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On Wallace Loci from the Projective Point of View

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Abstract. Let π_k be the projection of an n-dimensional projective space Σ $(2 \leq n < \infty)$ from the point B_k onto the hyperplane α_k, $k = 1, \ldots, n+1$, and assume that $\alpha_1, \ldots, \alpha_{n+1}$ are linearly independent. By the *Wallace locus of* $\pi_1, \ldots, \pi_{n+1}$ we mean the set of all points X of Σ whose images $\pi_1(X), \ldots, \pi_{n+1}(X)$ are linearly dependent. In a Pappian n-space each Wallace locus is either the entire space or an algebraic hypervariety whose degree is at most $n+1$. In a Pappian plane a triangle $\{B_1, B_2, B_3\}$ and a trilateral $\{\alpha_1, \alpha_2, \alpha_3\}$ determine the same Wallace locus as the triangle $\{\alpha_2 \cap \alpha_3, \alpha_3 \cap \alpha_1, \alpha_1 \cap \alpha_3\}$ and the trilateral $\{B_2 \vee B_3, B_3 \vee B_1, B_1 \vee B_2\}$. An analogous exchange rule for $3 \leq n < \infty$ is not valid. For Wallace loci of a Pappian plane with collinear centers B_1, B_2, B_3 we exhibit a theorem wherefrom we get the Wallace theorems for all degenerate Cayley-Klein planes by specialization. Thus we get the orthogonal and oblique Euclidean Wallace lines, the orthogonal and oblique pseudo-Euclidean Wallace lines, and the isotropic Wallace lines and, by duality, the Wallace points of the dual-Euclidean plane, of the dual-pseudo-Euclidean plane, and of the isotropic plane.

Key Words: triangle geometry, Wallace line, pedal line, Simson line, Wallace subspace

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1. Introduction

1.1. There are five degenerate Cayley-Klein planes: the Euclidean plane Σ_{2E}, the pseudo-Euclidean (=Minkowskian) plane Σ_{2P}, the isotropic (=Galilean) plane Σ_{2I}, the dual-Euclidean plane, and the dual-pseudo-Euclidean plane; cf. O. Giering [7, p. 137–138].

Assume $\Sigma_2 \in \{\Sigma_{2E}, \Sigma_{2P}, \Sigma_{2I}\}$. A triangle of Σ_2 is called *admissible*, if it is free of isotropic sides. If X is a point and λ a non-isotropic line of Σ_2, then the pencil of lines with vertex X contains exactly one line μ being orthogonal to λ (in symbols: $\lambda \perp \mu$). We call the unique common point of λ and μ the $\perp$-*pedal point of* X *on* λ.

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Wallace theorem. Let Δ be an admissible triangle and X be a point of the Euclidean plane. The three $\perp$-pedal points of X on the sides of Δ are collinear if, and only if, X is on the circumcircle of Δ. A line joining three collinear $\perp$-pedal points is called *orthogonal Euclidean Wallace line*; see Fig. 2 and cf. E.M. Schröder [16, p. 81].

This theorem is due to William Wallace (1768–1843). Also the notations Simson line and pedal line are in use (cf. [10], [12, p. 17–18]).

E.T. Steller [17] and also D. Wode [21] show that the Wallace theorem is valid in the pseudo-Euclidean plane, too. In this case we speak of *orthogonal pseudo-Euclidean Wallace lines.*

In the isotropic plane each point determines collinear $\perp$-pedal points on the sides of any admissible triangle; cf. [15, p. 20].

1.2. We endow the Euclidean plane Σ_{2E} with a sense of rotation. Then we can measure the angle from a line ξ to a line η in the sense of rotation using the interval $\{x \in \mathbb{R} \mid 0 < x < \pi\} =: \mathbb{A}_E$. Given $\varphi \in \mathbb{A}_E$, a point X, and a line λ, then the pencil of lines with vertex X contains exactly one line μ such that the angle from λ to μ equals φ. The unique common point of λ and μ is called the *φ-pedal point of X on λ*. By E.M. Schröder [16, p. 81], the Wallace theorem remains valid, if we replace the words "$\perp$-pedal points" with "φ-pedal points". Thus we get *Euclidean φ-Wallace lines.* For $\varphi \neq \frac{\pi}{2}$ we speak of *oblique Euclidean Wallace lines*; cf. R.H. Dye [6].

In the isotropic plane the measurement of the angle of two non-isotropic lines is done via their slopes; cf. H. Sachs [15, p. 16] or W. Benz [2, p. 35]. For $\varphi \in \mathbb{R} \setminus \{0\} =: \mathbb{A}_I$ the *isotropic φ-pedal point of a point X on a non-isotropic line* λ is defined analogously to the Euclidean case. By J. Lang [11], the Wallace theorem holds for the isotropic plane and φ-pedal points with $\varphi \in \mathbb{A}_I$. Thus we get *isotropic φ-Wallace lines.* See also J. Tölke [19].

In the pseudo-Euclidean plane the measure of an angle of two non-isotropic lines is an ordered pair taken from $\mathbb{R} \times \{+1, -1\}$; cf. W. Benz [2, p. 55–57]. For $\varphi \in (\mathbb{R} \times \{+1, -1\}) \setminus \{(0, +1)\} =: \mathbb{A}_P$ the *pseudo-Euclidean φ-pedal point of a point X on a non-isotropic line* λ is defined as in the Euclidean case. As a side-result of the present paper we get, that the Wallace theorem is valid for the pseudo-Euclidean plane and φ-pedal points with $\varphi \in \mathbb{A}_P$. Thus we have *pseudo-Euclidean φ-Wallace lines.* Pseudo-Euclidean orthogonality is characterized by the angle measure $(0, -1)$; for $\varphi \neq (0, -1)$ we speak of *oblique pseudo-Euclidean Wallace lines.*

In Section 3 we exhibit a theorem of plane projective geometry wherefrom we can deduce the Wallace theorems for all degenerate Cayley-Klein planes easily by specialization.

1.3. The Wallace theorem has been extended in numerous ways, for a survey see H. Martini [12, p. 17–18]; especially T. Takasu [18] gave some projective generalizations all starting from a given conic. M. de Guzman [5] and O. Giering [8] generalized via special triples of projections, we continue this way and admit arbitrary projections.

Definition 1. Let Σ_n be an n-dimensional projective space with point set $\mathcal{P}_n$ and let $A_1, \dots, A_{n+1} \in \mathcal{P}_n$ be linearly independent. Put $A_1 \vee \ldots \vee A_{k-1} \vee A_{k+1} \vee \ldots \vee A_{n+1} =: \alpha_k$, ($\vee$ denotes the join) $k = 1, \dots, n+1$, for the $n+1$ accompanying linearly independent hyperplanes; we call $\{A_1, \dots, A_{n+1}\} =: \mathcal{A}$ the *basis* $(n+1)$*-gon* and $\alpha_1, \dots, \alpha_{n+1}$ its *faces.*

By the *center* $(n+1)$*-gon* we mean a set $\mathcal{B} := \{B_1, \dots, B_{n+1}\} \subseteq \mathcal{P}_n$ of $n+1$ different points with $B_k \notin \alpha_k$ for $k = 1, \dots, n+1$. The projection of Σ_n from the center B_k onto the face α_k

is denoted by π_k, in symbols

$$\pi_k : \mathcal{P}_n \setminus \{B_k\} \to \alpha_k : X \mapsto (X \vee B_k) \cap \alpha_k \text{ for } k = 1, \ldots, n+1. \tag{1}$$

For $X \notin \{B_1, \ldots, B_{n+1}\}$ we put

$$J(X) := \bigvee_{k=1}^{n+1} \pi_k(X). \tag{2}$$

Additionally, for $X = B_1$ we define

$$J(B_1) := \bigvee_{k=2}^{n+1} \pi_k(X) \tag{3}$$

and analogously for $X = B_2, \ldots, X = B_{n+1}$. By the *Wallace locus of the basis* $(n+1)$*-gon* $\mathcal{A}$ *with respect to the center* $(n+1)$*-gon* $\mathcal{B}$ we mean the point set

$$\{X \in \mathcal{P}_n \mid \dim J(X) < n\} =: W(\mathcal{A}; \mathcal{B}). \tag{4}$$

If $\dim J(X) < n$, then we call $J(X)$ the *Wallace subspace of* X *with respect to the basis* $(n+1)$*-gon* $\mathcal{A}$ *and the center* $(n+1)$*-gon* $\mathcal{B}$.

Wallace loci of the real projective plane with a center triangle $\{B_1, B_2, B_3\}$ were investigated by P.D. Barry [1] and B. Orbán and M. Ţarină [13]. As an application of Wu's method of mechanical geometry theorem proving E. Roanes-Macías and E. Roanes-Lozano computed in [14] the Wallace locus of a tetrahedron of the Euclidean 3-space with respect to orthogonal (parallel) projections. In a Pappian n-space the Wallace locus is either an algebraic hypersurface whose degree is at most $n+1$ or the complete space (see Lemma 6). If the center $(n+1)$-gon spans a hyperplane, then the Wallace locus is either the entire space or it decomposes into this hyperplane and a remaining algebraic hypersurface whose degree is at most n (for details see Corollary 7). If $\mathcal{A}$ and $\mathcal{B}$ are triangles of a Pappian plane, then following exchange rule holds: $W(\mathcal{A}; \mathcal{B}) = W(\mathcal{B}; \mathcal{A})$ (see Theorem 9). An analogous exchange rule for dimensions greater than 2 is not valid (see Lemma 12).

The present article contains eight figures all created with Maple and LaTeX on a computer.

2. Elementary properties of Wallace loci

In the subsequent, more exactly from Lemma 2 up to Corollary 7, the assumptions and notations from Definition 1 are valid.

Lemma 2. *Then we have:*

(i) $\alpha_j \cap \alpha_k \subseteq W(\mathcal{A}; \mathcal{B})$ *for* $(j, k) \in \{1, 2, \ldots, n+1\}^2$ *and* $j \neq k$. *In particular,* $\mathcal{A} \subseteq W(\mathcal{A}; \mathcal{B})$.

(ii) $\mathcal{B} \subseteq W(\mathcal{A}; \mathcal{B})$.

(iii) Put $B_1 \vee \ldots \vee B_{k-1} \vee B_{k+1} \vee \ldots \vee B_{n+1} =: \beta_k$, *then* $\alpha_k \cap \beta_k \subseteq W(\mathcal{A}; \mathcal{B})$ *for* $k = 1, \ldots, n+1$.

Proof. From $X \in \alpha_j \cap \alpha_k \Rightarrow \pi_j(X) = \pi_k(X)$ follows (i) and (3) implies (ii).
For the proof of (iii) we may assume $k = 1$ without loss of generality. Clearly, $\dim \beta_1 = \dim(B_2 \vee \ldots \vee B_{n+1}) \leq n-1$. If $X \in \alpha_1 \cap \beta_1$, then $X = \pi_1(X) \in \beta_1$. Moreover, $\pi_2(X) \in \beta_1, \ldots, \pi_{n+1}(X) \in \beta_1$, hence $\pi_1(X) \vee \pi_2(X) \vee \ldots \vee \pi_{n+1}(X) \subseteq \beta_1$. Thus, $\dim(\pi_1(X) \vee \pi_2(X) \vee \ldots \vee \pi_{n+1}(X)) \leq n-1$. □

Lemma 3. *If the center $(n+1)$-gon $\mathcal{B}$ spans a subspace σ with $\dim\sigma < n$, then $\sigma \subseteq W(\mathcal{A};\mathcal{B})$.*

Proof. Let X be an arbitrary point of σ. If $X \notin \mathcal{B}$, then $\pi_k(X) \in B_k \vee X \subseteq \sigma$ for $k = 1,\dots,n+1$ and hence $J(X) \subseteq \sigma$, i.e., $\dim J(X) < n$, and consequently $X \in W(\mathcal{A};\mathcal{B})$. If $X = B_1$, then $\dim J(X) \leq n-1$ according to (3), i.e., $X \in W(\mathcal{A};\mathcal{B})$. □

The Wallace locus $W(\mathcal{A};\mathcal{B})$ can be the entire space even under the assumption that the center $(n+1)$-gon $\mathcal{B}$ consists of linearly independent points, as the following Lemma shows.

Lemma 4. *Assume that Σ_n is Pappian with (commutative) coordinatizing field $\mathbb{K}$ and that $A_k = B_k$ for $k = 1,2,\dots,n+1$. Put* $\operatorname{char}\mathbb{K} =: p$.
If $p \neq 0$ and p is a divisor of n, then $W(\{\mathcal{A}_1,\dots,A_{n+1}\};\{A_1,\dots,A_{n+1}\}) = \mathcal{P}_n$.
If $p = 0$ or if $p \neq 0$ and p is no divisor of n, then $W(\{A_1,\dots,A_{n+1}\};\{A_1,\dots,A_{n+1}\}) = \alpha_1 \cup \alpha_2 \ldots \cup \alpha_{n+1}$.

Proof. We may assume that Σ_n is the projective space on the right vector space $\mathbb{K}^n$ and

$$A_k = (\delta_{1k},\delta_{2k},\dots,\delta_{n+1,k})\mathbb{K}, \quad k = 1,2,\dots,n+1,$$

where δ_{jk} denotes the Kronecker symbol. For an arbitrary point

$$(\xi_1,\xi_2,\dots,\xi_{n+1})\mathbb{K} \in \mathcal{P}_n \setminus \{B_k\}$$

holds: $\pi_k(X) = (\xi_1,\dots,\xi_{k-1},0,\xi_{k+1},\dots,\xi_{n+1})\mathbb{K}$, $k = 1,2,\dots,n+1$. Thus $X \in W(A_1,\dots,A_{n+1};\ A_1,\dots,A_{n+1})$ is equivalent to the vanishing of the subsequent determinant:

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \dots & \xi_n & 0 \\ \xi_1 & 0 & \xi_3 & \xi_4 & \dots & \xi_n & \xi_{n+1} \\ \xi_1 & \xi_2 & 0 & \xi_4 & \dots & \xi_n & \xi_{n+1} \\ \dots & & & & & & \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \dots & 0 & \xi_{n+1} \\ 0 & \xi_2 & \xi_3 & \xi_4 & \dots & \xi_n & \xi_{n+1} \end{vmatrix} = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \dots & \xi_n & 0 \\ 0 & -\xi_2 & 0 & 0 & \dots & 0 & \xi_{n+1} \\ 0 & 0 & -\xi_3 & 0 & \dots & 0 & \xi_{n+1} \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -\xi_n & \xi_{n+1} \\ 0 & \xi_2 & \xi_3 & \xi_4 & \dots & \xi_n & \xi_{n+1} \end{vmatrix} =$$

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \dots & \xi_n & 0 \\ 0 & -\xi_2 & 0 & 0 & \dots & 0 & \xi_{n+1} \\ 0 & 0 & -\xi_3 & 0 & \dots & 0 & \xi_{n+1} \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -\xi_n & \xi_{n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & n\xi_{n+1} \end{vmatrix} = (-1)^{n-1} n\xi_1\xi_2\ldots\xi_{n+1}.$$ □

Remark 5. If Σ_n is an elliptic space with absolute polarity μ_n and if the basis $(n+1)$-gon is self polar, i.e., $\mu_n(A_k) = \alpha_k$ for $k = 1,\dots,n+1$, then the construction of orthogonal elliptic Wallace subspaces leads to the situation of Lemma 4.

Lemma 6. *Assume that Σ_n is Pappian. Then $W(\mathcal{A};\mathcal{B})$ is either the entire space or an algebraic hypervariety whose degree is at most $n+1$.*

Proof. We may assume that Σ_n is the projective space on the right vector space $\mathbb{K}^n$, $B_k = (b_{k1},b_{k2},\dots,b_{k(n+1)})\mathbb{K}$, and that the hyperplane α_k is described by $\Sigma_{j=1}^{n+1}\alpha_{kj}x_j = 0$, $k =$

$1, \dots, n+1$. Because of $B_k \notin \alpha_k$ we have $\Sigma_{j=1}^{n+1} \alpha_{kj} b_{kj} \neq 0$. For $X = (\xi_1, \xi_2, \dots, \xi_{n+1})\mathbb{K}$ we compute $\pi_k(X) = (\ell_{k1}, \ell_{k2}, \dots, \ell_{k(n+1)})\mathbb{K}$ with

$$\ell_{km} := -b_{km} \Sigma_{j=1}^{n+1} \alpha_{kj} \xi_j + \xi_m \Sigma_{j=1}^{n+1} \alpha_{kj} b_{kj} \quad \text{and} \quad (k, m) \in \{1, \dots, n+1\}^2.$$

The coordinates of $\pi_k(X)$ are linear homogeneous forms in $\xi_1, \xi_2, \dots, \xi_{n+1}$. From the theorem of Laplace follows that the determinant of the matrix $(\ell_{km})_{k=1,\dots,n+1}^{m=1,\dots,n+1}$ is a homogeneous form of degree $n+1$. □

From Lemma 3 and 6 follows

Corollary 7. *Assume that Σ_n is Pappian. If the center $(n+1)$-gon $\mathcal{B}$ spans a hyperplane σ, then the Wallace locus $W(\mathcal{A}; \mathcal{B})$ is either the entire space or it decomposes into σ and an algebraic hypervariety whose degree is at most n.*

Remark 8. If Σ_n is a Euclidean or pseudo-Euclidean space, then the construction of orthogonal Euclidean or pseudo-Euclidean Wallace subspaces leads to the situation of Corollary 7 with the hyperplane at infinity being σ. If $3 \leq n < \infty$, then the Wallace locus can not decompose into an irreducible quadric and the repeatedly counted hyperplane at infinity, since an irreducible quadric never contains all $(n-2)$-dimensional, i.e., at least 1-dimensional, "edges" $\alpha_j \cap \alpha_k$ of a basis $(n+1)$-gon $\mathcal{A}$; cf. Lemma 2.

Theorem 9 ("Exchange rule"). *Let $\mathcal{A} = \{A_1, A_2, A_3\}$ and $\mathcal{B} = \{B_1, B_2, B_3\}$ be triangles of a Pappian plane Σ_2 with $B_k \notin A_m \vee A_n$ and $A_k \notin B_m \vee B_n$ for all $(k, m, n) \in \{1, 2, 3\}^3$ with $\{k, m, n\} = \{1, 2, 3\}$. The Wallace locus of the basis triangle $\mathcal{A}$ with respect to the center triangle $\mathcal{B}$ coincides with the Wallace locus of the basis triangle $\mathcal{B}$ with respect to the center triangle $\mathcal{A}$, in symbols: $W(\mathcal{A}; \mathcal{B}) = W(\mathcal{B}; \mathcal{A})$.*

Proof. We may assume that Σ_2 is the projective space on the right vector space $\mathbb{K}^3$, $A_k = \mathbf{a}_k\mathbb{K}$ with $\mathbf{a}_k = (a_{k1}, a_{k2}, a_{k3})$, and $B_k = \mathbf{b}_k\mathbb{K}$ with $\mathbf{b}_k = (b_{k1}, b_{k2}, b_{k3})$, $k = 1, 2, 3$, and $X = \mathbf{x}\mathbb{K}$ with $\mathbf{x} = (\xi_1, \xi_2, \xi_3)$. Clearly, $A := \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \neq 0$ and $B := \det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \neq 0$. Using a computer we get: /def/breite1.5mm

$$\begin{aligned}
\pi_1(X) &= (-b_{11}E_1 + \xi_1 F_1,\ -b_{12}E_1 + \xi_2 F_1,\ -b_{13}E_1 + \xi_3 F_1)\mathbb{K} \quad \text{with} \\
&\qquad E_1 := \det(\mathbf{a}_2, \mathbf{a}_3, \mathbf{x}) \quad \text{and} \quad F_1 := \det(\mathbf{a}_2, \mathbf{a}_3, \mathbf{b}) \\
\pi_2(X) &= (-b_{21}E_2 + \xi_1 F_2,\ -b_{22}E_2 + \xi_2 F_2,\ -b_{23}E_2 + \xi_3 F_2)\mathbb{K} \quad \text{with} \\
&\qquad E_2 := \det(\mathbf{a}_3, \mathbf{a}_1, \mathbf{x}) \quad \text{and} \quad F_2 := \det(\mathbf{a}_3, \mathbf{a}_1, \mathbf{b}) \\
\pi_3(X) &= (-b_{31}E_3 + \xi_1 F_3,\ -b_{32}E_3 + \xi_2 F_3,\ -b_{33}E_3 + \xi_3 F_3)\mathbb{K} \quad \text{with} \\
&\qquad E_3 := \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{x}) \quad \text{and} \quad F_3 := \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}).
\end{aligned}$$

The Wallace locus $W(\mathcal{A}; \mathcal{B})$ is described by the vanishing of the determinant

$$W_{ab} := \begin{vmatrix} -b_{11}E_1 + \xi_1 F_1 & -b_{12}E_1 + \xi_2 F_1 & -b_{13}E_1 + \xi_3 F_1 \\ -b_{21}E_2 + \xi_1 F_2 & -b_{22}E_2 + \xi_2 F_2 & -b_{23}E_2 + \xi_3 F_2 \\ -b_{31}E_3 + \xi_1 F_3 & -b_{32}E_3 + \xi_2 F_3 & -b_{33}E_3 + \xi_3 F_3 \end{vmatrix}.$$

In W_{ab} we interchange a_{km} and b_{km} and get the determinant W_{ba} whose vanishing describes the other Wallace locus $W(\mathcal{B}; \mathcal{A})$. Finally, we check with the help of a computer: $BW_{ab} = AW_{ba}$. Because of $A \neq 0$ and $B \neq 0$ follows: $W_{ab} = 0 \iff W_{ba} = 0$. □

Equivalent to Theorem 9 is

Corollary 10. *Let $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ be triangles of a Pappian plane Σ_2, assume $B_k \notin A_m \vee A_n$ and $A_k \notin B_m \vee B_n$, and put π_k for the projection of Σ_2 from B_k onto $A_m \vee A_n$, and ρ_k for the projection of Σ_2 from A_k onto $B_m \vee B_n$ for all $(k, m, n) \in \{1, 2, 3\}^3$ with $\{k, m, n\} = \{1, 2, 3\}$.*
If the images $\pi_1(X)$, $\pi_2(X)$, $\pi_3(X)$ of a point X are collinear (look at the line $J_{AB}(X)$ of Fig. 1), *then the images $\rho_1(X)$, $\rho_2(X)$, $\rho_3(X)$ are also collinear* (look at the line $J_{BA}(X)$).

Remark 11. The irreducible cubic in Fig. 1 represents the Wallace locus $W(\mathcal{A};\mathcal{B}) = W(\mathcal{B};\mathcal{A})$.

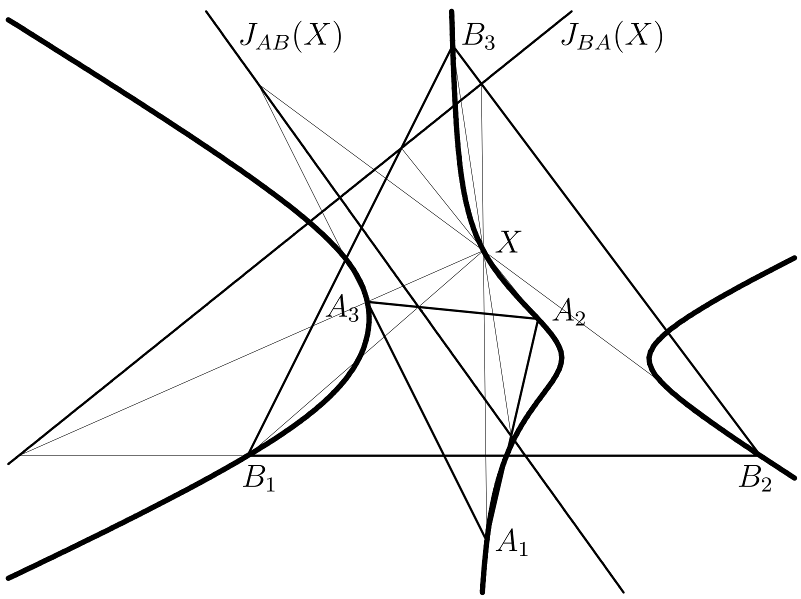

Figure 1: Wallace locus ($n = 2$, non-collinear centers) and exchange rule

Lemma 12. *Let $\mathcal{A}$ be a basis $(n+1)$-gon of a projective Pappian n-space Σ_n with $3 \leq n < \infty$. Then there exists a basis $(n+1)$-gon $\mathcal{B}$ of Σ_n such that $W(\mathcal{A};\mathcal{B}) \neq W(\mathcal{B};\mathcal{A})$.*

Proof. We may assume $A_k = (\delta_{1k}, \delta_{2k}, \ldots, \delta_{n+1,k})\mathbb{K}$, $k = 1, 2, \ldots, n+1$. We choose $B_1 = (1, 1, 1, \ldots, 1, 0, 0)\mathbb{K}$, $B_j = A_j$ for $j = 2, \ldots, n$, and $B_{n+1} = (0, 0, 0, 0, \ldots, 0, 1, 1)\mathbb{K}$. The reader checks easily that $\{B_1, \ldots, B_{n+1}\}$ is an $(n+1)$-gon. Put $\beta_2 := B_1 \vee B_3 \vee \ldots \vee B_n \vee B_{n+1}$ and $\beta_n := B_1 \vee B_2 \vee \ldots \vee B_{n-1} \vee B_{n+1}$. We have:

$$E := (1, 1, 1, \ldots, 1, 1, 1)\mathbb{K} \in B_1 \vee B_{n+1} \subseteq \beta_2 \cap \beta_n \overset{\text{Lemma 2(i)}}{\subseteq} W(\mathcal{B};\mathcal{A}).$$

It suffices to show

$$E \notin W(\mathcal{A};\mathcal{B}). \tag{5}$$

Obviously, $\pi_{n+1}(E) = B_1$, $\pi_2(E) = (1, 0, 1, \ldots, 1, 1, 1)\mathbb{K}$, ..., $\pi_n(E) = (1, 1, 1, \ldots, 1, 0, 1)\mathbb{K}$, and $\pi_1(E) = B_2$. From

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 1 & 1 & 1 \\ 1 & 1 & 0 & \ldots & 1 & 1 & 1 \\ \multicolumn{7}{c}{\dotfill} \\ 1 & 1 & 1 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 & 0 & 0 \\ 0 & -1 & 0 & \ldots & 0 & 1 & 1 \\ 0 & 0 & -1 & \ldots & 0 & 1 & 1 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{vmatrix} = (-1)^{n-3} \neq 0$$

follows (5). □

3. $n = 2$ and collinear centers B_1, B_2, B_3

3.1. Commuting projectivities. From [20, Ch. VIII, p. 226–227] we recall three items:

1) If p is a non-parabolic non-involutoric projectivity of a Pappian plane, then there exists exactly one projective involution commutative with p.

2) The projective involution commutative with a given non-parabolic non-involutoric projectivity p is called the *involution belonging to* p. A projective involution *belongs to itself.*

3) If a non-parabolic projectivity p of a Pappian plane has fixed points, then the involution belonging to p has the same fixed points.

Theorem 13. *Let* $\mathcal{A} = \{A_1, A_2, A_3\}$ *be a triangle of* $\mathrm{PG}(2,\mathbb{K})$, $\mathbb{K}$ *commutative and* $\operatorname{char}\mathbb{K} \neq 2$. *Put* $\alpha_1 := A_2 \vee A_3$, $\alpha_2 := A_3 \vee A_1$, $\alpha_3 := A_1 \vee A_2$. *Let* $\mathcal{B} = \{B_1, B_2, B_3\}$ *be a set of mutually different collinear points whose join* ω *does not contain* A_k *for* $k = 1, 2, 3$ *and assume* $B_j \notin \alpha_j$, $j = 1, 2, 3$. *By* p *we denote the unique autoprojectivity of* ω *with* $\alpha_j \cap \omega \mapsto B_j$ *for* $j = 1, 2, 3$.

(i) If p *is non-parabolic, then the Wallace locus* $W(\mathcal{A};\mathcal{B})$ *decomposes into* ω *and the circumconic* k *of* $\mathcal{A}$ *whose involution of conjugate points on the non-tangent* ω *belongs to* p.

(ii) If p *is parabolic, then the Wallace locus* $W(\mathcal{A};\mathcal{B})$ *decomposes into* ω *and the circumconic* k *of* $\mathcal{A}$ *being tangent to* ω *at the fixed point of* p.

Proof. Without loss of generality we may assume $A_1 = (0,0,1)\mathbb{K}$, $A_2 = (1,0,1)\mathbb{K}$, $A_3 = (0,1,1)\mathbb{K}$, and that $\omega = \{\mathbf{x}\mathbb{K} \in \mathcal{P}_2 |\, x_3 = 0\}$, $\mathbf{x} := (x_1, x_2, x_3)$. Thus $\alpha_1 = \{\mathbf{x}\mathbb{K} \in \mathcal{P}_2 |\, x_1 + x_2 - x_3 = 0\}$, $\alpha_2 = \{\mathbf{x}\mathbb{K} \in \mathcal{P}_2 |\, x_1 = 0\}$, $\alpha_3 = \{\mathbf{x}\mathbb{K} \in \mathcal{P}_2 |\, x_2 = 0\}$, and $\alpha_1 \cap \omega = (1,-1,0)\mathbb{K}$, $\alpha_2 \cap \omega = (0,1,0)\mathbb{K}$, $\alpha_3 \cap \omega = (1,0,0)\mathbb{K}$. For the projectivity p we have:

$$p:\ \omega \to \omega;\quad (x_1,x_2,0)\mathbb{K} \ \mapsto\ (p_{11}x_1 + p_{12}x_2, p_{21}x_1 + p_{22}x_2, 0)\mathbb{K},\quad p_{jk} \in \mathbb{K},\ \text{and}$$

$$\det(P) \neq 0,\quad P := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \tag{6}$$

i.e., $B_1 = (p_{11} - p_{12}, p_{21} - p_{22}, 0)\mathbb{K}$, $B_2 = (p_{12}, p_{22}, 0)\mathbb{K}$, $B_3 = (p_{11}, p_{21}, 0)\mathbb{K}$ and because of $B_j \notin \alpha_j$ holds

$$p_{11} - p_{12} + p_{21} - p_{22} \neq 0,\quad p_{12} \neq 0,\quad p_{21} \neq 0. \tag{7}$$

For an arbitrary point $X = (\xi_1, \xi_2, \xi_3)\mathbb{K}$ we compute:

$$\begin{aligned}
\pi_1(X) &= \big(\xi_1(p_{22} - p_{21}) + \xi_2(p_{11} - p_{12}) + \xi_3(p_{12} - p_{11}),\\
&\qquad \xi_1(p_{21} - p_{22}) + \xi_2(p_{12} - p_{11}) + \xi_3(p_{22} - p_{21}),\ \xi_3\,(p_{12} + p_{22} - p_{11} - p_{21})\big)\mathbb{K},\\
\pi_2(X) &= (0,\ \xi_1 p_{22} - \xi_2 p_{12},\ -\xi_3 p_{12})\mathbb{K},\\
\pi_3(X) &= (\xi_1 p_{21} - \xi_2 p_{11},\ 0,\ \xi_3 p_{21})\mathbb{K}.
\end{aligned}$$

Equivalent to $\dim\big(\pi_1(X) \vee \pi_2(X) \vee \pi_3(X)\big) < 2$ is the vanishing of the determinant of the matrix formed by the coordinates of the three points, i.e., $W(A_1, A_2, A_3; B_1, B_2, B_3)$ is described by

$$\xi_3\,\underbrace{(p_{11}p_{22} - p_{12}p_{21})}_{\neq 0 \text{ by } (6)}\,\underbrace{\big(p_{21}\xi_1^2 + (p_{22} - p_{11})\xi_1\xi_2 - p_{21}\xi_1\xi_3 - p_{12}\xi_2^2 + p_{12}\xi_2\xi_3\big)}_{=:\,K(X)} = 0.$$

Now $k := \{\mathbf{x}\mathbb{R} \in \mathcal{P}_2 | \, K(X) = 0\}$ is a conic since for the determinant d_k of the matrix of the bilinear form corresponding to $K(X)$ holds $d_k = 2^{-2} p_{12} p_{21} (p_{11} - p_{12} + p_{21} - p_{22}) \neq 0$ because of (7). The assertion $\{A_1, A_2, A_3\} \subseteq k$ follows either from Lemma 2 or can be checked directly by substitution.

By (7), $(1,0,0,)\mathbb{K} \notin k$ and $(0,1,0)\mathbb{K} \notin k$, hence the determination of $k \cap \omega$ is equivalent to the solution of the quadratic equation $p_{21}\left(\frac{\xi_1}{\xi_2}\right)^2 + (p_{22} - p_{11})\frac{\xi_1}{\xi_2} - p_{12} = 0$ in the unknown $\frac{\xi_1}{\xi_2}$; for its discriminant we have $D_k := (p_{22} - p_{11})^2 + 4p_{12}p_{21}$. We compute the characteristic polynomial of the matrix (p_{jk}):

$$\begin{vmatrix} p_{11} - \rho & p_{12} \\ p_{21} & p_{22} - \rho \end{vmatrix} = \rho^2 - (p_{11} + p_{22})\,\rho + p_{11}p_{22} - p_{12}p_{21}$$

and its discriminant $D_e := (p_{11} + p_{22})^2 - 4(p_{11}p_{22} - p_{12}p_{21})$. Immediately we check $D_k = D_e$. Consequently, p is parabolic if, and only if, k is tangent to ω.

Case i: $D_k \neq 0$. The points

$$(y_1, y_2, 0)\mathbb{K} \quad \text{and} \quad \left(\tfrac{1}{2}(p_{22} - p_{11})y_1 - p_{12}y_2,\ -p_{21}y_1 + \tfrac{1}{2}(p_{11} - p_{22})y_2,\ 0\right)\mathbb{K}$$

are conjugate with regard to k, hence

$$b:\ \omega \to \omega\,; \quad (y_1, y_2, 0)\mathbb{K} \mapsto (b_{11}y_1 + b_{12}y_2,\ b_{21}y_1 + b_{22}y_2,\ 0)\mathbb{K} \ \text{ with}$$

$$B := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(p_{22} - p_{11}) & -p_{12} \\ -p_{21} & \frac{1}{2}(p_{11} - p_{22}) \end{pmatrix}$$

is the involution of conjugate points on ω with regard to k. Finally, $PB = BP$ shows the assertion.

Case ii: $D_k = 0$. Now k is tangent to ω at $T := (p_{11} - p_{22}, 2p_{21}, 0)\mathbb{K}$. The point $p(T)$ coincides with T because of

$$\begin{vmatrix} p_{11} - p_{22} & 2p_{21} \\ p_{11}(p_{11} - p_{22}) + 2p_{12}p_{21} & p_{21}(p_{11} - p_{22}) + 2p_{22}p_{21} \end{vmatrix} = -p_{21}D_k = 0,$$

i.e., T is fixed by p. □

3.2. Giering's generalization. We compare Fig. 6 from O. Giering [8] and Fig. 2 which shows the ordinary Euclidean circumcircle of a triangle and the envelope of its Wallace lines. We assert that these two pictures correspond in a collineation. In order to prove the validity of this assertion, we use in PG$(2,\mathbb{R})$ the same coordinates for ω (corresponds to f in [8]) and $\{A_1, A_2, A_3\}$ as in 3.1 and choose the point $Z = (z_1, z_2, z_3)\mathbb{R}$ with $Z \notin \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \omega$, i.e.,

$$z_1 z_2 z_3 (z_1 + z_2 - z_3) \neq 0.$$

According to [8], the three centers of the projections are: $B_k = (Z \vee A_k) \cap \omega$ for $k = 1, 2, 3$, hence $B_1 = (z_1, z_2, 0)\mathbb{R}$, $B_2 = (z_1 - z_3, z_2, 0)\mathbb{R}$, $B_3 = (z_1, z_2 - z_3, 0)\mathbb{R}$. The projectivity p from 3.1 maps $\alpha_k \cap \omega$ to B_k, $k = 1, 2, 3$. Consequently, we get for the matrix describing p (compare (6)):

$$P_{\text{Gie}} := \begin{pmatrix} z_1 z_2 & -z_1(z_1 - z_3) \\ z_2(z_2 - z_3) & -z_1 z_2 \end{pmatrix}.$$

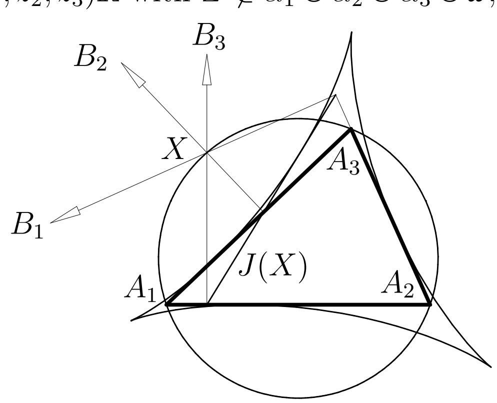

Figure 2: Orthogonal Euclidean Wallace line $J(X)$

We compute $P_{\text{Gie}}\, P_{\text{Gie}} = -z_1 z_2 z_3(z_1 + z_2 - z_3)\text{diag}(1,1)$, hence Giering's construction yields always involutoric projectivities p_{Gie}. For $z_1 z_2 z_3(z_1 + z_2 - z_3) < 0$ the involution p_{Gie} is elliptic and for $z_1 z_2 z_3(z_1 + z_2 - z_3) > 0$ hyperbolic. We endow $\text{PG}(2,\mathbb{R})$ with p_{Gie} as absolute involution; this yields a Euclidean or pseudo-Euclidean plane, respectively. From the structural point of view, Giering's article covers the orthogonal Euclidean and orthogonal pseudo-Euclidean Wallace lines, but not oblique and isotropic Wallace lines. See also O. Giering [9].

In the following three subsections we deal with the Euclidean, pseudo-Euclidean and isotropic plane. An introductory, simultaneous exposition of these three planes can be found in [15, p. 124–150].

3.3. Euclidean and dual-Euclidean case. Put

$$A_E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P_E := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad \varphi \in \mathbb{R}, \quad 0 < \varphi < \pi.$$

We assume that $\omega = \{\mathbf{x}\mathbb{R} \in \mathcal{P}_2 | \, x_3 = 0\}$ is the line at infinity. The matrix A_E describes an elliptic involution a_E of ω which we use as absolute involution. Because of $A_E P_E = P_E A_E$, the involution a_E belongs to the projectivity p_E corresponding to P_E. In Theorem 13 we use $B_k = p_E(\alpha_k \cap \omega)$ as centers, $k = 1,2,3$. The projection π_k is a parallel projection of angle φ onto α_k, $k = 1,2,3$, since p_E is induced by a rotation through an angle of measure φ. This shows that the Euclidean Wallace theorem can be derived from Theorem 13 by specialization.

Fig. 3 shows the triangle $\{A_1, A_2, A_3\}$ and its circumcircle in the ordinary Euclidean plane. For the point X of the circumcircle the φ-Wallace line $J(X)$ for $\varphi = \frac{\pi}{3}$ is drawn. The centers of projection are the points B_1, B_2, B_3 at infinity. As X varies in the circumcircle, the associated φ-Wallace lines $J(X)$ envelope a curve with three cusps which is also given in Fig. 3.

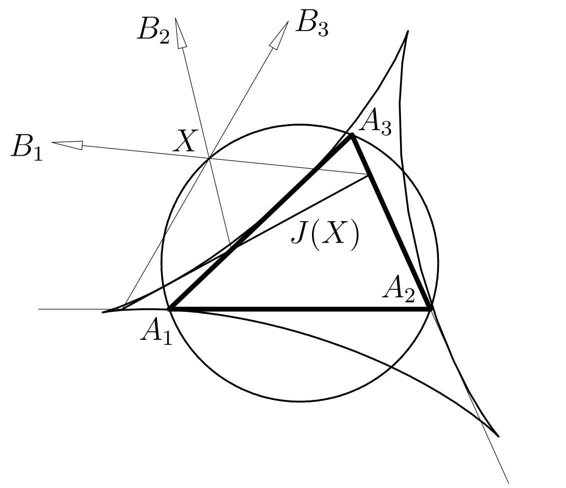

Figure 3: Euclidean φ-Wallace line $J(X)$

Fig. 4 shows the dual-Euclidean case, illustrated in the ordinary Euclidean plane. Consider the Euclidean rotation ρ about the point ω^* through the Euclidean angle $\frac{\pi}{2}$. As absolute involution we choose the restriction of ρ to the pencil of lines with vertex ω^*.

The conic in Fig. 4 is the dual-Euclidean circumcircle of the sides A_1^*, A_2^*, A_3^* of the triangle $\{\alpha_1^*, \alpha_2^*, \alpha_3^*\}$. (From the Euclidean point of view ω^* is the Euclidean focus of the conic). For the tangent X^* of the conic the φ-Wallace point $J(X^*)$ for $\varphi = \frac{\pi}{3}$ is given. If X^* varies in the tangent set of the conic, then the associated φ-Wallace points $J(X^*)$ describe a curve e^* which is also drawn in Fig. 4.

Remark 14. The curve e^* contains the points α_j^* and $A_j^* \cap B_j^*$ for $j = 1,2,3$. The reader easily checks that in Fig. 2 and 3 the lines $A_j \vee B_j$ are tangent to the envelope, $j = 1,2,3$.

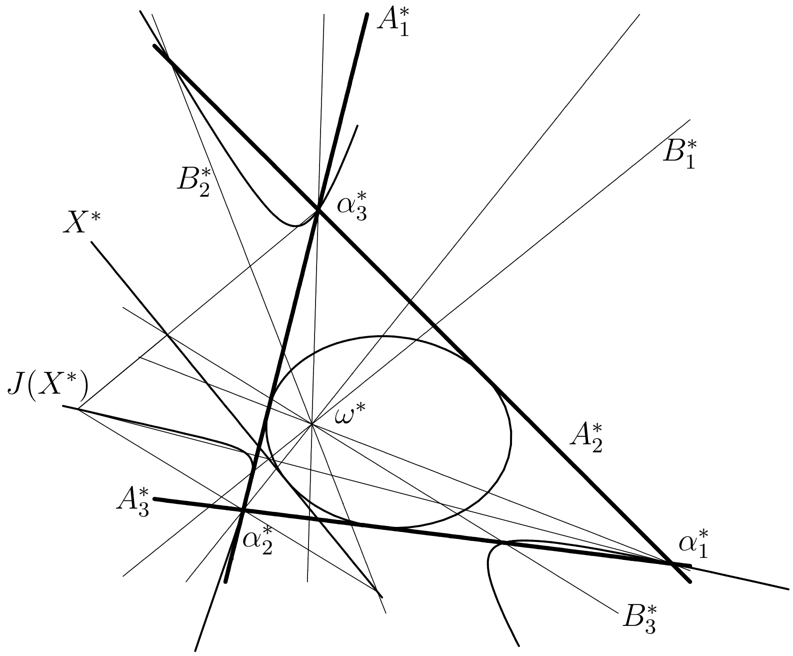

Figure 4: Dual-Euclidean φ-Wallace point $J(X^*)$

3.4. Pseudo-Euclidean and dual-pseudo-Euclidean case. We choose $I_1 := (1,1,0)\mathbb{R}$ and $I_2 := (1,-1,0)\mathbb{R}$ as absolute points such that the absolute involution is described by

$$A_P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denote by $\mathcal{L}_O$ the pencil of lines with vertex $(0,0,1)\mathbb{R}$. To a non-isotropic line $\ell \in \mathcal{L}_O$ there exists exactly one line $d_\varphi(\ell) \in \mathcal{L}_O$ such that the angle measure from ℓ to $d_\varphi(\ell)$ equals a given $\varphi \in \mathbb{R} \times \{+1, -1\}$.

Firstly, we discuss the case $\varphi = (\Phi, +1)$ with $\Phi \in \mathbb{R} \setminus \{0\}$. If $x_2 = \tanh\alpha \cdot x_1$ is the

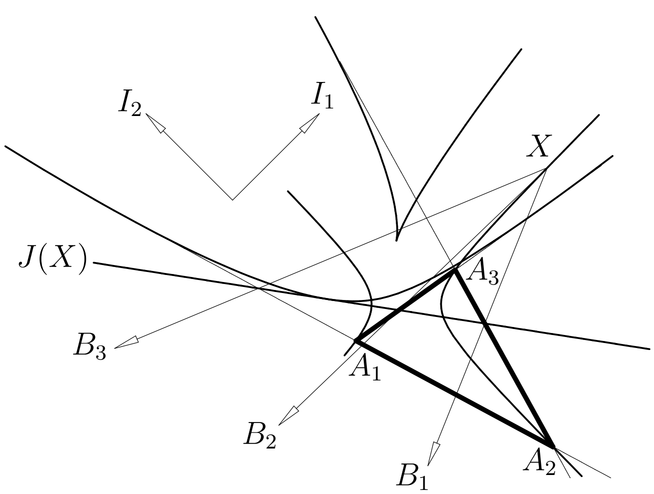

Figure 5: Pseudo-Euclidean φ-Wallace line $J(X)$

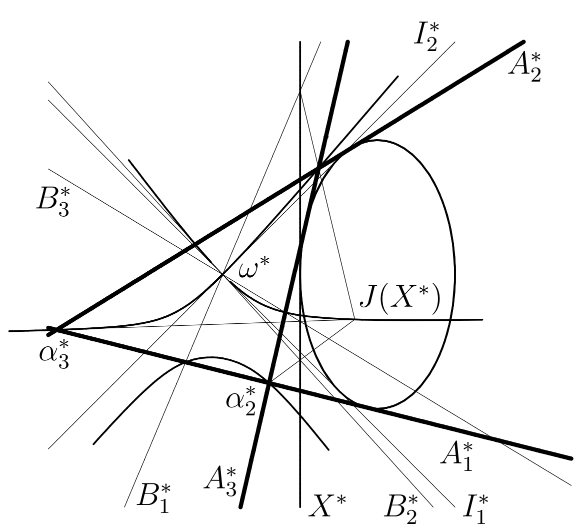

Figure 6: Dual-pseudo-Euclidean φ-Wallace point $J(X^*)$

equation of ℓ, then $d_\varphi(\ell)$ is described by $x_2 = \tanh(\alpha+\Phi)\cdot x_1$; cf. [2, p. 55–57]. Hence

$$\omega \cap \ell = (\cosh\alpha, \sinh\alpha, 0)\mathbb{R} \quad \text{and} \quad \omega \cap d_\varphi(\ell) = \big(\cosh(\alpha+\Phi),\ \sinh(\alpha+\Phi),\ 0\big)\mathbb{R} =$$
$$(\cosh\alpha\cosh\Phi + \sinh\alpha\sinh\Phi,\ \sinh\alpha\cosh\Phi + \cosh\alpha\sinh\Phi,\ 0)\,\mathbb{R},$$

i.e., $\omega\cap\ell$ is mapped to $\omega\cap d_\varphi(\ell)$ by the autoprojectivity p_P of ω corresponding to

$$P_P := \begin{pmatrix} \cosh\Phi & \sinh\Phi \\ \sinh\Phi & \cosh\Phi \end{pmatrix}, \qquad \Phi \in \mathbb{R}\setminus\{0\};$$

cf. [15, p. 138,(8.32a)]. Now $A_P P_P = P_P A_P$ shows the validity of the Wallace theorem in the pseudo-Euclidean plane and for φ-pedal points with $\varphi \in (\mathbb{R}\setminus\{0\})\times\{+1\}$.

Secondly, $\varphi = (\Phi, -1)$ with $\Phi\in\mathbb{R}$. Now d_φ maps $x_1 = \tanh\alpha\cdot x_2$ onto $x_1 = \tanh(\alpha+\Phi)\cdot x_2$ and we have

$$\omega \cap d_\varphi(\ell) = \big(\sinh(\alpha+\Phi),\ \cosh(\alpha+\Phi),\ 0\big)\mathbb{R}.$$

In the second case $\omega\cap\ell$ is mapped to $\omega\cap d_\varphi(\ell)$ by the autoprojectivity q_P of ω corresponding to

$$Q_P := \begin{pmatrix} \sinh\Phi & \cosh\Phi \\ \cosh\Phi & \sinh\Phi \end{pmatrix}, \qquad \Phi \in \mathbb{R};$$

cf. [15, p. 139,(8.32b)]. Again we check $A_P Q_P = Q_P A_P$.

The pseudo-Euclidean case is demonstrated in Fig. 5, the dual pseudo-Euclidean in Fig. 6. The reader easily interprets the figures by virtue of the labels.

Remark 15. Also in these two cases Remark 14 is valid. Furthermore, we see in Fig. 6 that the curve e^* is tangent to the absolute lines I_1^* and I_2^* in the double point ω^*. Consequently, the envelope e in Fig. 5 has the line ω at infinity as double tangent with the absolute points I_1 and I_2 as points of contact.

3.5. Isotropic case. We assume that $\omega = \{\mathbf{x}\mathbb{R}\in\mathcal{P}_2 \,|\, x_3 = 0\}$ and $O := (0,1,0)\mathbb{R}$ form the absolute flag. Consider the motion (cf. [15, p. 17]):

$$d_\varphi:\ \mathcal{P}_2 \to \mathcal{P}_2;\quad (x_1,x_2,x_3)\mathbb{R} \mapsto \big(x_1 + a x_3,\ \varphi x_1 + x_2 + b x_3,\ x_3\big)\mathbb{R},\quad a,b,\varphi\in\mathbb{R},\ \varphi\neq 0.$$

Using [15, p. 15–16,(2.13)–(2.15)] the reader proves easily: If ℓ is a non-isotropic line, then the angle measure from ℓ to $d_\varphi(\ell)$ equals φ. Put

$$P_I := \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}.$$

The projectivity p_I corresponding to P_I is parabolic and fixes O. As centers in Theorem 13 we use $B_k = p_I(\alpha_k\cap\omega)$ for $k = 1,2,3$. The projection π_k is a parallel projection of angle φ onto α_k, $k = 1,2,3$, since p_I is the restriction of the motion d_φ. This shows that also the isotropic Wallace theorem can be derived from Theorem 13 by specialization.

Due to the metric duality of the isotropic plane (cf. [15, Chpt. 6]) there exist isotropic Wallace lines and isotropic Wallace points. Fig. 7 shows the φ-Wallace line of the point X of the circumcircle of the triangle $\{A_1, A_2, A_3\}$ and the envelope of all φ-Wallace lines; for details on the envelopes of isotropic Wallace lines see [19, Section 3 and Fig. 2]. Fig. 8 demonstrates the construction of a φ-Wallace point and the curve described by all φ-Wallace points. Again the reader checks the validity of Remark 14.

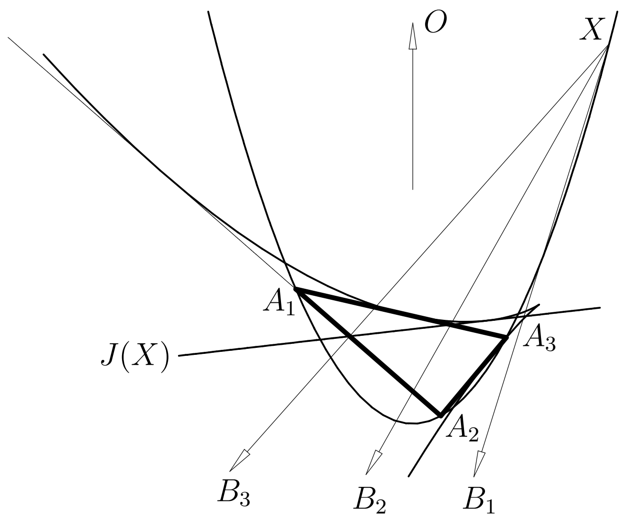

Figure 7: Isotropic φ-Wallace line $J(X)$

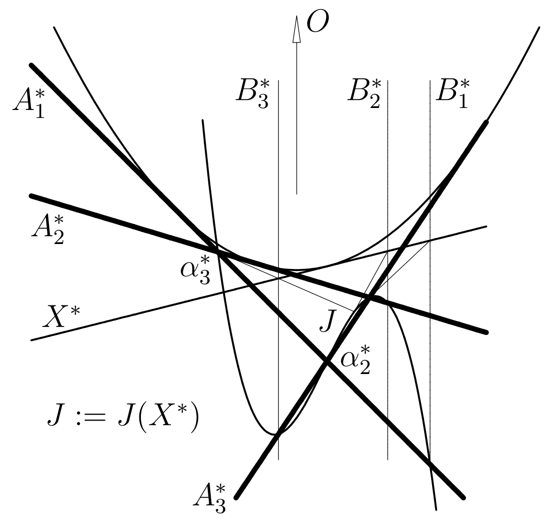

Figure 8: Isotropic φ-Wallace point J

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