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Generalized Apollonian Circles

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Abstract. Given a triangle, we construct a one-parameter family of triads of coaxal circles orthogonal to circumcircle and mutually tangent to each other at a point on the circumcircle. This is a generalization of the classical Apollonian circles which intersect at the two isodynamic points.

Key Words: Apollonian circles, barycentric coordinates, circumconics

MSC 2000: 51M04

1. Generalization of Apollonius circles

The classical Apollonian circles of a triangle are the three circles each with diameter the segment between the feet of the two bisectors of an angle on its opposite sides. It is well known that the centers of the three circles are the intercepts of the Lemoine axis on the sidelines,[1] and that they have two points in common, the isodynamic points ([1, p. 218]). The feet of the internal angle bisectors are the traces of the incenter. Those of the external angle bisectors are the intercepts of the trilinear polar of the incenter. More generally, given a point P, with traces X, Y, Z on the sidelines BC, CA, AB of triangle ABC, the harmonic conjugates of X in BC, Y in CA, Z in AB are the intercepts X', Y', Z' of the trilinear polar of P on these sidelines. The circles with diameters XX', YY', ZZ' are the *Apollonian circles associated with* P. The isodynamic points are the common points of the triad of Apollonian circles associated with the incenter. In this note we characterize the points P for which the three Apollonian circles are mutually tangent to each other, and provide an elegant euclidean construction for such circles.

We work with barycentric coordinates with respect to triangle ABC, and their homogenization (see, for example, [5, 6]).[2] If P has homogeneous barycentric coordinates $(u : v : w)$, its traces are the points

$$X = (0 : v : w), \qquad Y = (u : 0 : w), \qquad Z = (u : v : 0).$$

[1] The Lemoine axis of a triangle is the trilinear polar of the Lemoine symmedian point. See, for example, [1, p. 253].

[2] For a treatment of the Apollonian circles using complex numbers, see [4].

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The harmonic conjugates of these points on the respective sidelines are

$$X' = (0 : v : -w), \quad Y' = (-u : 0 : w), \quad Z' = (u : -v : 0).$$

These are the intercepts of the line

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0\,,$$

the trilinear polar of P (see Fig. 1). The centers of the Apollonian circles, being the midpoints of the segments XX', YY', ZZ' respectively, are the points

$$A' = (0 : v^2 : -w^2), \qquad B' = (-u^2 : 0 : w^2), \qquad C' = (u^2 : -v^2 : 0).$$

These are the intercepts of the line

$$\mathcal{L}: \qquad \frac{x}{u^2} + \frac{y}{v^2} + \frac{z}{w^2} = 0\,,$$

the trilinear polar of the point $(u^2 : v^2 : w^2)$. It is clear that the Apollonian circles associated with the four points $(\pm u : \pm v : \pm w)$ are the same. We may therefore speak of the *triad of Apollonian circles* associated with a harmonic quadruple $(\pm u : \pm v : \pm w)$.[3]

Since X, X' divide BC harmonically, so do B, C divide XX'. It follows that B and C are inversive with respect to $\mathcal{C}_\mathrm{a}$, and any circle through them, in particular the circumcircle of ABC, is orthogonal to $\mathcal{C}_\mathrm{a}$. Similarly, the circumcircle $\mathcal{C}$ is orthogonal to $\mathcal{C}_\mathrm{b}$ and $\mathcal{C}_\mathrm{c}$. Since the centers of $\mathcal{C}_\mathrm{a}$, $\mathcal{C}_\mathrm{b}$, $\mathcal{C}_\mathrm{c}$ are collinear, this triad of circles generates a pencil of circles orthogonal to $\mathcal{C}$. The radical axis contains the circumcenter O of triangle ABC, and is therefore the perpendicular from O to the line $\mathcal{L}$ of centers of the circles of the triad. The circle $\mathcal{C}_\mathrm{a}$ with diameter XX' is the locus of points M satisfying

$$|BM| : |CM| = |BX| : |CX| = |BX'| : |CX'| = |w| : |v| = \frac{1}{|v|} : \frac{1}{|w|}\,.$$

It follows that if the triad has a common point M, then

$$|AM| : |BM| : |CM| = \frac{1}{|u|} : \frac{1}{|v|} : \frac{1}{|w|}\,.$$

Let Q be the pole of the line $\mathcal{L}$ (of centers of the Apollonian circle) with respect to the circumcircle $\mathcal{C}$. Since the equation of the circumcircle $\mathcal{C}$ is

$$a^2yz + b^2zx + c^2xy = 0, \tag{1}$$

this means that[4]

$$\begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} Q = \begin{pmatrix} 1/u^2 \\ 1/v^2 \\ 1/w^2 \end{pmatrix},$$

and

$$Q = \left(a^2\left(\frac{b^2}{v^2} + \frac{c^2}{w^2} - \frac{a^2}{u^2}\right) : \ b^2\left(\frac{c^2}{w^2} + \frac{a^2}{u^2} - \frac{b^2}{v^2}\right) : \ c^2\left(\frac{a^2}{u^2} + \frac{b^2}{v^2} - \frac{c^2}{w^2}\right)\right). \tag{2}$$

[3] A harmonic quadruple is a quadruple of points which is invariant under harmonic homologies with any vertex as center and the corresponding opposite side as axis.

[4] Here is an abuse of notations in which we treat Q as a 3×1 matrix.

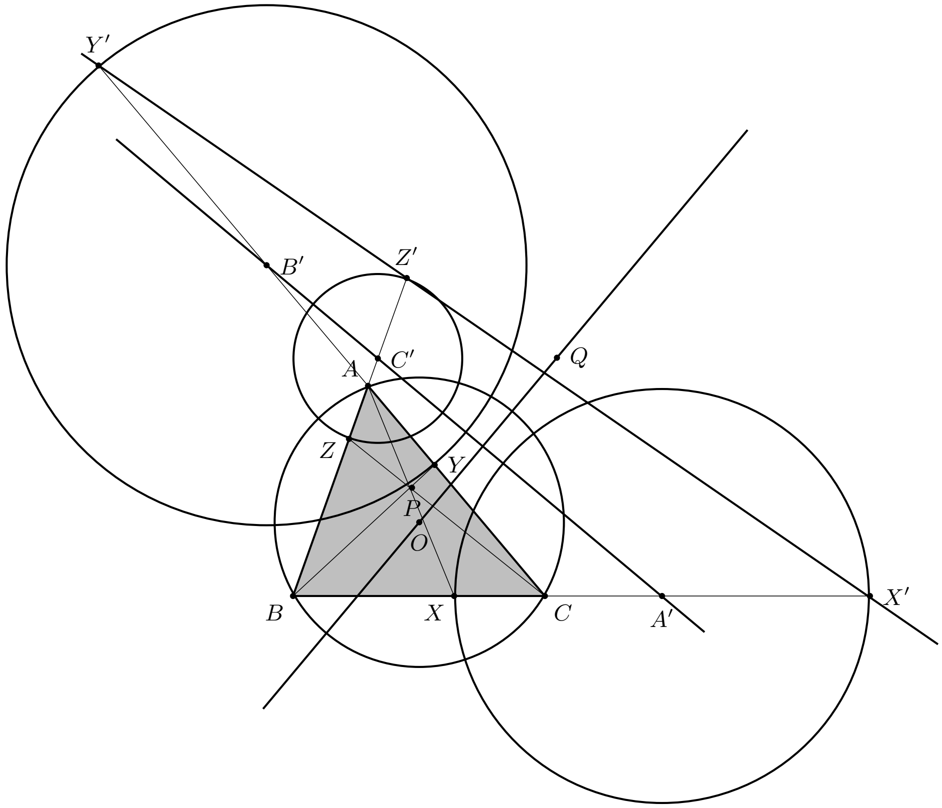

Figure 1: Apollonian circles associated with P

2. Theorems

Theorem 1. *The Apollonian circles of P intersect at real points if and only if the point Q lies inside or on the circumcircle.*

Proof. Since the radical axis contains the circumcenter O, the point Q is the inversive image of the intersection of the radical axis and $\mathcal{L}$. The Apollonian circles have real common points if and only if $\mathcal{L}$ and $\mathcal{C}$ do not, *i.e.*, Q lies inside or on the circumcircle. The circles in the triad are mutually tangent to each other if and only if Q lies on the circumcircle. In this case, the common point is Q. □

A point with coordinates $(x : y : z)$ lies inside or on the circumcircle $\mathcal{C}$ if and only if [5]

$$a^2yz + b^2zx + c^2xy \geq 0.$$

The Steiner circum-ellipse $\mathcal{E}$ (with center at the centroid G of triangle ABC) has the equation

$$yz + zx + xy = 0.$$

A point $(x : y : z)$ is inside or on $\mathcal{E}$ if and only if

$$yz + zx + xy \geq 0.$$

[5] For a finite point $P = (x : y : z)$, the *power* of P with respect to $\mathcal{C}$ is given by $\dfrac{a^2yz + b^2zx + c^2xy}{(x+y+z)^2}$. This is the product $XP \cdot PY$ of signed lengths for an arbitrary line through P intersecting the circumcircle at X and Y.

From the coordinates given in (2), it is clear that Q lies inside or on $\mathcal{C}$ if and only if the point

$$Q' = \left(\frac{b^2}{v^2}+\frac{c^2}{w^2}-\frac{a^2}{u^2} : \frac{c^2}{w^2}+\frac{a^2}{u^2}-\frac{b^2}{v^2} : \frac{a^2}{u^2}+\frac{b^2}{v^2}-\frac{c^2}{w^2}\right)$$

lies inside or on $\mathcal{E}$, *i.e.*,

$$\begin{aligned} 0 \le\ & \left(\frac{c^2}{w^2}+\frac{a^2}{u^2}-\frac{b^2}{v^2}\right)\left(\frac{a^2}{u^2}+\frac{b^2}{v^2}-\frac{c^2}{w^2}\right)+\left(\frac{a^2}{u^2}+\frac{b^2}{v^2}-\frac{c^2}{w^2}\right)\left(\frac{b^2}{v^2}+\frac{c^2}{w^2}-\frac{a^2}{u^2}\right)\\ & +\left(\frac{b^2}{v^2}+\frac{c^2}{w^2}-\frac{a^2}{u^2}\right)\left(\frac{c^2}{w^2}+\frac{a^2}{u^2}-\frac{b^2}{v^2}\right)\\ =\ & 2\left(\frac{b}{v}\right)^2\left(\frac{c}{w}\right)^2+2\left(\frac{c}{w}\right)^2\left(\frac{a}{u}\right)^2+2\left(\frac{a}{u}\right)^2\left(\frac{b}{v}\right)^2-\left(\frac{a}{u}\right)^4-\left(\frac{b}{v}\right)^4-\left(\frac{c}{w}\right)^4\\ =\ & \left(\frac{a}{u}+\frac{b}{v}+\frac{c}{w}\right)\left(-\frac{a}{u}+\frac{b}{v}+\frac{c}{w}\right)\left(\frac{a}{u}-\frac{b}{v}+\frac{c}{w}\right)\left(\frac{a}{u}+\frac{b}{v}-\frac{c}{w}\right). \end{aligned}$$

From this, the following characterization is evident for points with mutually tangent Apollonian circles.

Theorem 2. *The following statements are equivalent.*

(a) The Apollonian circles of P are tangent to each other.

(b) The point Q lies on the circumcircle.

(c) One of the points in the harmonic quadruple containing P lies on the circum-ellipse[6]

$$\mathcal{E}_{\rm m} : \qquad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0\,.$$

If these conditions are satisfied, Q is the point of tangency of the Apollonian circles.

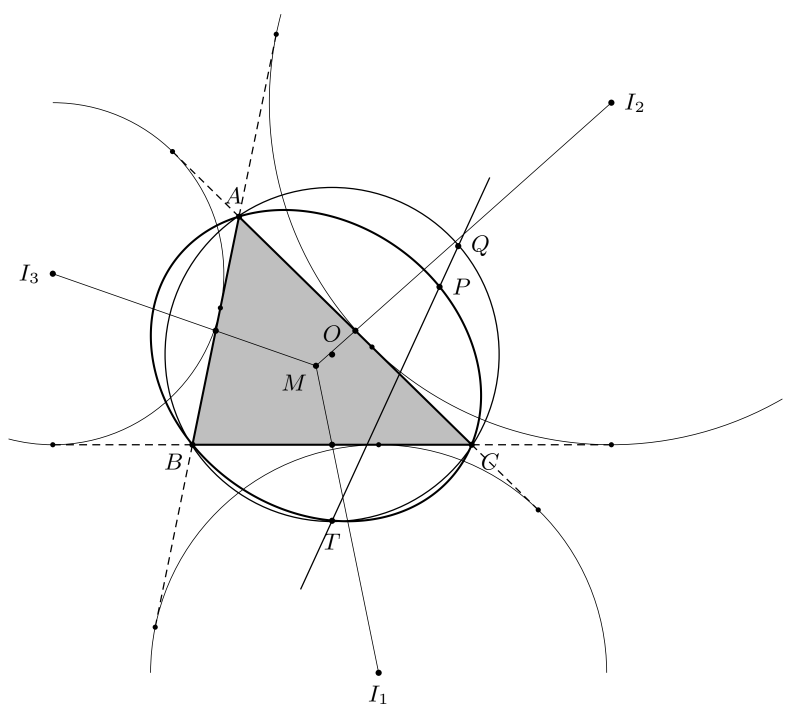

Figure 2: Circum-ellipse with center at the Mittenpunkt

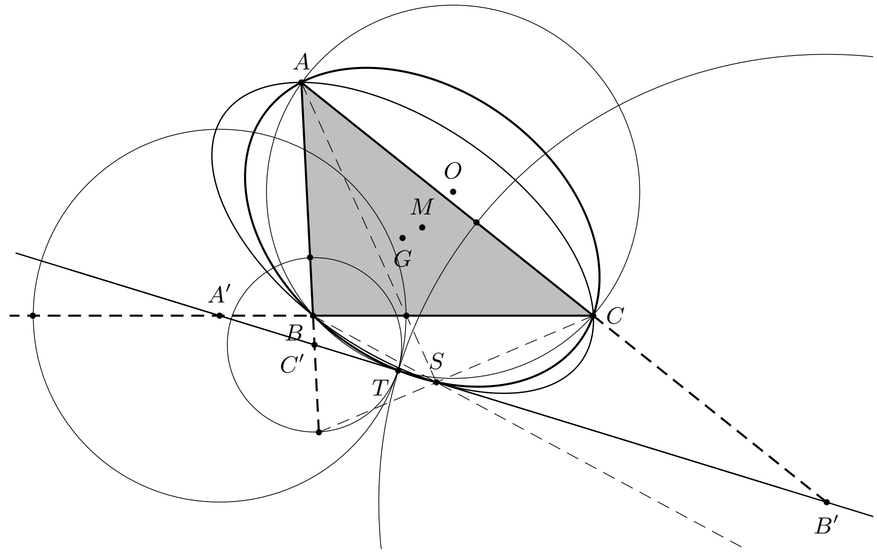

Figure 3: Apollonian circles of S mutually tangent at T

The circum-ellipse $\mathcal{E}_{\rm m}$ has its center at the Mittenpunkt with coordinates

$$\Big(a(b+c-a):\ b(c+a-b):\ c(a+b-c)\Big).$$[7]

This is the intersection of the three lines each joining an excenter to the midpoint of the corresponding side of ABC (see Fig. 2). The circum-ellipse $\mathcal{E}_{\rm m}$ intersects the circumcircle $\mathcal{C}$ at the point[8]

$$T=\left(\frac{a}{b-c}:\frac{b}{c-a}:\frac{c}{a-b}\right).$$

If a line through T intersects the circumcircle at Q and the ellipse $\mathcal{E}_{\rm m}$ at P, then the Apollonian circles associated with P are mutually tangent at Q. This leads to a remarkable animation picture showing a moving triad of circles orthogonal to the circumcircle, mutually tangent to each other at a point on $\mathcal{C}$. As an interesting special case, if P is the intersection of the ellipses $\mathcal{E}_{\rm m}$ and $\mathcal{E}$, namely, the point[9]

$$S=\left(\frac{1}{b-c}:\frac{1}{c-a}:\frac{1}{a-b}\right),$$

then the Apollonian circles are tangent to each other at T (see Fig. 3). Furthermore, the line of centers is tangent to $\mathcal{E}$ at S and contains T.[10]

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[6](c) is also equivalent to (c′): The point P lies on one of the circumconics $\frac{a}{x}\pm\frac{b}{y}\pm\frac{c}{z}=0$.

[7]This is the point X_9 in [3].

[8]This is X_{100} in [3]. It can be constructed as the point of tangency of the circumcircle and the incircle of the superior triangle, the one bounded by the parallels to the sides of ABC through the opposite vertices.

[9]This is X_{190} in [3].

[10]This is the line $(b-c)^2x+(c-a)^2y+(a-b)^2z=0$.

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