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On Arne Dür's Equation Concerning Central Axonometries

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Abstract. It is a classical Descriptive Geometry problem in the Euclidean n-space to characterize the central projections among collinear transformations with rank deficiency. Recently A. Dür presented for $n = 3$ a characterization in form of an equation in complex coordinates — the central axonometric counterpart of the Gauss equation for orthogonal axonometries. Here two new proofs for Dür's equation are given combined with equivalent statements. And its n-dimensional generalization is addressed which characterizes two-dimensional orthogonal central views among central axonometries.

Key Words: central projection, central axonometry

MSC 2000: 51N05

1. The axonometric principle

At the beginning we summarize some results on n-dimensional axonometry.

1.1. Parallel projections

Let $(O; E_1, E_2, E_3)$ be a cartesian basis of the Euclidean 3-space $\mathbb{E}^3$. Then for arbitrarily given noncollinear points $(O^p; E_1^p, E_2^p, E_3^p)$ in an image plane Π there is an unique affine transformation

$$\alpha\colon\ \mathbb{E}^3 \to \Pi \ \text{ with } \ O \mapsto O^p,\ E_i \mapsto E_i^p, \quad i = 1, 2, 3.$$

The point $X \in \mathbb{E}^3$ with coordinates $(x_1, x_2, x_3)^T$ is mapped onto its *'axonometric view'*

$$X^p = \alpha(X) \ \text{ with } \ \overrightarrow{O^pX^p} = x_1\,\overrightarrow{O^pE_1^p} + x_2\,\overrightarrow{O^pE_2^p} + x_3\,\overrightarrow{O^pE_3^p}$$

(see Fig. 1). The famous Pohlke theorem claims that α is the product of a 3D similarity and a parallel projection. Hence *any axonometric view is similar to a parallel view.*

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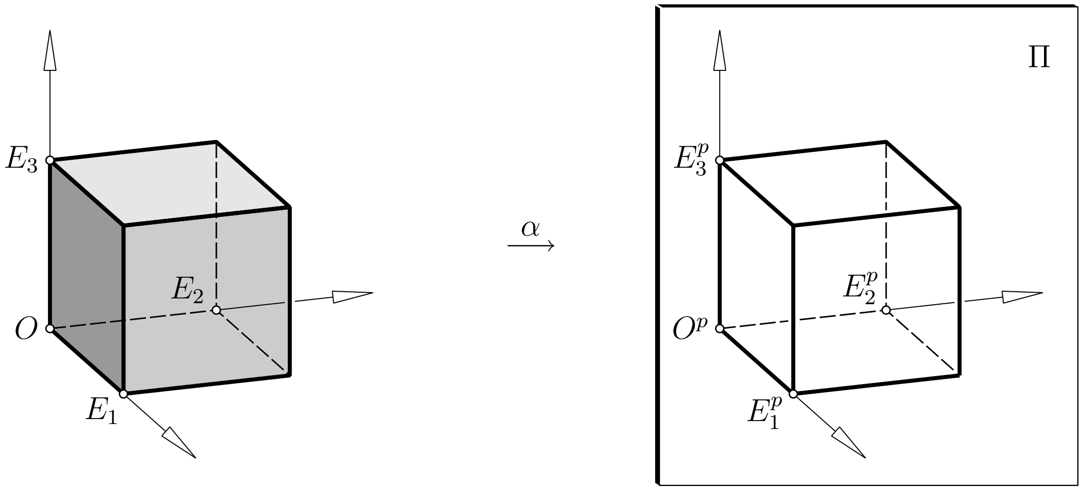

Figure 1: The axonometric principle in $\mathbb{E}^3$

More general, an m-dimensional axonometric view of $\mathbb{E}^n$, $m < n$, is given by an *axonometric reference system* $(O^p; E_1^p, \ldots, E_m^p)$ in $\mathbb{E}^m$ as the image under the affine transformation

$$\alpha\colon\ \mathbb{E}^n \to \mathbb{E}^m \ \text{ with } \ O \mapsto O^p,\ E_i \mapsto E_i^p, \quad i = 1, \ldots, n. \tag{1}$$

Point $X = (x_1, \ldots, x_n)$ is mapped onto $\alpha(X) \in \mathbb{E}^m$ with cartesian coordinates $(x'_1, \ldots, x'_m)$ obeying

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_m \end{pmatrix} = \begin{pmatrix} a_{10} \\ \vdots \\ a_{m0} \end{pmatrix} + A\cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}. \tag{2}$$

And the multi-dimensional version of Pohlke's Theorem reads (see, e.g., [17, 7, 2, 12])

Theorem 1. *1. The affine transformation α defined in (1) with the coordinate representation (2) is the product of a similarity and a surjective parallel projection if and only if either $2m \le n+1$ or the smallest singular value λ of matrix A, i.e., the smallest eigenvalue of $A\cdot A^T$, has a multiplicity $\ge 2m-n$. For $\lambda = 1$ the axonometric view is congruent to a parallel view.*

2. The projection is orthogonal if and only if the row vectors of matrix A are of equal length and pairwise orthogonal, i.e., if there is one singular value only.

The columns in A are cartesian coordinates of the vectors $\overrightarrow{O^pE_i^p}$ in $\mathbb{E}^m$. According to L. Schläfli [11, p. 134 resp. 298], in the case of an *orthogonal* projection the images $E_1^p, \ldots, E_n^p$ of the unit points are called *eutactic* with respect to O^p (see also [6], or [3, p. 251]). Such points in $\mathbb{E}^m$ are characterized by the property that for any hyperplane Γ' through O^p the squared distances $\overline{E_i^p\Gamma'}$ have a sum λ^2 independent from Γ;[1] λ is the scaling factor of the involved similarity. This results from the fact that Γ' can be seen as 'edge view' of a hyperplane Γ in $\mathbb{E}^n$; the distances from Γ are preserved under the orthogonal projection; and the unit points of a cartesian basis in $\mathbb{E}^n$ are eutactic with respect to the origin.

[1]This is equivalent to the statement that for $E_1^p, \ldots, E_n^p$ the ellipsoid of inertia centered at O^p is a sphere. In [12, Satz 6] an iterative procedure is given for obtaining eutactic points in $\mathbb{E}^m$. Eutactic points define 'almost orthonormal' vector systems with various properties (see [5]).

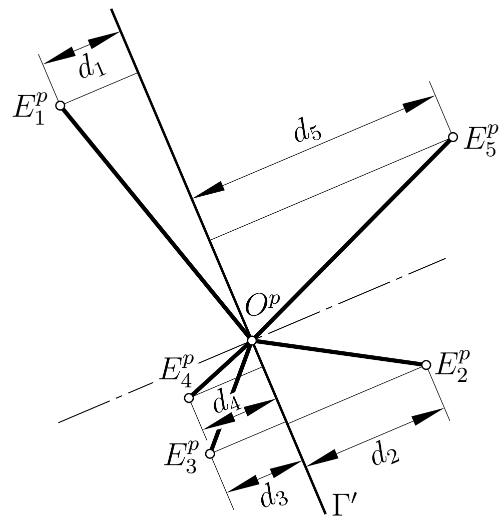

Figure 2: $E_1^p, \ldots, E_5^p$ are *eutactic* with respect to O^p, i.e., they are similar to an orthogonal view of a cartesian basis. For $d_i = \overline{E_i^p \Gamma'}$ the sum $\lambda^2 = \sum_{i=1}^5 d_i^2$ is independent from Γ' through O^p

In the case $m = 2$ the coordinates (x'_1, x'_2) of each image point can be combined to a complex number $\mathbf{x}' := x'_1 + ix'_2$. Then the second part of Theorem 1 gives the n-dimensional version [14] of the Gauss theorem:

$\mathbf{e}_1^p, \ldots, \mathbf{e}_n^p$ *are complex coordinates of points being eutactic with respect to the origin* $\iff$

$$\mathbf{e}_1^{p\,2} + \ldots + \mathbf{e}_n^{p\,2} = 0\,.^2 \tag{3}$$

1.2. Central axonometry

For handling central projections we extend $\mathbb{E}^n$ and the image space $\mathbb{E}^m$ by their points at infinity to projective spaces $\mathbb{E}^{n*}$ and $\mathbb{E}^{m*}$, respectively:

Let $U_1, \ldots, U_n$ denote the points at infinity of the axes of the cartesian basis $O; E_1, \ldots, E_n$. Then any $(2n+1)$-tupel $(O^c; E_1^c, \ldots, E_n^c; U_1^c, \ldots, U_n^c)$ in $\mathbb{E}^{m*}$, $m < n$, with pairwise different and collinear $\{O^c, E_i^c, U_i^c\}$ for $i = 1, \ldots, n$ is called a *central axonometric reference system* in $\mathbb{E}^{m*}$, provided these points span $\mathbb{E}^{m*}$ and $O^c, E_1^c, \ldots, E_n^c$ are finite as well as at least one U_i^c.

There is a unique surjective collinear transformation

$$\kappa\colon\ \mathbb{E}^{n*} \to \mathbb{E}^{m*} \quad \text{with} \quad O \mapsto O^c,\ E_i \mapsto E_i^c,\ U_i \mapsto U_i^c, \quad i = 1, \ldots, n. \tag{4}$$

We homogenize the cartesian coordinates in $\mathbb{E}^{n*}$ and $\mathbb{E}^{m*}$ and indicate this by the symbol '$*$'. E.g., for a finite point $X \in \mathbb{E}^{n*}$ with the coordinate vector $\mathbf{x} = (x_1, \ldots, x_n)$ a particular homogeneous coordinate vector reads

$$\mathbf{x}^* = (x_0^*, \ldots, x_n^*) = (1, x_1, \ldots, x_n) = (1, \mathbf{x}) \quad \text{and} \quad X = \mathbb{R}\mathbf{x}^*.$$

[2] In any case the points with complex coordinates $\pm\mathbf{f}$ obeying $\mathbf{f}^2 = \mathbf{e}_1^{p\,2} + \ldots + \mathbf{e}_n^{p\,2}$ are the focal points of the visual contour of the unit sphere of $\mathbb{E}^n$.

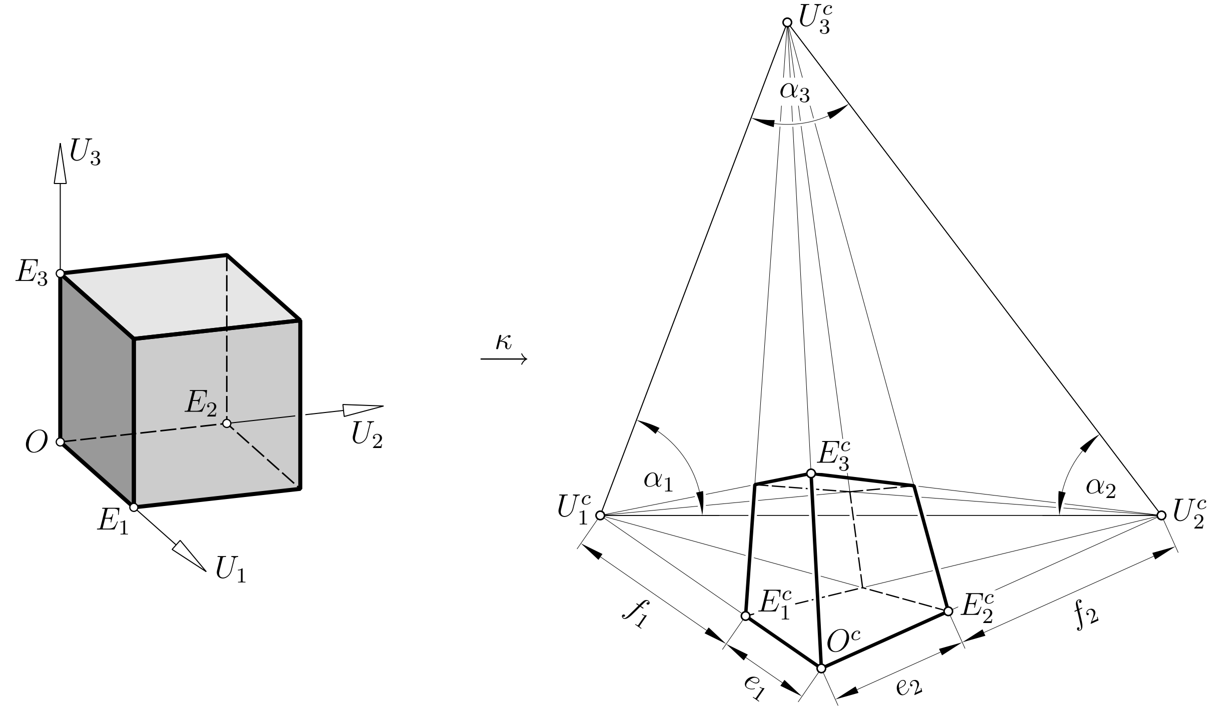

Figure 3: Central axonometric principle

Then $\kappa\colon\ \mathbb{E}^{n*} \to \mathbb{E}^{m*}$ can be expressed as the linear map

$$\mathbf{x}'^{*} = \begin{pmatrix} x_0'^{*} \\ x_1'^{*} \\ \vdots \\ x_m'^{*} \end{pmatrix} = l(\mathbf{x}^*) = A\cdot \begin{pmatrix} x_0^{*} \\ x_1^{*} \\ \vdots \\ x_n^{*} \end{pmatrix}, \quad A = \left(\begin{array}{c|ccc} a_{00} & a_{01} & \dots & a_{0n} \\ \hline a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m0} & a_{m0} & \dots & a_{mn} \end{array}\right). \tag{5}$$

Due to the *central axonometric principle* a central axonometric reference system can be arbitrarily specified, and the image of $\mathbb{E}^{n*}$ under κ is called *central axonometric view*.

According to [7, 8] the central axonometric analogon of Theorem 1 needs some preparatory steps: From the $(m+1 \times n+1)$-matrix A in (5) we compute a $(m \times n)$-matrix $\widetilde{A}$ as follows:

- drop the first column and the first row,
- replace for $i = 1, \dots, m$ the row vector $\mathbf{a}_i$ by the component which is orthogonal to the row vector $\mathbf{a}_0$. (6)

Theorem 2. *1. The collinear transformation κ in (4) with coordinate representation (5) is the product of a surjective central projection and an isometry if and only if either $2m \le n$ or the smallest singular value of the derived matrix $\widetilde{A}$ has a multiplicity $\ge (2m-n+1)$.*

2. This central projection is orthogonal, i.e., the $(n-m-1)$-dimensional center is totally orthogonal to the image space $\mathbb{E}^{m*}$ if and only if the row vectors of the derived matrix $\widetilde{A}$ are of equal length and pairwise orthogonal.

Remark 1: [13] reveals why these conditions look similar to that in Theorem 1: Any central projection is *associated* to a parallel projection with $(n-m)$-dimensional fibres parallel to the center and to the common perpendicular p between the center and the image space. $\widetilde{A}$ is proportional to the coordinate matrix of the associated parallel projection. Exactly for orthogonal central projections

the associated projection is orthogonal.
And for any finite point $X = (x_1, \dots, x_n)$ the central projection X^c and the associated parallel projection X^p are aligned with the *principal point* H which is the central (and parallel) view of the common perpendicular p. The ratio $(X^p X^c H) := \overline{HX^p}/\overline{HX^c}$ (signed lengths) equals $\overline{X\Pi_v}/\overline{H\Pi_v}$ where Π_v is the *vanishing hyperplane* of κ, i.e., the hyperplane through the center and parallel to the image space (see Fig. 4).

However, Theorem 2 says nothing about how the central axonometric reference systems for central projections can be charactized. For the case $(n,m) = (3,2)$ some characterizations are known (e.g., [10]). We pick out J. Szabó's condition in [16] which works only for the case that all points of the reference system are finite: $(O^c; E_1^c, \dots, U_3^c)$ is the central view of a cartesian reference system if and only if with the notation of Fig. 3

$$\left(\frac{e_1}{f_1}\right)^2 : \left(\frac{e_2}{f_2}\right)^2 : \left(\frac{e_3}{f_3}\right)^2 = \tan\alpha_1 : \tan\alpha_2 : \tan\alpha_3 \,. \tag{7}$$

According to [9] the limit of this condition for one point U_i^c tending to infinity equals that given in [15].

Recently A. Dür presented in [4] a new characterization which works without restrictions on U_i^c. He uses the ratios

$$\rho_i := (O^c E_i^c U_i^c) = \overline{O^c U_i^c}/\overline{E_i^c U_i^c} \quad \text{(signed distances)} \quad \text{and} \quad \rho_i' := 1 - \rho_i \,. \tag{8}$$

Like in the Gauss equation (3) point O^c is the origin of the coordinate system in $\mathbb{E}^2$ and the cartesian coordinates of E_i^c are combined in the complex number $\mathbf{e}_i^c$. Then we obtain

Theorem 3. (A. Dür [4]) *Any planar central view of a three-dimensional cartesian reference system is characterized by*

$$(\rho_2'\rho_1\mathbf{e}_1^c - \rho_1'\rho_2\mathbf{e}_2^c)^2 + (\rho_3'\rho_2\mathbf{e}_2^c - \rho_2'\rho_3\mathbf{e}_3^c)^2 + (\rho_1'\rho_3\mathbf{e}_3^c - \rho_3'\rho_1\mathbf{e}_1^c)^2 = 0, \quad \mathbf{e}_1^c, \dots, \mathbf{e}_3^c \in \mathbb{C} \,. \tag{9}$$

2. A new proof of Dür's equation

We start with a central projection in $\mathbb{E}^{3*}$ with center Z, image plane $\Pi = \mathbb{E}^2$ and principal point H. Due to standard formulas from Projective Geometry the ratios ρ_i and ρ_i' from (8) can be expressed as cross ratios[3] (see Fig. 4). For this purpose we insert on the coordinate axis OE_i the *vanishing point* V_i which under κ is mapped into infinity. All vanishing points in space are located in the *vanishing plane* Π_v through Z parallel Π.

$$\rho_i = (O^c E_i^c U_i^c) = (O^c E_i^c U_i^c V_i^c) = (OE_i U_i V_i), \tag{10}$$

$$\rho_i' = 1 - \rho_i = (OU_i E_i V_i) = (E_i V_i O U_i) = (E_i V_i O). \tag{11}$$

For $\rho_i' = 0$ point U_i^c is at infinity; otherwise the vanishing point V_i on the axis OE_i obeys $\overline{OV_i} = 1/\rho_i'$. Due to our assumption for central axonometric reference systems there is at least one $\rho_i' \neq 0$. The equation of the vanishing plane spanned by $V_1, \dots, V_3$ reads

$$\Pi_v\colon\ \rho_1' x_1 + \ \dots\ + \rho_3' x_3 = 1 \,. \tag{12}$$

[3]The ratio $(X_1X_2X_3)$ is equal to the cross ratio $(X_1X_2X_3U)$ with the aligned point U at infinity.

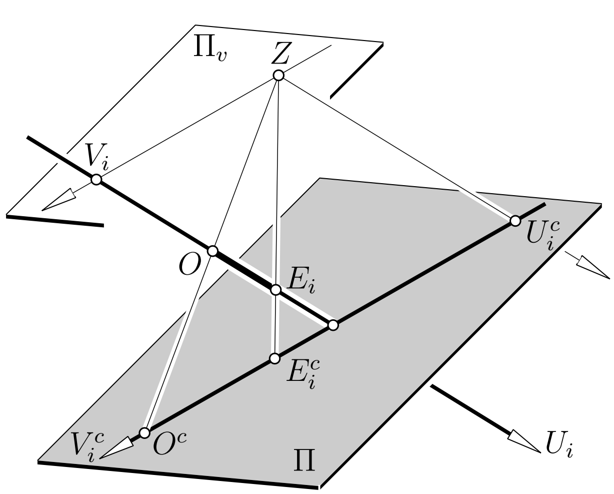

Figure 4: DÜR's ratios $\rho_i = (O^c E_i^c U_i^c)$ and $\rho'_i = 1 - \rho_i$, seen as cross ratios

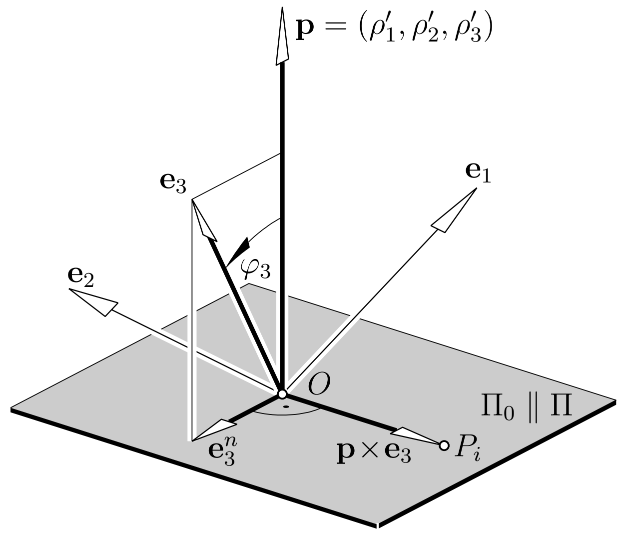

Figure 5: Proof of A. DÜR's equation

Now the coordinate representation (5) of our central projection $\kappa:\ \mathbb{E}^{3*} \to \mathbb{E}^{2*}$ is already available. The following matrix equation looks unusual as for points in the image space two of the three homogeneous coordinates are combined in a complex number.

$$\mathbf{x}'^* = \left(\frac{x_0'^*}{x_0'^*\mathbf{z}'}\right) = l(\mathbf{x}^*) = \left(\begin{array}{c|ccc} 1 & -\rho'_1 & \dots & -\rho'_3 \\ \hline \mathbf{o} & \rho_1\mathbf{e}_1^c & \dots & \rho_3\mathbf{e}_3^c \end{array}\right) \cdot \left(\begin{array}{c} x_0^* \\ \hline x_1^* \\ \vdots \\ x_3^* \end{array}\right). \tag{13}$$

Proof. Exactly the points of Π_v give $x_0'^* = 0$ and are therefore mapped onto points at infinity. On the other hand E_i is mapped onto the point with the inhomogeneous complex coordinate

$$\frac{1}{1-\rho'_i}\,\rho_i\,\mathbf{e}_i^c = \mathbf{e}_i^c,$$

which is E_i^c as required. □

The normal vector $\mathbf{p} := (\rho'_1, \rho'_2, \rho'_3) \neq \mathbf{o}$ of the vanishing plane has the direction of the *principal ray* $p = ZH$. The cross products with the unit vectors $\mathbf{e}_i$ in direction of the coordinate axes are

$$\mathbf{p}\times\mathbf{e}_1 = (0, \rho'_3, -\rho'_2)\,, \quad \mathbf{p}\times\mathbf{e}_2 = (-\rho'_3,\ 0,\ \rho'_1)\,, \quad \mathbf{p}\times\mathbf{e}_3 = (\rho'_2,\ -\rho'_1,\ 0)\,.$$

These are 3D coordinates of points P_1, P_2, P_3 in a plane Π_0 parallel zu Π. The geometric meaning of cross products (see Fig. 5)

$$\|\mathbf{p}\times\mathbf{e}_i\| = |\sin\varphi_i| \cdot \|\mathbf{p}\| = \|\mathbf{e}_i^n\| \cdot \|\mathbf{p}\|$$

implies that P_1, P_2, P_3 are related to the orthogonal views E_1^n, E_2^n, E_2^n of the unit points by a dilation from O with factor $\|\mathbf{p}\|$ and a rotation about O through 90°. Hence P_1, P_2, P_3 are eutactic, too, and this is preserved under the projection from Z into Π as Π_0 is parallel to Π.

By (13) the images P_1^c, P_2^c, P_3^c have the complex coordinates

$$\mathbf{p}_1^c = (\rho_3'\rho_2\mathbf{e}_2^c - \rho_2'\rho_3\mathbf{e}_3^c)^2, \quad \mathbf{p}_2^c = (\rho_1'\rho_3\mathbf{e}_3^c - \rho_3'\rho_1\mathbf{e}_1^c)^2, \quad \mathbf{p}_3^c = (\rho_1'\rho_3\mathbf{e}_3^c - \rho_3'\rho_1\mathbf{e}_1^c)^2, \tag{14}$$

and the Gauss equation (3) $\mathbf{p}_1^{c\,2} + \mathbf{p}_2^{c\,2} + \mathbf{p}_3^{c\,2} = 0$ coincides with (9).

Conversely, we note that for any central axonometric reference system in $\mathbb{E}^2$ the linear map (13) describes the underlying collinear transformation κ defined in (4) because collinear transformations preserve cross ratios on each line which is not mapped onto a single point. And (13) assigns to each collinear triple (O, E_i, V_i) the required images (O^c, E_i^c, V_i^c).

Now, let the given central axonometric reference system $(O^c; E_1^c, \ldots, U_3^3)$ in the plane Π obey (9) and let P_1^c, P_2^c, P_3^c be the eutactic points with coordinates $\mathbf{p}_i^c$ by (14) with ρ_i, ρ_i' by (8). We embed Π into $\mathbb{E}^{3*}$ and erect a normal vector $\mathbf{p}$ of length $\|\mathbf{p}\| = \sqrt{\rho_1'^2 + \cdots + \rho_3'^2}$. Then we reverse the procedure displayed in Fig. 5: We set $O = O^c$, $P_i = P_i^c$, $i = 1, 2, 3$, and obtain an unique cartesian frame $(O; E_1, \ldots, E_3)$ with $\mathbf{p}_i = \overrightarrow{OP_i} = \mathbf{p}\times\mathbf{e}_i$. There are at least two linearly independent vectors, say $\mathbf{p}_1, \mathbf{p}_2$. With respect to this particular cartesian frame the plane $\Pi = \Pi_0$ has the equation

$$\rho_1' x_1 + \ldots + \rho_3' x_3 = 0.$$

It remains to prove that the corresponding collinear transformation κ defined in (4) and represented by the linear map $\mathbf{x}^* \mapsto \mathbf{x}'^* = l(\mathbf{x}^*)$ in (13) is a projection:
First we note that besides O and P_i *all finite points* $X \in \Pi$ *remain fixed under* κ because we can set up the homogeneous coordinate vector of X as $\mathbf{x}^* = (1,\ \alpha_1\mathbf{p}_1 + \alpha_2\mathbf{p}_2)$ with $\alpha_1, \alpha_2 \in \mathbb{R}$, and this implies $l(\mathbf{x}^*) = (1,\ \alpha_1\mathbf{p}_1^c + \alpha_2\mathbf{p}_2^c)$, hence $\kappa(X) = X$.
κ has rank deficiency 1. Therefore there is a center Z with coordinate vector $\mathbf{z}^*$ in the kernel of l, to say $l(\mathbf{z}^*) = \mathbf{o}^*$.[4] For any point $Y \neq Z$ let X denote the point of intersection $YZ \cap \Pi$. We can set up $\mathbf{y}^* = \beta_1\,\mathbf{z}^* + \beta_2\,\mathbf{x}^*$ with $\beta_2 \neq 0$. Then $l(\mathbf{y}^*) = \beta_2\, l(\mathbf{x}^*)$ means $\kappa(Y) = \kappa(X) = X$. Hence, κ is a projection.

3. Analoga of Dür's equation

We now concentrate on two-dimensional central-axonometric views of $\mathbb{E}^{n*}$, $n \geq 3$, i.e., on collinear transformations $\kappa\colon\ \mathbb{E}^{n*} \to \mathbb{E}^{2*}$. We still use the ratios ρ_i and ρ_i' from (8); their interpretations (10), (11) as cross ratios are still valid. We obtain the linear map l describing κ when we replace the subscript 3 by n in (13). The image of U_i under (13) has the complex coordinate

$$\mathbf{u}_i^c = -\frac{\rho_i}{\rho_i'}\,\mathbf{e}_i^c, \quad i = 1, \ldots, n. \tag{15}$$

This is in accordance with $\rho_i = (O^c E_i^c U_i^c)$ in (8).

Replacing 3 by n converts (12) into the equation of the *vanishing hyperplane* of κ. Its normal vector $\mathbf{p} := (\rho_1', \ldots, \rho_n')$ defines a point at infinity $(0, \mathbf{p})\mathbb{R}$ which is mapped under κ onto the *principal point* H with the complex coordinate

$$\mathbf{h} = \frac{-1}{\|\mathbf{p}\|^2}\,(\rho_1\rho_1'\mathbf{e}_1^c + \ldots + \rho_n\rho_n'\mathbf{e}_n^c) = \frac{1}{\rho_1'^2 + \ldots + \rho_n'^2}\left(\rho_1'^2\mathbf{u}_1^c + \ldots + \rho_n'^2\mathbf{u}_n^c\right). \tag{16}$$

This expresses $\mathbf{h}$ as a weighted mean of $\mathbf{u}_1^c, \ldots, \mathbf{u}_n^c$ — with nonnegative weights.

[4] Z is the point of intersection between Π_v and the line p through the principal point H by (16) orthogonal to Π.

3.1. Case $n = 3$:

Corollary 4. SZABÓ*'s condition (7) is equivalent to the statement that the principal point H given by (16) coincides with the orthocentre of $U_1^c U_2^c U_3^c$.*

Proof. A straightforward computation reveals that for a non-rectangular triangle the orthocentre is the weighted mean of the vertices with weights $\tan\alpha_i$. The ratios on the left hand side of (7) obey $e_i/f_i = -\rho_i'$. Hence (7) states proportional weights for H and the orthocentre. □

Remark 2: For central projections this coincidence is obvious. Conversely, if for a central axonometry in Π the principal point H coincides with the orthocentre of $U_1^c U_2^c U_3^c$, then by standard methods of Descriptive Geometry a center Z relative to Π can be reconstructed. Now there are four points in the plane at infinity for which the axonometric view coincides with their projection via Z into Π. This turns out to be sufficient for the identity between κ and this projection.

Theorem 5. *The characterization (9) of central projections among central axonometries due to* A. DÜR *is equivalent to*

$$(\rho_1'\mathbf{h} + \rho_1\mathbf{e}_1^c)^2 + (\rho_2'\mathbf{h} + \rho_2\mathbf{e}_2^c)^2 + (\rho_3'\mathbf{h} + \rho_3\mathbf{e}_3^c)^2 = 0\,. \tag{17}$$

For finite U_i^c it is also equivalent to

$$\frac{1}{\rho_1'^2}(\mathbf{u}_2^c - \mathbf{u}_3^c)^2 + \frac{1}{\rho_2'^2}(\mathbf{u}_3^c - \mathbf{u}_1^c)^2 + \frac{1}{\rho_3'^2}(\mathbf{u}_1^c - \mathbf{u}_2^c)^2 = 0\,. \tag{18}$$

Proof. For $\rho_1'\rho_2'\rho_3' \neq 0$ we substitute in (9) $\mathbf{e}_i^c$ by $\mathbf{u}_i^c$ according to (15) and obtain (18).
Eq. (17) is related to Remark 1: For any point E_i in $\mathbb{E}^{3*}$ the central view E_i^c and its associated parallel view E_i^p (which is an orthogonal view here) are alined with the principal point H. The dilation with center H mapping E_i^c onto E_i^p has the scaling factor $f = \overline{E_i\Pi_v}/\overline{H\Pi_v}$. Without loss of generality we can replace the image plane by the parallel plane Π_0 through point O as the translation of Π in direction of the principal ray $p = ZH$ acts on the central view like a dilation from H. Then the scaling factor reads

$$f = \overline{E_iV_i}/\overline{OV_i} = (E_iOV_iU_i) = \rho_i$$

according to (10). Hence

$$\mathbf{e}_i^p = \mathbf{h} + \rho_i(\mathbf{e}_i^c - \mathbf{h}) = \rho_i'\mathbf{h} + \rho_i\mathbf{e}_i^c \tag{19}$$

is the complex coordinate of an orthogonal view of E_i. So, E_1^p, E_2^p, E_3^p are eutactic with respect to $O^p = O^c$, and (18) results from the Gauss equation (3).[5] The equivalence between (9) and (18) will be demonstrated for each $n \geq 3$ in the proof of Theorem 6, and this ends a second new proof for DÜR's equation. □

3.2. Case $n \geq 4$:

From Theorem 2 we learn that for $n \geq 4$ any central axonometric image is congruent to a central view. So, there is no restriction on central axonometric reference systems. However, we will confine ourselves to *orthogonal central views*, i.e., the center of the projection is supposed to be totally orthogonal to the image plane. Then there are higher-dimensional analoga to A. DÜR's equation (9):

[5] When we apply the procedure (6) to the matrix in (13) then we get $\widetilde{A}$ with column vectors $(\mathbf{e}_1^p, \mathbf{e}_2^p, \mathbf{e}_3^p)$.

Theorem 6. *For any central axonometric reference system* $(O^c; E_1^c, \dots, U_n^c)$ *of* $\mathbb{E}^{n*}$ *in the plane* $\mathbb{E}^{2*}$ *the collinear transformation* κ *defined by (4) is the product of a surjective orthogonal central projection and an isometry if and only if*

$$(\rho_1'\mathbf{h} + \rho_1\mathbf{e}_1^c)^2 + \dots + (\rho_n'\mathbf{h} + \rho_n\mathbf{e}_n^c)^2 = 0$$

with the complex number $\mathbf{h}$ *being defined by (16). This equation is equivalent to*

$$\sum_{\substack{i,j=1\\ i<j}}^{n} \left(\rho_i\rho_j'\mathbf{e}_i^c - \rho_j\rho_i'\mathbf{e}_j^c\right)^2 = 0 \quad \text{and under } \rho_1'\dots\rho_n' \neq 0 \text{ also to} \quad \sum_{\substack{i,j=1\\ i<j}}^{n} \rho_i'^2\rho_j'^2\left(\mathbf{u}_i^c - \mathbf{u}_j^c\right)^2 = 0\,.$$

Proof. We follow exactly the arguments in the proof of Theorem 5, eq. (17), (see also [13]) and obtain the first equation as Gauss equation for the associated (and now again orthogonal) views $E_1^p, \dots, E_n^p$ with the complex coordinates (19).
The equivalence to the second and the third equation is proved straightforward:

$$\begin{aligned}
&\textstyle\sum_i(\rho_i'\mathbf{h} + \rho_i\mathbf{e}_i^c)^2 = \mathbf{h}^2\|\mathbf{p}\|^2 - 2\mathbf{h}^2\|\mathbf{p}\|^2 + \sum_i \rho_i^2\mathbf{e}_i^{c\,2} = \\
&= \tfrac{1}{\|\mathbf{p}\|^2}\left[-\left(\textstyle\sum_i \rho_i\rho_i'\mathbf{e}_i^c\right)^2 + \|\mathbf{p}\|^2\sum_i \rho_i^2\mathbf{e}_i^{c\,2}\right] = \tfrac{1}{\|\mathbf{p}\|^2}\Big[-\textstyle\sum_i \rho_i^2\rho_i'^2\mathbf{e}_i^{c\,2} - \\
&- 2\textstyle\sum_{i<j}\rho_i\rho_i'\rho_j\rho_j'\mathbf{e}_i^c\mathbf{e}_j^c + \sum_i \rho_i^2\rho_i'^2\mathbf{e}_i^{c2} + \sum_{i<j}\left(\rho_i^2\rho_j'^2\mathbf{e}_i^{c2} + \rho_j^2\rho_i'^2\mathbf{e}_j^{c2}\right)\Big] = \\
&= \tfrac{1}{\|\mathbf{p}\|^2}\left[\textstyle\sum_{i<j}\left(\rho_i\rho_j'\mathbf{e}_i^c - \rho_j\rho_i'\mathbf{e}_j^c\right)^2\right] = \tfrac{1}{\|\mathbf{p}\|^2}\left[\textstyle\sum_{i<j}\rho_i'^2\rho_j'^2\left(\mathbf{u}_i^c - \mathbf{u}_j^c\right)^2\right]
\end{aligned}$$

by (15) and (16). □

The following version is valid also for a higher-dimensional image space $\mathbb{E}^{m*}$, provided $\mathbf{e}_i^c$ and $\mathbf{h}$ denote cartesian coordinate vectors of E_i^c and the principal point H.

Corollary 7. *The central axonometric reference system* $(O^c; E_1^c, \dots, U_n^c)$ *in* $\mathbb{E}^{m*}$, $2 \le m < n$, *defines an orthogonal central view of* $\mathbb{E}^{n*}$ *if and only if the points* E_i^p *with cartesian coordinate vectors* $\mathbf{e}_i^p = \rho_i'\mathbf{h} + \rho_i\mathbf{e}_i^c$ *by (8) and (16) are eutactic with respect to* O^c.

As already mentioned in Footnote 5, the affine combinations $\mathbf{e}_i^p = \rho_i'\mathbf{h} + \rho_i\mathbf{e}_i^c$ are the columns of the 'reduced' matrix $\widetilde{A}$ according to (6).

Remark 3: We finally recall that due to [13, Satz 3] for $m < n/2$ the *orthogonal* central views of $\mathbb{E}^{n*}$ in $\mathbb{E}^{m*}$ cannot be distinguished from *isocline* central views, where the center is supposed to be isocline to the image space. This is an analogue to the fact that for $m \le n/2$ orthogonal views are similar to oblique views with fibres being isocline to the image space (cf. [12, p. 164]).

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