_i - 3$ ridges. These ridges are different, therefore all s hipped roof ends together give $\sum_{i=1}^{p}(m_i - 3)$ distinct disappearing ridges. The obtained number must be less than or equal to $v - 3$. Hence we easily obtain the inequality (3). □

The particular specification $m_i = m$ for $i = 1, 2, \ldots, s$ yields

Corollary 7. *If a regular simply connected roof generated by a v-gon has s m-gonal hipped roof ends* ($3 \leq m \leq v$) *such that no two share a disappearing ridge, then*

$$v \geq s(m-3)+3. \tag{17}$$

3. An existence analysis of the shapes of roofs generated by simply connected polygons

In this section we will consider roofs generated by simply connected polygons. So we put $k = 1$. Then the equations (1), (2) assume the following form

$$m_3 + m_4 + \ldots + m_v = v, \tag{18}$$

$$3m_3 + 4m_4 + \ldots + vm_v = 5v - 6. \tag{19}$$

Due to Corollary 6 every simply connected roof generated by a v-gon for $v > 4$ may contain at most one v-gonal hipped roof end. The above remark is very useful for an algebraic examination of the equations (18), (19). It eliminates a lot of solutions of these equations.

First, we note that for every v-gon we have the following solution

$$m_3 = 2, \quad m_4 = v-3, \quad m_5 = 0, \quad \ldots, \quad m_{v-1} = 0, \quad m_v = 1. \tag{20}$$

Indeed, the equality

$$3 \cdot 2 + 4 \cdot (v-3) + 5 \cdot 0 + \ldots + (v-1) \cdot 0 + v \cdot 1 = 5v - 6 \tag{21}$$

holds. Such a roof contains as one hipped roof end a polygon which has the maximal number (equal v) of sides. The remaining hipped roof ends are triangles and quadrangles. The above solution of a roof exists for all v with $v \geq 3$. Moreover, note that the assumption of an existing v-gonal hipped roof end determines the roof up to an isomorphism:

Indeed, the v-gonal hipped roof end has two corner ridges and $v - 3$ disappearing ridges (each hipped roof end has exactly one edge and exactly two corner ridges independent of a regularity of a roof). These $v - 3$ ridges form an unicoursal polygonal arc and they must be ridges of $v - 3$ polygons. Each a such polygon must have exactly one disappearing ridge. So these polygons must be quadrangles.

Indeed, if we assumed that these polygons were triangles or n-gones with $4 < n < v$, then triangles which have no disappearing ridges would induce a top point with degree 4, any n-gon with $4 < n < v$ would have more than 1 common ridge with a v-gonal hipped roof end, contrary to the regularity of the roof.

Such a roof will be called *universal*. Geometric shapes of such roofs are, e.g., "pie" or "stairs" in Fig. 2.

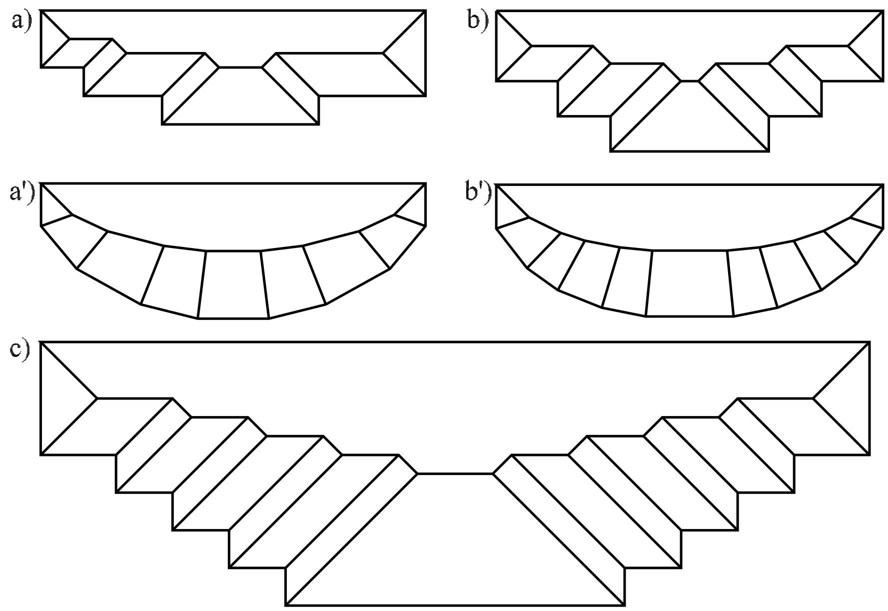

Figure 2: Universal shapes of regular roofs generated by a v-gon (for $v = 10, 12, 20$):
a) "stairs" ($v = 10$), a') "pie" ($v = 10$);
b) "stairs" ($v = 12$), b') "pie" ($v = 12$); c) "stairs" ($v = 20$)

3.1. Regular roofs generated by a v-gon, v=3,4,5,6,7,8

We will use the properties of roofs proved above in an examination of shapes of roofs. We adopt the following *convention*: If a roof has n m-gonal hipped roof ends then we will write $m(n)$.

In order to study the shapes of roofs we will examine the equations of a roof (1), (2) for $k = 1$. It appears that every roof spread over a v-gon for $v \geq 9$ can be decomposed into several roofs determined by v-gons with $v = 5, 6, 7, 8$. We restrict our considerations to the discussion of only one (most representative) case for $v = 8$. The results of the analysis of the remaining cases (for $v = 3, 4, 5, 6, 7$) are omitted in this paper; we will describe them comprehensively in appropriate figures and in Table 1.

C8) Let $v = 8$.
For $v = 8$ the equations (1), (2) assume the form

$$\begin{cases} m_3 + m_4 + m_5 + m_6 + m_7 + m_8 = 8, \\ 3m_3 + 4m_4 + 5m_5 + 6m_6 + 7m_7 + 8m_8 = 34. \end{cases}$$

Consider the subcases:

C81) For $m_8 = 1$ due to (20) we have only one geometrically correct solution 8(1), 7(0), 6(0), 5(0), 4(5), 3(2) (see Fig. 7c, c').

C82) Let $m_8 = 0$. We have subcases:

C821) $m_7 = s$ with $s \geq 2$. This case due to Corollary 7 is impossible. It suffices to assume $v = 8$, $m = 7$, $s \geq 2$ in (17). Then we have one possibility.

C822) $m_7 = 1$, then we have

C8221) $m_6 = m$ with $m \geq 1$. This case is impossible. Indeed, due to Theorem 2 substituting

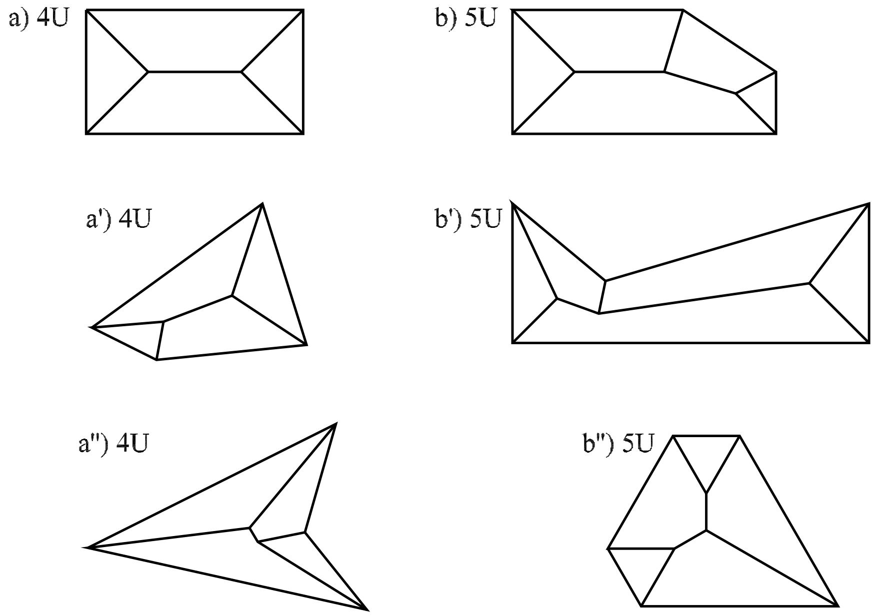

Figure 3: a), a'), a") seemingly different realizations of roofs determined by a quadrangle; b), b'), b") seemingly different realizations of roofs determined by a pentagon

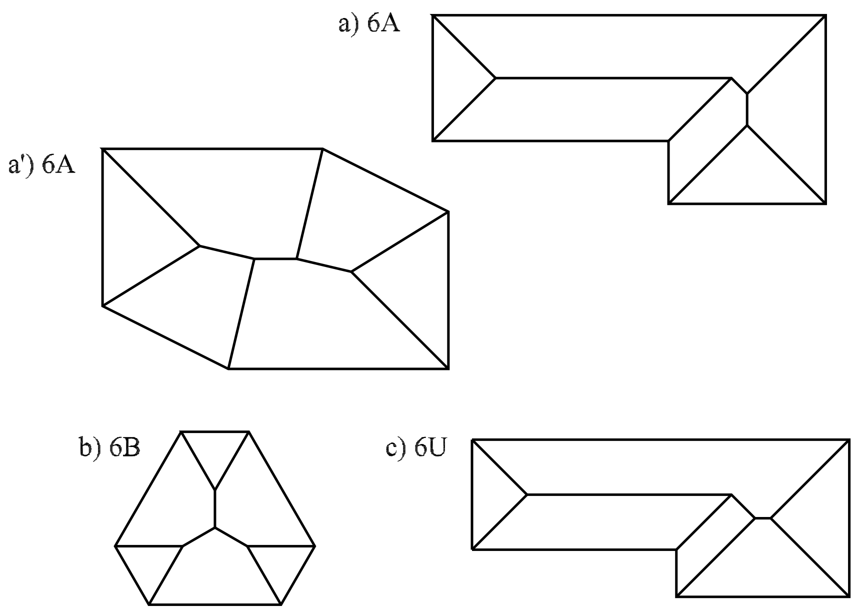

Figure 4: Different topological kinds of roofs a), b), c) determined by a hexagon; a), a') seemingly different realizations of roofs determined by a hexagon

$v = 8$, $p = 2$, $n = 1$, $s_1 = 1$, $s_2 = m$ into (3) we obtain an inequality

$$m + 1 \ \leq \ 8 \ \leq \ 5 + \frac{1}{m} + \frac{m+1}{2}\,. \tag{22}$$

The left inequality (22) is satisfied by $m = 1, 2, \ldots, 7$. The right inequality (22) is not satisfied by $m = 1, 2, 3, 4$. For $m = 5, 6, 7$ the equation (2) with $k = 1$, $v = 8$ is not satisfied. Then we may consider the case 6(0) only.

C8222) Let $m_6 = 0$.

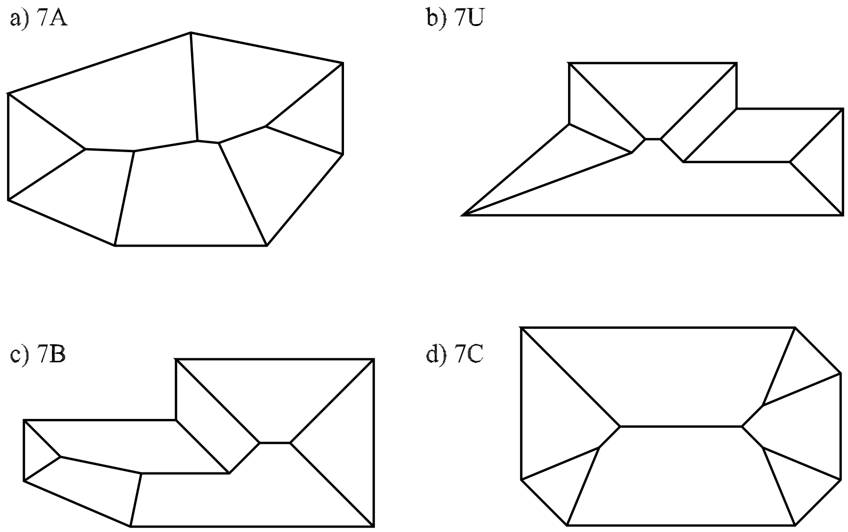

Figure 5: Different topological kinds of roofs determined by a 7-gon

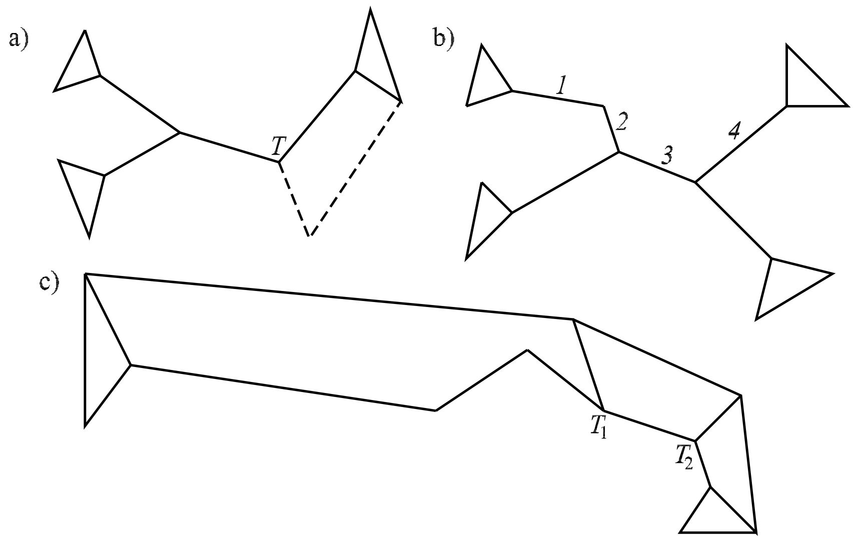

Figure 6: A sketch of the analysis of some roofs

Let us consider the following subcases

C82221) $m_5 = 5$. This case is impossible, because $3 \cdot 2 + 4 \cdot k + 5 \cdot 5 + 7 \cdot 1 > 34$ (we must regard at least two triangles) for every integer positive k.

C82222) $m_5 = 4$. The case 5(4) is impossible because we have $3 \cdot 2 + 4 \cdot k + 5 \cdot 4 + 7 \cdot 1 \neq 34$ and $3 \cdot 3 + 4 \cdot k + 5 \cdot 4 + 7 \cdot 1 \neq 34$ and $3 \cdot 4 + 4 \cdot k + 5 \cdot 4 + 7 \cdot 1 \neq 34$ for every positive integer k.

C82223) $m_5 = 3$. Then we have a solution 8(0), 7(1), 6(0), 5(3), 4(0), 3(4) and a roof must contain four triangles as hipped roof ends. A line of disappearing ridges has a form which is displayed in Fig. 6b (a line with four components 1, 2, 3, 4 derives from the 7-gon). Then a number of disappearing ridges is equal to 6, which is impossible ($d = 8 - 3 \neq 6$, cf. Theorem 1 for $k = 1$, $v = 8$).

C82224) $m_5 = 2$. Then we have a solution 8(0), 7(1), 6(0), 5(2), 4(2), 3(3). The roof which satisfies such parameters is displayed in Fig. 7a.

C82225) $m_5 = 1$. Then we have a solution 8(0), 7(1), 6(0), 5(1), 4(4), 3(2). The roof which

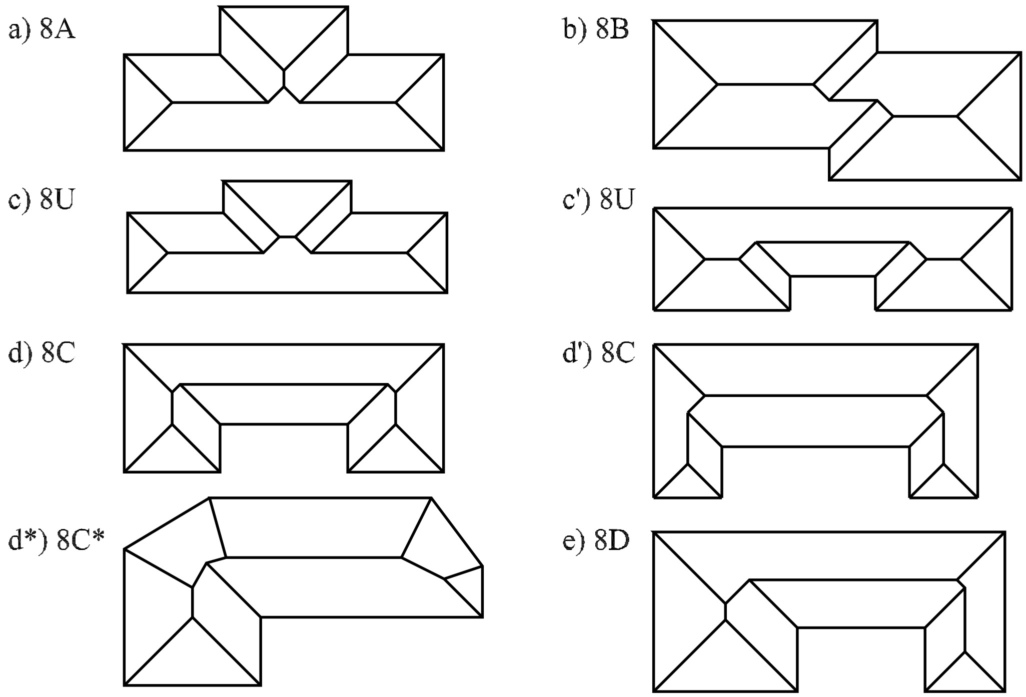

Figure 7: Different topological kinds of roofs determined by an octagon

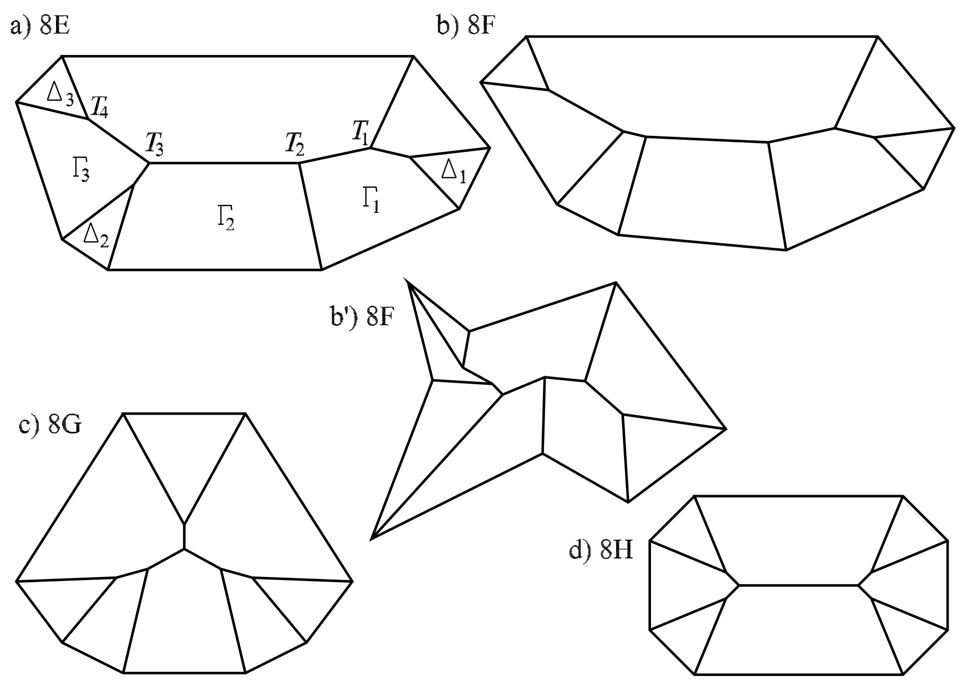

Figure 8: Remaining topological kinds of roofs determined by an octagon

satisfies such parameters is displayed in Fig. 8b, b'.

Let us consider

C823) Let $m_7 = 0$. Then we have the following cases:

C8231) $m_6 = s$ with $m \geq 3$. This case due to Corollary 4 with parameters $n = 2$, s is

Table 1: Classification of solutions of shapes of a regular roof generated by a simply connected v-gon, $v = 3, 4, 5, 6, 7, 8$. The symbol '+' indicates the existence of a roof (in the algebraic sense this symbol shows the existence of a solution of the equations of a roof). The symbols '$^{n)}$' and '−' stand for the non-existence of a geometrical (i.e., real) roof

Code	*Base of the roof*	*Number of v-gons* 3	4	5	6	7	8	*Kind of solution* *algebraic*	*geometrical*
T01	triangle	3						+	+ (universal roof)
Q01	quadrangle	2	2					+	+ (universal roof, Figs. 3a, a', a")
P01$^{n)}$	pentagon	3	0	2				+	− (Statement 1)
P02$^{n)}$		1	4	0				+	− (Corollary 3)
P03		3	0	2				+	+ (universal roof, Figs. 3b, b', b")
H01$^{n)}$	hexagon	4	0	0	2			+	− (Statement 1)
H02$^{n)}$		3	1	1	1			+	− (Statement 2)
H03		2	3	0	1			+	+ (universal roof, Fig. 4c)
H04		2	2	2	0			+	+ (Figs. 4a, a')
H05		3	0	3	0			+	+ (Fig. 4b)
S01	7-gon	2	4	0	0	1		+	+ (universal roof, Fig. 5b)
S02		3	1	2	1	0		+	+ (Fig. 5d)
S03		2	3	1	1	0		+	+ (Fig. 5c)
S04		2	2	3	0	0		+	+ (Fig. 5a)
S05$^{n)}$		3	2	0	2	0		+	− (Corollary 7)
S06$^{n)}$		4	0	1	2	0		+	− (Statement 1)
O01$^{n)}$	octagon	6	0	0	0	0	2	+	− (Statement 1)
O02$^{n)}$		5	0	1	1	0	1	+	− (Statement 1)
O03		2	5	0	0	0	1	+	+ (universal roof, Figs. 7c, c')
O04		3	2	2	0	1	0	+	+ (Fig. 7a)
O05		2	4	1	0	1	0	+	+ (Figs. 8b, b')
O06		4	0	2	2	0	0	+	+ (Fig. 8d)
O07		3	2	1	2	0	0	+	+ (Fig. 8c)
O08		2	4	0	2	0	0	+	+ (Fig. 7b)
O09		2	3	2	1	0	0	+	+ (Figs. 7d, d'; d*)
O10		3	1	3	1	0	0	+	+ (Fig. 8a)
O11$^{n)}$		3	0	5	0	0	0	+	− (the case C82342)
O12		2	2	4	0	0	0	+	+ (Fig. 7e)

impossible. Indeed, then we have an inequality

$$8 \ \leq \ 5 + \frac{s}{2} + \frac{2}{s-1}. \tag{23}$$

The numbers $s = 3, 4, 5$ do not satisfy the above inequality. For $s = 6, 7$ at least two hexagonal hipped roof ends must be adjacent. Then a line of disappearing ridges would contain at least six components, which is impossible (cf. Theorem 1).

C8232) $m_6 = 2$. Then we have

C82321) $m_5 = s$ with $s \geq 3$. Assuming that $m_3 \geq 2$ this case is algebraically impossible. The equation $3m_3 + 4m_4 + 5s + 6 \cdot 2 = 34$ has not got any solution for $s \geq 3$.

C82322) $m_5 = 2$. Then we have the solution 8(0), 7(0), 6(2), 5(2), 4(0), 3(4). Due to Theorem 1 two hexagononal hipped roof ends must have a common disappearing ridge and with regard to an existence of four triangles the roof must have a form displayed in Fig. 8d.

C82323) $m_5 = 1$. We have a solution 8(0), 7(0), 6(2), 5(1), 4(2), 3(3). Due to Theorem 1 two hexagononal hipped roof ends must have a common disappearing ridge and with regard to the existence of three triangles these hexagons must be adjacent along the first disappearing ridge. Then we obtain the roof which is displayed in Fig. 8c.

C82324) $m_5 = 0$. We have a solution 8(0), 7(0), 6(2), 5(0), 4(4), 3(2). Notice that three quadrangles may not be adjacent in a sequence and two hexagons may not have any common corner ridge. In both cases a line of disappearing ridges would contain too many components. Therefore the roof has a form displayed in Fig. 7b.

C8233) $m_6 = 1$. Let us consider the following cases:

C82331) $m_5 = s$ with $s \geq 4$. Such a case is impossible because the solution $3m_3 + 4m_4 + 5s + 6 \cdot 1 = 34$ has no solution for $m_3 \geq 2$. Then we may examine in turn:

C82332) $m_5 = 3$. Then we have a solution 8(0), 7(0), 6(1), 5(3), 4(1), 3(3). With regard to a number of leaves of graph of a line of disappearing ridges a hexagon must be adjacent with a triangle and with a quadrangle. Such conditions in a roof displayed in Fig. 8a hold.

C82333) $m_5 = 2$. Then we have a solution 8(0), 7(0), 6(1), 5(2), 4(3), 3(2). A hexagon and one pentagon must be adjacent or a hexagon must be adjacent with two quadrangles because a line of disappearing ridges must be unicoursal (there exist exactly two triangles). Then we have two different shapes of roofs. In the first case we have a roof in Fig. 7d*, in the second case we have a roof presented in Fig. 7d, d'. But a configuration in which a sequence (hexagon, quadrangle, quadrangle) exists is impossible. Indeed, if one of two adjacent quadrangles is adjacent to a hexagon we would have the case displayed in Fig. 6c. In such a case a fitting two adjacent pentagons and one quadrangle is impossible because from the points T_1, T_2 may not start any corner ridge.

C8234) $m_6 = 0$. Let us consider the following subcases:

C82341) $m_5 = s$ with $s \geq 6$. This case is obviously algebraically impossible.

C82342) $m_5 = 5$. Then we have a solution 8(0), 7(0), 6(0), 5(5), 4(0), 3(3). From five pentagons two must be adjacent and a configuration in which there exists a sequence of three adjacent pentagons is impossible. Let us denote by Γ_1, Γ_2 two such adjacent pentagons. Next, this configuration must be extended as follows: We place two triangles Δ_1, Δ_2, the first adjacently to Γ_1, the second to Γ_2. Next, we place the third pentagon Γ_3 (we have only one possibility) beside to one of two triangles, e.g, Δ_2 such that the pentagons Γ_2 and Γ_3 have any common disappearing ridge. Next, beside Γ_3 we place the third triangle Δ_3. In Fig. 8a the obtained configuration is displayed. Notice that two remaining pentagons may not be inserted. Then four ridges would meet in a point T_2, contrary to the assumption of the regularity of a roof. Such a configuration may be completed only by a hexagon and a quadrangle (see Fig. 8a). So the above solution is geometrically incorrect. We have the interesting nontrivial algebraically correct case of a "roof" which does not exist.

C82343) $m_5 = 4$. Then we have a solution 8(0), 7(0), 6(0), 5(4), 4(2), 3(2). In Fig. 7e such roof is displayed.

C82344) $m_5 = s$ with $s = 0, 1, 2, 3$. This case is algebraically impossible. □

In Table 1 we list the above results — including the cases for $v = 3, 4, 5, 6, 7$ which were not proved here.

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