Journal for Geometry and Graphics
Volume 8 (2004), No. 2, 143–150.

Reconstructing an Ellipsoid from its Limbs' Projections

Mohamed Ali Said

Engineering Mathematics & Physics Dept.,
Faculty of Engineering, Zagazig University, Zagazig, Egypt
email: mmamsaid@hotmail.com

Abstract. In this paper it is shown that the projection of an ellipsoid's limb in a direction perpendicular to a principal plane is an ellipse in that plane. Then a method for reconstructing an ellipsoid from three projections of its limbs on the three principal planes is introduced. This supports the work of reconstructing a shape from its projections, views, or images which is of special interest in application areas like pattern recognition, computer vision, and medical imaging.

Key Words: Ellipsoid, limb, projection, reconstruction

MSC 2000: 51N05

1. Introduction

Reconstructing a 3D object from its orthographic projections, perspective views, an image sequence, or a collection of planar contours representing cross-sections through the object is a task that emerges in various applications. Biologists try to understand the shape of microscopic objects from serial sections through the object. In clinical medicine, the data generated by various imaging techniques such as computed axial tomography (CAT), ultrasound, and nuclear magnetic resonance (NMR) provide a series of slices through the object of study. In computer aided design (CAD), lofting techniques specify the geometry of an object by means of a series of contours. Also, this task has important applications in solid modelling, industrial inspection, computer vision, video coding, and pattern recognition.

In this paragraph a sample of the work in this area is given. In [1] it is shown that the curved surfaces can be reconstructed from their apparent contours viewed from an uncalibrated camera with linear translation, up to a 3D affine ambiguity. In [2] a least squares estimation of 3D shapes from their orthographic projections is considered. In [3] the shape reconstruction problem is considered on two bases: First is to use a minimum number of images, i.e., three images. Second is to use a weak perspective projection which is a scaled orthographic projection. It is different from orthographic projection by a scaling of the image

ISSN 1433-8157/\$ 2.50

according to the distance of the object. In [4] the concern is with the problem of reconstructing the surface of 3D objects, given a collection of planar contours representing cross-sections through the objects. In [5] a method of extracting, classifying and modelling non-rigid shapes from an image sequence is presented. A non-rigid shape is described by a fixed number of points.

While the work for the general problem is in progress, important special cases should be treated separately. This is the viewpoint reflected in this paper where the reconstruction of an ellipsoid from its limbs' projections is considered. Ellipsoids are of the most commonly used primitive models in various applications, see for example [6], [7], [8], and [9].

2. Preliminaries

2.1. The limb

Following [10], the limb of a surface is outlined. From a particular viewpoint, a surface "fold in back of itself". The locus of points where this happens is the limb. That is the limb is the virtual edge of a surface seen from a particular viewpoint. More precisely, the limb is the locus of points on a surface where the normal to the surface is perpendicular to the line of sight. These points must satisfy the surface equation $f(x, y, z) = 0$ and the perpendicularity condition $\vec{s} \cdot \nabla f = 0$, where $\vec{s}$ is the line of sight vector and ∇f is the surface normal.

For an ellipsoid we have

$$f(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J \tag{1}$$

where A, B, C, $T_1 = (AB + BC + AC) - (D^2 + E^2 + F^2)/4$, and $T_2 = (ABC + DEF/4) - (AE^2 + BF^2 + CD^2)/4$ are all positive.

To find the equation of the limb of the ellipsoid, let the line of sight vector be $\vec{s} = (p, q, r)$ and the normal to the ellipsoid be

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2Ax + Dy + Fz + G,\ 2By + Dx + Ez + H,\ 2Cz + Ey + Fx + I),$$

then

$$\vec{s} \cdot \nabla f = p(2Ax + Dy + Fz + G) + q(2By + Dx + Ez + H) + r(2Cz + Ey + Fx + I).$$

Thus the equation of the limb is given by

$$f(x, y, z) = \vec{s} \cdot \nabla f = 0. \tag{2}$$

2.2. Projection of a space curve on a principal plane

Following [11], let a space curve be given by the two algebraic equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, then the orthogonal projection of this curve on a principal plane is the resultant of f and g taken in direction d perpendicular to this plane. This is expressed symbolically as

$$R_d(f, g) = 0. \tag{3}$$

For instance, the resultant in the z-direction is the equation obtained from f and g after eliminating z between them.

3. Projection of an ellipsoid limb on a principal plane

The ellipsoid limb is direction dependent, so the ellipsoid limb in z-direction will be denoted by $Limb_z$, that in y-direction by $Limb_y$, and so on. Due to Eqs. (1) and (2) $Limb_z$, $Limb_y$, and $Limb_x$ are defined, respectively, by

$$f(x,y,z) = 2Cz + Ey + Fx + I = 0, \tag{4}$$

$$f(x,y,z) = 2By + Dx + Ez + H = 0, \tag{5}$$

$$f(x,y,z) = 2Ax + Dy + Fz + G = 0. \tag{6}$$

Using Eq. (3), the projections of $Limb_z$ on the xy-plane, of $Limb_y$ on the xz-plane, and of $Limb_x$ on the yz-plane are defined, respectively, by

$$\begin{gathered}(4AC - F^2)x^2 + (4BC - E^2)y^2 + 2(2CD - EF)xy + \\ + 2(2CG - IF)x + 2(2CH - IE)y + (4CJ - I^2) = 0,\end{gathered} \tag{7}$$

$$\begin{gathered}(4AB - D^2)x^2 + (4BC - E^2)z^2 + 2(2BF - DE)xz + \\ + 2(2BG - HD)x + 2(2BI - HE)z + (4BJ - H^2) = 0,\end{gathered} \tag{8}$$

$$\begin{gathered}(4AB - D^2)y^2 + (4AC - F^2)z^2 + 2(2AE - DF)yz + \\ + 2(2AH - DG)y + 2(2AI - FG)z + (4AJ - G^2) = 0.\end{gathered} \tag{9}$$

The nature of the projection of the ellipsoid limb on a principal plane is shown in the following theorem.

Theorem 1. *The projection of an ellipsoid limb in a direction perpendicular to a principal plane is an ellipse obeying (7), (8), and (9), respectively.*

Proof. For Eq. (4) to represent an ellipse, its discriminant δ should be negative, where

$$\begin{aligned}\delta &= (2CD - EF)^2 - (4AC - F^2)(4BC - E^2) = \\ &= -16C\left((ABC + DEF/4) - (AE^2 + BF^2 + CD^2)/4\right)\end{aligned}$$

and refer to the conditions put on Eq. (1) to represent an ellipsoid, then $\delta = -16CT_2$ and note that C and T_2 are positive, then δ is negative as required. Similarly, it can be proved that the projection $Limb_y$ on the xz-plane and $Limb_x$ on the yz-plane are ellipses. □

4. The proposed ellipsoid reconstruction

Rewrite Eqs. (7), (8), and (9) as follows:

$$\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 xy + \alpha_4 x + \alpha_5 y + \alpha_6 = 0, \tag{10}$$

$$\beta_1 x^2 + \beta_2 z^2 + \beta_3 xz + \beta_4 x + \beta_5 z + \beta_6 = 0, \tag{11}$$

$$\gamma_1 y^2 + \gamma_2 z^2 + \gamma_3 yz + \gamma_4 y + \gamma_5 z + \gamma_6 = 0. \tag{12}$$

This results in the following three sets of equations

$$\left.\begin{array}{rrr} 4AC - F^2 = \alpha_1 & 4CD - 2EF = \alpha_3 & 4CH - 2EI = \alpha_5 \\ 4BC - E^2 = \alpha_2 & 4CG - 2FI = \alpha_4 & 4CJ - I^2 = \alpha_6 \end{array}\right\} \tag{13}$$

$$\left.\begin{array}{lll} 4AB - D^2 = \beta_1 & 4BF - 2DE = \beta_3 & 4BI - 2EH = \beta_5 \\ 4BC - E^2 = \beta_2 & 4BG - 2DH = \beta_4 & 4BJ - H^2 = \beta_6 \end{array}\right\} \tag{14}$$

$$\left.\begin{array}{lll} 4AB - D^2 = \gamma_1 & 4AE - 2DF = \gamma_3 & 4AI - 2FG = \gamma_5 \\ 4AC - F^2 = \gamma_2 & 4AH - 2DG = \gamma_4 & 4AJ - G^2 = \gamma_6 \end{array}\right\} \tag{15}$$

The objective in what follows is to express $A, B, C, \ldots, J$ in terms of α_i, β_i, and γ_i, $i = 1, 2, \ldots, 6$. Thus the ellipsoid is reconstructed from the projections of its limbs. This will be done in three stages:

The first stage is described in Theorem 2 where Eqs. (10), (11) are the input and the output is two ellipsoids.

The second stage is described in Theorem 3 where Eqs. (11), (12) are the input and the output is two ellipsoids.

The third stage is to compare the output of the first and second stages and the common ellipsoid is the required one.

Now, rewrite Eqs. (13), (14), (15), respectively, as follows

$$\left.\begin{array}{lll} A = (F^2 + \alpha_1)/(4C) & D = (2EF + \alpha_3)/(4C) & H = (2EI + \alpha_5)/(4C) \\ B = (E^2 + \alpha_2)/(4C) & G = (2FI + \alpha_4)/(4C) & J = (I^2 + \alpha_6)/(4C) \end{array}\right\} \tag{16}$$

$$\left.\begin{array}{lll} A = (D^2 + \beta_1)/(4B) & F = (2DE + \beta_3)/(4B) & I = (2EH + \beta_5)/(4B) \\ C = (E^2 + \beta_2)/(4B) & G = (2DH + \beta_4)/(4B) & J = (H^2 + \beta_6)/(4B) \end{array}\right\} \tag{17}$$

$$\left.\begin{array}{lll} B = (D^2 + \gamma_1)/(4A) & E = (2DF + \gamma_3)/(4A) & I = (2FG + \gamma_5)/(4A) \\ C = (F^2 + \gamma_2)/(4A) & H = (2DG + \gamma_4)/(4A) & J = (G^2 + \gamma_6)/(4A) \end{array}\right\} \tag{18}$$

Merge (16) with (17) to get

$$\left.\begin{array}{lll} B = (E^2 + \alpha_2)/(4C) & C = (E^2 + \beta_2)/(4B) & (F^2 + \alpha_1)/C = (D^2 + \beta_1)/B \\ D = (2EF + \alpha_3)/(4C) & F = (2DE + \beta_3)/(4B) & (2FI + \alpha_4)/C = (2DH + \beta_4)/B \\ H = (2EI + \alpha_5)/(4C) & I = (2EH + \beta_5)/(4B) & (I^2 + \alpha_6)/C = (H^2 + \beta_6)/B \end{array}\right\} \tag{19}$$

Merge (17) with (18) to get

$$\left.\begin{array}{lll} A = (D^2 + \beta_1)/(4B) & B = (D^2 + \gamma_1)/(4A) & (E^2 + \beta_2)/B = (F^2 + \gamma_2)/A \\ F = (2DE + \beta_3)/(4B) & E = (2DF + \gamma_3)/(4A) & (2EH + \beta_5)/B = (2FG + \gamma_5)/A \\ G = (2DH + \beta_4)/(4B) & H = (2DG + \gamma_4)/(4A) & (H^2 + \beta_6)/B = (G^2 + \gamma_6)/A \end{array}\right\} \tag{20}$$

Consider (19) to express D, F, H, and I in terms of B, C, and E:
Solve $D = (2EF + \alpha_3)/(4C)$ with $F = (2DE + \beta_3)/(4B)$ then

$$D = (2B\alpha_3 + E\beta_3)/(8BC - 2E^2) \quad \text{and} \quad F = (2C\beta_3 + E\alpha_3)/(8BC - 2E^2).$$

Also, solve $H = (2EI + \alpha_5)/(4C)$ with $I = (2EH + \beta_5)/(4B)$ then

$$H = (2B\alpha_5 + E\beta_5)/(8BC - 2E^2) \quad \text{and} \quad I = (2C\beta_5 + E\alpha_5)/(8BC - 2E^2).$$

Note that $B = (E^2 + \alpha_2)/(4C)$, that is $4BC - E^2 = \alpha_2$ then

$$\left.\begin{array}{ll} D = (2B\alpha_3 + E\beta_3)/(2\alpha_2) & H = (2B\alpha_5 + E\beta_5)/(2\alpha_2) \\ F = (2C\beta_3 + E\alpha_3)/(2\alpha_2) & I = (2C\beta_5 + E\alpha_5)/(2\alpha_2) \end{array}\right\} \tag{21}$$

Now substitute for D, F, H, and I from Eq. (21) in $(F^2+\alpha_1)/C = (D^2+\beta_1)/B$, $(2FI+\alpha_4)/C = (2DH+\beta_4)/B$, and $(I^2+\alpha_6)/C = (H^2+\beta_6)/B$ to get, after simplification

$$\left.\begin{aligned} B/C &= (4\beta_1\beta_2-\beta_3^2)/(4\alpha_1\alpha_2-\alpha_3^2)\\ B/C &= (2\beta_2\beta_4-\beta_3\beta_5)/(2\alpha_2\alpha_4-\alpha_3\alpha_5)\\ B/C &= (4\beta_2\beta_6-\beta_5^2)/(4\alpha_2\alpha_6-\alpha_5^2). \end{aligned}\right\} \tag{22}$$

Finally, from $B=(E^2+\alpha_2)/(4C)$ and $C=(E^2+\beta_2)/(4B)$ it can be noted that

$$4BC - E^2 = \alpha_2 = \beta_2. \tag{23}$$

Now the first stage is described in the following

Theorem 2. *To reconstruct the ellipsoid* $Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Iz+J=0$ *from the two ellipses* $\alpha_1x^2+\alpha_2y^2+\alpha_3xy+\alpha_4x+\alpha_5y+\alpha_6=0$ *and* $\beta_1x^2+\beta_2z^2+\beta_3xz+\beta_4x+\beta_5z+\beta_6=0$, *do the following four steps:*

1st step: *Check the compatibility of the two ellipses through*

i. $\alpha_2=\beta_2$, *if not satisfied then rescale one of the two equations of the ellipses,*

ii. $\dfrac{4\beta_1\beta_2-\beta_3^2}{4\alpha_1\alpha_2-\alpha_3^2} = \dfrac{2\beta_2\beta_4-\beta_3\beta_5}{2\alpha_2\alpha_4-\alpha_3\alpha_5} = \dfrac{4\beta_2\beta_6-\beta_5^2}{4\alpha_2\alpha_6-\alpha_5^2}$ *; if not satisfied then stop.*

2nd step:

i. Let $\dfrac{B}{C} = \dfrac{4\beta_1\beta_2-\beta_3^2}{4\alpha_1\alpha_2-\alpha_3^2}$ *then* $BC = \left(\dfrac{4\beta_1\beta_2-\beta_3^2}{4\alpha_1\alpha_2-\alpha_3^2}\right)C^2$,

ii. Choose C *arbitrarily such that* E *is real, where* $E^2 = 4BC-\alpha_2$ *(two solutions),*

iii. Determine B *from* $B = \left(\dfrac{4\beta_1\beta_2-\beta_3^2}{4\alpha_1\alpha_2-\alpha_3^2}\right)C$.

3rd step: *Determine* D, F, H, *and* I *from the following equations:*

$$D=\frac{2B\alpha_3+E\beta_3}{2\alpha_2},\ F=\frac{2C\beta_3+E\alpha_3}{2\alpha_2},\ H=\frac{2B\alpha_5+E\beta_5}{2\alpha_2},\ I=\frac{2C\beta_5+E\alpha_5}{2\alpha_2}.$$

4th step: *Compute* A, G, *and* J *from the following equations:*

$$A=\frac{F^2+\alpha_1}{4C},\ G=\frac{2FI+\alpha_4}{4C},\ J=\frac{I^2+\alpha_6}{4C}.$$

Proof. See the previous derivations. □

The second stage is described in the following theorem that will be stated without proof since it is similar to Theorem 2 but with a different input ellipse

Theorem 3. *To reconstruct the ellipsoid* $Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Iz+J=0$ *from the two ellipses* $\beta_1x^2+\beta_2z^2+\beta_3xz+\beta_4x+\beta_5z+\beta_6=0$ *and* $\gamma_1y^2+\gamma_2z^2+\gamma_3yz+\gamma_4y+\gamma_5z+\gamma_6=0$ *do the following four steps:*

1st step: *Check check the compatibility of the two ellipses through*

i. $\gamma_1=\beta_1$, *if not satisfied then rescale one of the two equations of the ellipses,*

ii. $\dfrac{4\gamma_1\gamma_2-\gamma_3^2}{4\beta_1\beta_2-\beta_3^2} = \dfrac{2\gamma_1\gamma_5-\gamma_3\gamma_4}{2\beta_1\beta_5-\beta_3\beta_4} = \dfrac{4\gamma_1\gamma_6-\gamma_4^2}{4\beta_1\beta_6-\beta_4^2}$ *; if not satisfied then stop.*

2nd step:

i. Let $\dfrac{A}{B} = \dfrac{4\gamma_1\gamma_2 - \gamma_3^2}{4\beta_1\beta_2 - \beta_3^2}$ *then* $AB = \left(\dfrac{4\gamma_1\gamma_2 - \gamma_3^2}{4\beta_1\beta_2 - \beta_3^2}\right) B^2$.

ii. Choose B *arbitrarily such that* D *be real, where* $D^2 = 4AB - \gamma_1$ *(two solutions),*

iii. Determine A *from* $A = \left(\dfrac{4\gamma_1\gamma_2 - \gamma_3^2}{4\beta_1\beta_2 - \beta_3^2}\right) B$.

3rd step: Determine E, F, G, *and* H *from the following equations:*

$$E = \frac{D\beta_3 + 2B\gamma_3}{2\gamma_1},\ F = \frac{D\gamma_3 + 2A\beta_3}{2\gamma_1},\ G = \frac{D\gamma_4 + 2A\beta_4}{2\gamma_1},\ \text{and } H = \frac{D\beta_4 + 2B\gamma_4}{2\gamma_1}.$$

4th step: Compute C, I, *and* J *from the following equations:*

$$C = \frac{F^2 + \gamma_2}{4A},\ I = \frac{2FG + \gamma_5}{4A},\ J = \frac{G^2 + \gamma_6}{4A}.$$

The third stage: Compare the two ellipsoids obtained from the second stage with the two obtained in the third stage and the common one is the required ellipsoid.

5. Example

For the ellipsoid

$$6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx - 6\sqrt{5}x - 8\sqrt{5}y + 4\sqrt{5}z + \frac{1}{2} = 0 \tag{24}$$

the coefficients are $A = 6$, $B = 3$, $C = 3$, $D = -4$, $E = -2$, $F = 4$, $G = -6\sqrt{5}$, $H = -8\sqrt{5}$, $I = 4\sqrt{5}$, $J = \frac{1}{2}$.

The projection of $Limb_z$, $Limb_y$, and $Limb_x$ onto the corresponding principal planes is

$$\begin{aligned} 56x^2 + 32y^2 - 32xy - 104\sqrt{5}x - 80\sqrt{5}y - 74 &= 0 && (25)\\ 56x^2 + 32z^2 + 32xz - 136\sqrt{5}x + 16\sqrt{5}z - 314 &= 0 && (26)\\ 56y^2 + 56z^2 - 16yz - 240\sqrt{5}y + 144\sqrt{5}z - 168 &= 0 && (27) \end{aligned}$$

that is:

$$\begin{array}{llllll} \alpha_1 = 56, & \alpha_2 = 32, & \alpha_3 = -32, & \alpha_4 = -104\sqrt{5}, & \alpha_5 = -80\sqrt{5}, & \alpha_6 = -74, \\ \beta_1 = 56, & \beta_2 = 32, & \beta_3 = 32, & \beta_4 = -136\sqrt{5}, & \beta_5 = 16\sqrt{5}, & \beta_6 = -314, \\ \gamma_1 = 56, & \gamma_2 = 56, & \gamma_3 = -16, & \gamma_4 = -240\sqrt{5}, & \gamma_5 = 144\sqrt{5}, & \gamma_6 = -168. \end{array}$$

Now use Eqs. (25) and (26) as input data and apply Theorem 2 to reconstruct the underlying ellipsoid:

1st step: Check the compatibility of the two ellipses through

i. $\alpha_2 = \beta_2 = 32$, o.k.,

ii. $\dfrac{4\beta_1\beta_2 - \beta_3^2}{4\alpha_1\alpha_2 - \alpha_3^2} = \dfrac{2\beta_2\beta_4 - \beta_3\beta_5}{2\alpha_2\alpha_4 - \alpha_3\alpha_5} = \dfrac{4\beta_2\beta_6 - \beta_5^2}{4\alpha_2\alpha_6 - \alpha_5^2} = 1$, o.k.

2nd step: i. $B/C = 1$ then $BC = C^2$, ii. choose $C = 3$ then $E = \pm 2$, iii. $B = 3$.

3rd step: If $B = 3$, $C = 3$, $E = 2$, then $D = -2$, $F = 2$, $H = -7\sqrt{5}$, and $I = -\sqrt{5}$, and if $B = 3$, $C = 3$, $E = -2$, then $D = -4$, $F = 4$, $H = -8\sqrt{5}$, and $I = 4\sqrt{5}$.

4th step: If $B = 3$, $C = 3$, $E = 2$, then $A = 5$, $G = -9\sqrt{5}$, and $J = -\frac{23}{4}$, and if $B = 3$, $C = 3$, $E = -2$, then $A = 6$, $G = -6\sqrt{5}$, and $J = \frac{1}{2}$.

Thus the two reconstructed ellipsoids are

$$5x^2 + 3y^2 + 3z^2 - 2xy + 2yz + 2zx - 9\sqrt{5}x - 7\sqrt{5}y - \sqrt{5}z - \frac{23}{4} = 0, \tag{28}$$

$$6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx - 6\sqrt{5}x - 8\sqrt{5}y + 4\sqrt{5}z + \frac{1}{2} = 0. \tag{29}$$

Similarly the two reconstructed ellipsoids using Theorem 3 are

$$6x^2 + 3y^2 + \frac{131}{49}z^2 + 4xy + \frac{2}{7}yz + \frac{20}{7}zx - \frac{162\sqrt{5}}{7}x - \frac{124\sqrt{5}}{7}y + \frac{24\sqrt{5}}{49}z + \frac{10249}{98} = 0, \tag{30}$$

$$6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx - 6\sqrt{5}x - 8\sqrt{5}y + 4\sqrt{5}z + \frac{1}{2} = 0. \tag{31}$$

The common solution given by Eqs. (29) and (31) agrees with the source ellipsoid.

6. Conclusions

The problem of reconstructing a shape from its projections, views, or images is of special importance in application fields like pattern recognition, computer vision, and medical imaging. In this paper a trial is made to solve this problem for ellipsoids.

Two concepts play a central role in the treatise. The first is the surface limb and the second is the algebraic resultant and its relation with the concept of projection.

A method is described to reconstruct an ellipsoid from projections of its limbs on the principal planes. To start from a solid ground a theorem is proved stating that projection of an ellipsoid limb, in a direction perpendicular to a principal plane, is an ellipse in that plane. Then having three ellipses in the three principal planes, the ellipsoid reconstruction is performed in three stages. In the first stage two out of the three ellipses are used as input and the output is two ellipsoid. In the second stage the third ellipse together with one of the two ellipses used in the first stage are used as input and the output is also two ellipsoids. In the third stage a comparison is made between the output of the first and second stages and the common ellipsoid is the final solution of the problem. The method is demonstrated by a numerical example.

The method can be viewed as a linear 2D alternative to the non-linear 3D method of fitting an ellipsoid to a set of data.

References

[1] J. Sato, R. Cipolla: *Affine reconstruction of curved surfaces from uncalibrated views of apparent contours*. IEEE Transaction on Pattern Analysis and Machine Intelligence **21**, no. 11, Nov. 1999.

[2] Y. Xirouhakis, A. Delopoulos: *Least squares estimation of 3D shape and motion of rigid objects from their orthographic projections*. IEEE Transaction on Pattern Analysis and Machine Intelligence **22**, no. 4, Apr. 2000.

[3] G. Xu, N. Sugioto: *A linear algorithm for motion from three weak perspective images using Euler angles*. IEEE Transaction on Pattern Analysis and Machine Intelligence **21**, no. 1, Jan. 1999.

[4] D. Meyers et al.: *Surface from contours*. ACM Transactions on Graphics **11**, no. 3, July 1992.

[5] D.R. Magee, R.D. Boyle: *Building shape models from image sequences using piece-wise linear approximations*. Proc. British Machine Vision Conference 1998.

[6] S. Bischoff, Kobbelt: *Ellipsoid decomposition of 3D-models*. 1st Internat. Symp. on 3D Data Processing Visualization and Transmission, Padova/Italy 2002.

[7] S. Jaggi et al.: *Estimation of dynamically evolving ellipsoid with applications to medical imaging*. IEEE Transactions on Medical Imaging **14**, no. 2, June 1995.

[8] E. Rimon, S.P. Boyd: *Efficient distance computation using best ellipsoid fit*. Technical Report, Stanford University, Dept. of Electrical Engineering, Feb. 10, 1992.

[9] D.A. Turner et al.: *An algorithm for fitting an ellipsoid to data*. citeseer.nj.nec.com/322908.html.

[10] J. Levin: *A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces*. Communications of the ACM **19**, no. 10, Oct. 1976.

[11] M. Ali Said et al.: *A mathematical model for determining the intersection of two algebraic surfaces*. Scientific Bulletin, Fac. of Engineering, Ain Shams University, Egypt, **31**, no. 3, Sept. 1996.

Received November 3, 2003; final form December 10, 2004