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Equifaciality of Tetrahedra whose Incenter and Fermat-Torricelli Center Coincide

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Abstract. In this note, we show that if the incenter and the Fermat-Torricelli center of a tetrahedron coincide, then the tetrahedron is equifacial (or isosceles) in the sense that all its faces are congruent. The proof is intended to replace the incorrect proof given in [8] for the same statement.

Key Words: barycentric coordinates, Fermat-Torricelli center, isosceles tetrahedron, equifacial tetrahedron

MSC 2000: 51M04, 51M20, 52B10

1. Introduction

The *centroid*, the *incenter*, and the *circumcenter* of a non-planar tetrahedron, and in fact of any non-degenerate n-simplex, are defined, in analogy with those of a triangle, as the point of intersection of the medians, the center of the insphere, and the center of the circumsphere, respectively. The *orthocenter*, however, does not exist for all tetrahedra, since the altitudes of an arbitrary tetrahedron are not necessarily concurrent.

The *Fermat-Torricelli center* of a non-degenerate n-simplex $S = A_0A_1 \ldots A_n$ is defined as the point whose distances from the vertices have a minimal sum. It is known [13, Theorem 1.1] that if the norm of the sum of the $n-1$ unit vectors from one of the vertices A_i to the remaining vertices does not exceed 1, then the Fermat-Torricelli center F of S coincides with that vertex. Otherwise, F is interior and is characterized by the property that the sum of the n unit vectors from F to the vertices is 0, i.e., by

$$\sum_{i=0}^{n} \frac{PA_i}{\|PA_i\|} \quad = \quad 0. \tag{1}$$

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It follows that the Fermat-Torricelli center F of a triangle ABC is interior if and only if the measure of each angle of ABC is less than $2\pi/3$, in which case F is is characterized by the property that

$$\angle BPC = \angle CPA = \angle APB = 2\pi/3\,.$$

Similarly, the Fermat-Torricelli center F of a tetrahedron $ABCD$ is interior if and only if the measure of each solid vertex angle of $ABCD$ is less than π, where the measure $|\langle P; X, Y, Z\rangle|$ of a solid angle $\langle P; X, Y, Z\rangle$ with vertex P and arms PX, PY, and PZ is the area on the surface of the unit sphere centered at P of the spherical triangle enclosed by PX, PY, and PZ. In this case, F is characterized by the property that the solid angles $\langle F; A, B, C\rangle$, $\langle F; B, C, D\rangle$, $\langle F; C, D, A\rangle$, and $\langle F; D, A, B\rangle$ have equal measures, which in turn is equivalent to the conditions that

$$\angle AFB = \angle CFD, \quad \angle AFC = \angle BFD, \quad \angle AFD = \angle BPC\,; \tag{2}$$

see [1]. Rewriting (1) in the form

$$\frac{FA}{\|FA\|} + \frac{FB}{\|FB\|} + \frac{FC}{\|FC\|} + \frac{FD}{\|FD\|} = 0\,,$$

taking the norms of both sides, and using (2), we obtain

$$\cos\angle AFB + \cos\angle BFC + \cos\angle CFA = -1\,. \tag{3}$$

It is easy to see that if any two of the above-mentioned five triangle centers (namely, the centroid, the incenter, the circumcenter, the orthocenter, and the Fermat-Torricelli center) coincide for a certain triangle, then it is equilateral; see [15, exercise 1, p. 37], [14, pp. 78–79], and [7]. Theorem 4 of [8] states that if any two of the four relevant tetrahedron centers coincide for a certain tetrahedron, then it is equifacial (or isosceles, in the terminology of [2]), i.e., has congruent faces. Of the six statements that constitute that theorem, the one pertaining to the incenter and the Fermat-Torricelli center is the most intricate. Regrettably, the proof given in [8] of that statement is incorrect. More specifically, the conclusion made in the last two lines of [8, (FI)] regarding the congruence of the tetrahedra $OABC$ and $OBAD$ cannot be justified. The theorem below provides a correct version, and the note that follows elaborates on why the proof in [8] is faulty.

We remind the reader that equifaciality of a tetrahedron $ABCD$ is equivalent to the seemingly weaker condition that the faces have equal areas; see [11, Chapter 9, pages 90–97] and [6]. Also, it is trivially equivalent to the requirements that

$$AB = CD, \quad AC = BD, \quad AD = BC. \tag{4}$$

2. The Main Theorem

Theorem. *Let $T = ABCD$ be a non-planar tetrahedron. If the incenter I and the Fermat-Torricelli center F of T coincide, then T is equifacial.*

Proof: We think of A, B, C, and D as position vectors in some coordinate system whose origin O coincides with F and I. Let

$$a = \|A\|, \quad b = \|B\|, \quad c = \|C\|, \quad d = \|D\| \tag{5}$$

and let x, y, and z be defined by

$$x = \cos\angle BOC = \cos\angle DOA, \quad y = \cos\angle COA = \cos\angle DOB, \quad z = \cos\angle AOB = \cos\angle DOC.$$

Then (3) can be rewritten as

$$x + y + z + 1 = 0. \tag{6}$$

Let the areas of the faces BCD, CDA, DAB, and ABC be denoted by α, β, γ, and δ, respectively. Since O is the incenter, it follows that the tetrahedra $OBCD$, $OCDA$, $ODAB$ and $OABC$ have the same altitude at O. Therefore the barycentric coordinates of O with respect to $ABCD$, being proportional to the volumes of these tetrahedra, are proportional to the base areas α, β, γ, and δ. On the other hand, they are proportional to a^{-1}, b^{-1}, c^{-1}, and d^{-1}, since

$$\frac{A}{a} + \frac{B}{b} + \frac{C}{c} + \frac{D}{d} = O \tag{7}$$

by (1). Therefore $(a^{-1}, b^{-1}, c^{-1}, d^{-1})$ and $(\alpha, \beta, \gamma, \delta)$ are proportional, and hence

$$a\alpha = b\beta = c\gamma = d\delta\,. \tag{8}$$

It remains to compute α, β, γ, and δ in terms of a, b, c, and d, and solve the resulting equations.

If u, v, and w denote the side-lengths of ABC, then the Law of Cosines gives

$$u^2 = a^2 + b^2 - 2abz, \quad v^2 = b^2 + c^2 - 2bcx, \quad w^2 = c^2 + a^2 - 2cay.$$

Substituting these in Heron's Formula

$$16\delta^2 = 2(u^2v^2 + v^2w^2 + w^2u^2) - (u^4 + v^4 + w^4),$$

we obtain (after laborious calculations verified by Maple) that

$$\begin{aligned} 4\delta^2 &= (a^2b^2 + b^2c^2 + c^2a^2) - (a^2b^2z^2 + b^2c^2x^2 + c^2a^2y^2) \\ &\quad +2abc(ayz + bxz + cxy - ax - by - cz). \end{aligned}$$

Similarly, we see that

$$\begin{aligned} 4\gamma^2 &= (a^2b^2 + b^2d^2 + d^2a^2) - (a^2b^2z^2 + b^2d^2y^2 + d^2a^2x^2) \\ &\quad +2abd(byz + axz + dxy - ay - bx - dz). \end{aligned}$$

Therefore

$$4d^2\delta^2 - 4c^2\gamma^2 = (z+1)(bcd + cda + dab + abc)H, \tag{9}$$

where

$$H = \frac{-x+y}{a} + \frac{x-y}{b} + \frac{-x-y-2}{c} + \frac{x+y+2}{d}\,, \tag{10}$$

and where we have used the relation (6). Note that if $z+1=0$, then $\angle AOB = \angle COD = 180°$, and $ABCD$ would be planar. Also $bcd + cda + dab + abc > 0$. Thus $d\delta = c\gamma$ is equivalent to $H = 0$. It follows that the system (8) is equivalent to the matrix equation $QX = O$, where

$$Q = \begin{bmatrix} -x+y & x-y & -x-y-2 & x+y+2 \\ x-1 & -x-2y-1 & x+2y+1 & -x+1 \\ -2x-y-1 & y-1 & 2x+y+1 & -y+1 \end{bmatrix}, \quad X = \begin{bmatrix} a^{-1} \\ b^{-1} \\ c^{-1} \\ d^{-1} \end{bmatrix}.$$

The leftmost 3×3 submatrix of Q has determinant $-4(x+1)(y+1)(z+1) \neq 0$ and therefore the rank of Q is 3. Therefore the solution space of the system $QX = 0$ has dimension 1 and is necessarily generated by the obvious solution

$$(a^{-1},\ b^{-1},\ c^{-1},\ d^{-1}) = (1,\ 1,\ 1,\ 1).$$

Thus $a = b = c = d$. Therefore O is the centroid (by (7)) (and the circumcenter by (5)). It follows now from (the valid parts of) [8, Theorem 4] that T is equifacial, as desired. Alternatively, one may use [12, Theorem 4.10], or directly prove (4) using $a = b = c = d$ and the law of cosines. □

Note 1. The proof in [8] claims in effect that the single equation $d\delta = c\gamma$ implies that $c = d$. We have just seen that $d\delta = c\gamma$ is equivalent to

$$\frac{-x+y}{a} + \frac{x-y}{b} + \frac{-x-y-2}{c} + \frac{x+y+2}{d} = 0,$$

which would imply $c = d$ if and only if $a = b$.

Note 2. Theorem 4 of [8], now correct by the Theorem above, considers the centroid, the incenter, the circumcenter, and the Fermat-Torricelli center of a tetrahedron and proves that the coincidence of any two of these centers implies equifaciality. It is easy to see that the Monge point can be added to this list [4, Theorem 2.1], and it would be interesting to explore whether other centers, such as the 1- and 2-centroids [3] and the Gergonne and Nagel points [10], can be added too. It is also worth mentioning that the aforementioned theorem [8, Theorem 4] does not admit an exact generalization to higher dimensional simplices; see [9] and [4]. However, if one restricts oneself to orthocentric simplices, i.e., simplices whose altitudes concur, then one gets the pleasant theorem that if any two of the centroid, incenter, circumcenter, and orthocenter of an orthocentric simplex S coincide then S is regular; see [5].

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