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The Monge Point and the 3(n+1) Point Sphere of an n-Simplex

Małgorzata Buba-Brzozowa

Department of Mathematics and Information Sciences, Warsaw University of Technology
Pl. Politechniki 1, PL 00-661 Warsaw, Poland
email: buba@mini.pw.edu.pl

Abstract. The hyperplanes through the centroids of the $(n-2)$-dimensional faces of an n-simplex and perpendicular to the respectively opposite 1-dimensional edges have a point in common. As a consequence, we define an analogue of the nine-point circle for any n-simplex.

Key words: nine-point circle, Feuerbach circle, Monge point, Euler line, $3(n+1)$ point sphere, n-simplex, Menelaus theorem, Manheim theorem, Stewart theorem.

MSC 2000: 51M04

1. Introduction

The theorem about the nine-point circle (also called the Feuerbach circle) can be generalized to the n-dimensional Euclidean space E^n. In [2] some properties connected with this theorem and the Euler line of orthocentric n-simplexes are given. Here we prove some of these theorems for the general case. For our consideration we will use the equation fulfilled on the Euler line of an orthocentric n-simplex Θ_{ort}, namely $H^nG^n : G^nO^n = 2 : (n-1)$, $n \geq 2$, where H^n is the orthocenter, G^n the centroid and O^n the circumcenter of Θ_{ort}.

2. Quasi-medians and medians

Let Θ be a given, nondegenerate n-simplex, with the vertices $A_1, \ldots, A_{n+1}$ in E^n. For $i = 1$ to $n+1$ the symbol Θ_i denotes the hyperplane $A_1 \ldots A_{i-1}A_{i+1} \ldots A_{n+1}$ and, at the same time, the corresponding $(n-1)$-dimensional face of Θ (which is opposite to the vertex A_i). Let G^n be the centroid of Θ and G_i^{n-1} be the centroid of the face Θ_i, $i = 1, \ldots, n+1$.

Definition 1 A line joining the centroid of any $(n-2)$-dimensional face of an n-simplex Θ with the midpoint of the opposite 1-dimensional edge of Θ will be called a *quasi-median* of the n-simplex Θ. An n-simplex has $\binom{n+1}{2}$ quasi-medians (a tetrahedron has only three quasi-medians because the opposite edges of it are both 1-dimensional).

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For our next consideration we show the following

Lemma 1 *The quasi-medians of Θ are concurrent at the centroid of Θ and intersect each other at the ratio* $2:(n-1)$.

Proof: Without loss of generality, let us examine the face Θ_n and Θ_{n+1} of Θ. Let $G^{n-2}_{1..n-1}$ be the centroid of the $(n-2)$-dimensional subspace $A_1 \dots A_n$ of Θ_{n+1}. The median of Θ_{n+1} passing through this point is $A_n G^{n-2}_{1..n-1}$. Obviously, the centroid G^{n-1}_{n+1} of Θ_{n+1} belongs to this median. The median of Θ from the vertex A_{n+1} meets Θ_{n+1} at the point G^{n-1}_{n+1} and G^n belongs to $A_{n+1}G^{n-1}_{n+1}$. Similarly, for the face Θ_n the line $A_{n+1}G^{n-2}_{1..n-1}$ is the median of Θ_n and the centroid G^{n-1}_n of Θ_n belongs to $A_{n+1}G^{n-1}_{1..n}$. But the median of Θ from the vertex A_n meets Θ_n in G^{n-1}_n and G^n belongs to this line (Fig. 1). Let $B_{nn+1} = G^n G^{n-2}_{1..n-1} \cap A_n A_{n+1}$.

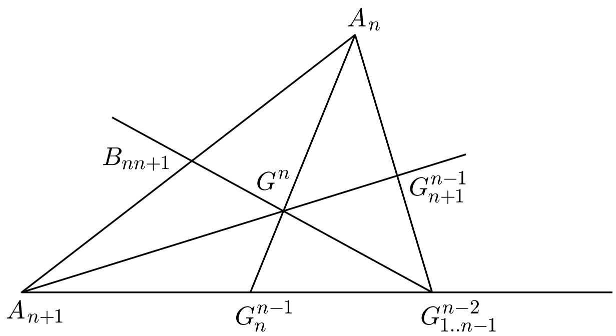

Figure 1: $B_{nn+1}G^{n-2}_{1..n-1}$ is a quasi-median

Thus, using the Area Principle (see [4]), we get

$$B_{nn+1}G^n : B_{nn+1}G^{n-2}_{1..n-1} = B_{nn+1}G^n : (B_{nn+1}G^n + G^n G^{n-2}_{1..n-1}) = (n-1):(n+1)$$

and

$$A_n B_{nn+1} : B_{nn+1} A_{n+1} = 1.$$

Thus

$$G^n G^{n-2}_{1..n-1} : B_{nn+1}G^n = 2:(n-1).$$

Therefore the line $B_{nn+1}G^{n-2}_{1..n-1}$ must be the quasi-median joining the centroid of the subspace $A_1 \dots A_{n-1}$ and the mid-point of $A_n A_{n+1}$. Conducting the same observations for all faces of Θ we get the proposition. $\square$

We conclude this section by noting a simple lemma (the proof is by induction applying the Stewart theorem).

Lemma 2 *Let a_{ij} denote the length of the* 1*-dimensional edge of Θ joining the vertices A_i and A_j, and m_i the length of the median of Θ from the vertex A_i $(i, j = 1, \dots, n+1,\ i \neq j)$. Then*

$$(m_k)^2 = \frac{1}{n} \sum_{\substack{i=1,\dots,n;\\ i \neq k}} (a_{ik})^2 - \frac{1}{n^2} \sum_{\substack{i=1,\dots,n;\\ j=i+1;\ i,j\neq k}} (a_{ij})^2 \quad \textit{with} \;\; n+2 \equiv 1 \pmod{n+1}$$

where $k = 1, \dots, n+1$.

3. The Monge point and the Euler line of an n-simplex

Definition 2 A hyperplane through the mid-point of a given segment and perpendicular to that segment will be called the *mediator* of the segment.

Now we may prove the generalization of the Monge theorem for the n-dimensional space E^n (for the 3-dimensional case see [1] and [3]).

Theorem 3 *The hyperplanes through the centroids of the* $(n-2)$*-dimensional faces of an* n*-simplex* Θ *and perpendicular to the* 1*-dimensional edges respectively opposite have a point in common.*

Proof: Let α_{nn+1} be the mediator of the segment A_nA_{n+1} and α'_{nn+1} be the hyperplane through the centroid $G^{n-2}_{1..n}$ of the $(n-2)$-dimensional face $A_1 \dots A_{n-1}$ perpendicular to A_nA_{n+1} (i.e. parallel to α_{nn+1}). The quasi-median between A_nA_{n+1} and $G^{n-2}_{1..n}$ (containing G^n) joins two points in these two hyperplanes, and these two hyperplanes are parallel, hence the segment intercepted by these hyperplanes on any line passing through the centroid G^n will be intersected by G^n at the ratio $2:(n-1)$. Now the hyperplane α_{nn+1} contains the circumcenter O^n of Θ, hence α'_{nn+1} meets the line O^nG^n in a point M^n such that

$$G^nM^n : G^nO^n = 2 : (n-1).$$

But the points O^n and G^n are independent of the particular edge A_iA_j considered, hence the proposition holds true. □

Definition 3 The hyperplanes perpendicular to the 1-dimensional edges of an n-simplex Θ and passing through the centroids of the respective opposite $(n-2)$-dimensional faces are called the *Monge hyperplanes* of Θ. The point common to these hyperplanes is the *Monge point* of the n-simplex Θ.

Remark In the Euclidean plane E^2 this theorem is also true: the Monge point is the orthocenter and the quasi-medians are the medians of a triangle.

Definition 4 The line joining the centroid and the circumcenter of an n-simplex (and containing the Monge point) will be referred to as the *Euler line* of the n-simplex.

An immediate consequence from the above and [2] is

Corollary 4 *In an orthocentric* n*-simplex* Θ *the Monge point coincides with the orthocenter of* Θ.

Corollary 5 *The Monge point of* Θ *belongs to the Euler line and*

$$M^nG^n : G^nO^n = 2 : (n-1), \quad n \geq 2.$$

The second theorem related to the Monge point of the n-simplex is the Mannheim theorem (see also [3]):

Theorem 6 *The* 2*-dimensional planes determined by the* $n+1$ *altitudes of an* n*-simplex* Θ *and the Monge points of the corresponding faces pass through the Monge point of* Θ.

Proof: We consider the face Θ_{n+1} and the vertex A_{n+1} of Θ. Let H_{n+1} be the foot of the altitude from A_{n+1} and G' the projection of the centroid G^n of Θ upon Θ_{n+1}. Then G' belongs to $H_{n+1}G^{n-1}_{n+1}$. Using the Area Principle we have $H_{n+1}G' : H_{n+1}G^{n-1}_{n+1} = n : (n+1)$. Let O^{n-1}_{n+1} be the circumcenter and M^{n-1}_{n+1} the Monge point of the face Θ_{n+1}. Therefore, using Corollary 2 and some properties of the Euler line (see [2]) we get (Fig. 2)

$$M^{n-1}_{n+1}G^{n-1}_{n+1} : G^{n-1}_{n+1}O^{n-1}_{n+1} = 2 : (n-2) \quad \text{and} \quad M^{n-1}_{n+1}G^{n-1}_{n+1} : M^{n-1}_{n+1}O^{n-1}_{n+1} = 2 : n.$$

The lines $M^{n-1}_{n+1}O^{n-1}_{n+1}G^{n-1}_{n+1}$ and $H_{n+1}G'$ intersect in G^{n-1}_{n+1}; so there exists a common point of $M^{n-1}_{n+1}H_{n+1}$ and $G'O^{n-1}_{n+1}$, say C. The Menelaus theorem for the triangle $G'G^{n-1}_{n+1}O^{n-1}_{n+1}$ and the line $H_{n+1}CM^{n-1}_{n+1}$ states that

$$(H_{n+1}G' : H_{n+1}G^{n-1}_{n+1}) \cdot (M^{n-1}_{n+1}G^{n-1}_{n+1} : M^{n-1}_{n+1}O^{n-1}_{n+1}) \cdot (O^{n-1}_{n+1}C : CG') = 1.$$

Hence $CG' : O^{n-1}_{n+1}C = 2 : (n+1)$. The point O^{n-1}_{n+1} is the projection of the circumcenter O^n and

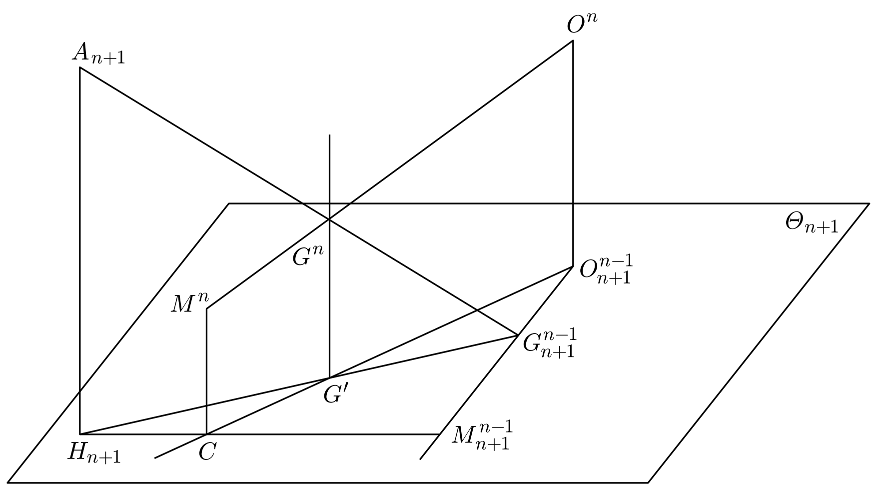

Figure 2: The 2-plane $A_{n+1}H_{n+1}M^{n-1}_{n+1}$ passes through the Monge point M^n

G' the projection of the centroid upon Θ_{n+1}, hence the perpendicular at C to Θ_{n+1} will meet the Euler line of Θ in a point M^n such that $M^nG^n : M^nO^n = 2 : (n+1)$. Therefore M^n must be the Monge point of the n-simplex Θ. But CM^n lies in the plane $A_{n+1}H_{n+1}M^{n-1}_{n+1}$ ($A_{n+1}H_{n+1}$ and CM^n are perpendicular to Θ_{n+1} and there exists only one direction perpendicular to the hyperplane so they are parallel), hence this plane passes through M^n, the Monge point of the n-simplex. Conducting the same observations for the others points A_i ($i = 1, \ldots, n$) we get our result. □

4. The 3(n+1) point sphere related to an n-simplex

Now we introduce the analogue of the nine-point circle for any n-simplex (see [1] and [3] for a tetrahedron and [2] for an orthocentric n-simplex): Let S' be the sphere determined by the $n+1$ centroids of the faces of Θ. From [2] we have that S' is the homothetic of the circumsphere S of Θ with respect to the centroid G^n of Θ, the homothetic ratio being $-1 : n$. Hence the center O' of S' lies on the Euler line G^nO^n of Θ and $G^nO' : G^nO^n = 1 : n$. S' is called the $3(n+1)$ *point sphere*.

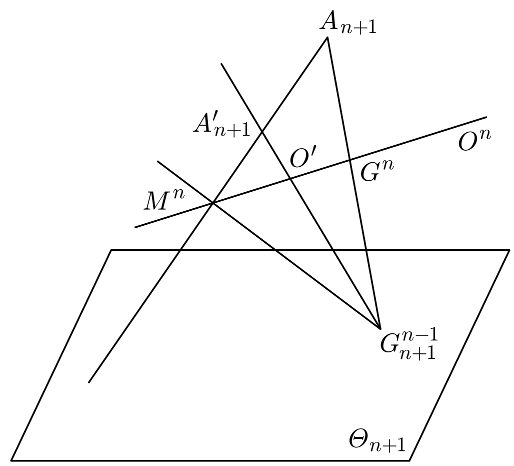

Figure 3: The $3(n+1)$ point sphere with center O' passes through A'_{n+1}

We consider the face Θ_{n+1} of Θ (Fig. 3).$M^nO'G^nO^n$ is the Euler line and $A_{n+1}G_{n+1}^{n-1}$ the median of Θ. Therefore $G^nO' : O'M^n = (n-1) : (n+1)$ and $A_{n+1}G_{n+1}^{n-1} : G_{n+1}^{n-1}G^n = (n+1) : 1$ (see [2]). Let A'_{n+1} be the common point of the lines $A_{n+1}M^n$ and $G_{n+1}^{n-1}O'$. Applying Menelaus' theorem to the triangle $M^nA_{n+1}G^n$ and the transversal $A'_{n+1}O'$ we have

$$(M^nA'_{n+1} : A'_{n+1}A_{n+1}) \cdot (G^nO' : O'^nM^n) \cdot (A_{n+1}G_{n+1}^{n-1} : G_{n+1}^{n-1}G^n) = 1.$$

Hence

$$M^nA'_{n+1} : A'_{n+1}A_{n+1} = 1 : (n-1)$$

and

$$M^nA'_{n+1} : M^nA_{n+1} = M^nA'_{n+1} : (M^nA'_{n+1} + A'_{n+1}A_{n+1}) = 1 : n.$$

On the other hand, by applying Menelaus' theorem to the triangle $A_{n+1}A'_{n+1}G_{n+1}^{n-1}$ and the transversal $M^nO'G^n$ we obtain

$$(A_{n+1}G^n : G^nG_{n+1}^{n-1}) \cdot (M^nA'_{n+1} : A'_{n+1}A_{n+1}) \cdot (O'G_{n+1}^{n-1} : A'_{n+1}O') = 1.$$

And finally

$$A'_{n+1}O' : O'G_{n+1}^{n-1} = 1.$$

Thus the sphere S' passes through the point A'_{n+1} such that $M^nA'_{n+1} : M^nA_{n+1} = 1 : n$, and through the n analogous points A'_i, respectively relative to the vertices A_i, $i = 1, \dots, n$. Moreover, the point A'_i and the centroid G_i^{n-1} of the face Θ_i are diametrically opposite on the sphere S' ($i = 1, \dots, n+1$). Therefore S' passes through the projection A''_i of A'_i upon Θ_i, $i = 1, \dots, n+1$. Thus we have $3(n+1)$ points on S', which justifies the name given to this sphere.

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