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Symmetric alteration of two knots of B-spline curves

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Abstract. B-spline curves are defined in a piecewise way over a closed interval, where the interval section points are called knots. In some recent publications geometrical properties of the modification of one knot value are discussed. The aim of this paper is to describe an effect of the symmetric modification of two knots.

Key Words: B-spline curve, knot modification, path

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1. Introduction

B-spline curves are well-known tools in geometric modeling and their properties and capabilities are always of importance in computer aided geometric design. They are polynomial curves defined as linear combination of the control points by some polynomial functions called basis functions. These basis functions are defined in a piecewise way over a closed interval and the subdivision values of this interval are called knots. These knots as well as their modification are in the focus of this paper. The basic definitions of the basis functions and the curve are the following.

Definition 1 *The recursive function $N_j^k(u)$ given by the equations*

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j,\, u_{j+1}), \\ 0 & \text{otherwise,} \end{cases}$$
$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u)$$

is called normalized B-spline basis function of order k (degree $k-1$). The numbers $u_j \leq u_{j+1} \in \mathbb{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

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Definition 2 *The curve* $\mathbf{s}(u)$ *defined by*

$$\mathbf{s}(u) = \sum_{l=0}^{n} \mathbf{d}_l \, N_l^k(u), \quad u \in [u_{k-1}, \, u_{n+1}]$$

is called B-spline curve of order $k \leq n$ *(degree* $k-1$*), where* $N_l^k(u)$ *is the* l^{th} *normalized B-spline basis function, for the evaluation of which the knots* $u_0, u_1, \ldots, u_{n+k}$ *are necessary. The points* $\mathbf{d}_l$ *are called control points or de Boor-points, while the polygon formed by these points is called control polygon.*

The j^{th} arc can be written as

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l \, N_l^k(u), \quad u \in [u_j, \, u_{j+1}) \, .$$

As one can observe, the data structure of these polynomial curves include their order, control points and knot values. Obviously, any modification of these data yields a change of the shape of the curve. This effect is widely studied in case of control point repositioning (cf. [2], [8] or [9]). In some recent publications [3, 5, 6, 7] the authors studied the effect of the alteration of a single knot on the shape of the curve. When a knot u_i is altered, the points of the curve move on special curves $\mathbf{s}\,(u, u_i)$ called *paths*. In [5] the authors proved that these paths are rational curves. Symmetric alteration of two knots can also be studied.

Definition 3 *By a symmetric alteration of knots* u_i *and* u_j*,* $(i < j)$ *we mean the* $u_i + \lambda$*,* $u_j - \lambda$*,* $\lambda \in \mathbb{R}$*, type modification. In order to preserve the monotony of knot values* λ *can not take any value but it has to be within the range* $[-c, \, c]$ *for*

$$c = \min\{u_i - u_{i-1}, \; u_{i+1} - u_i, \; u_j - u_{j-1}, \; u_{j+1} - u_j\}.$$

In [4] a very special case of simultaneous modification of two knots has already been discussed. This type of modification also yields rational curves as paths of points of the B-spline curve.

Paths obtained by the modification of one or more knots are relatively short arcs due to the limited domain of modification. In order to get more information about their characteristics we can extend their domain, i.e., modifying the knot u_i we can allow $u_i < u_{i-1}$ and $u_i > u_{i+1}$. More precisely, let us consider the path

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l \, N_l^k(u, \, u_i),$$

where u is a fixed parameter value, and u_i is the variable. As one can observe in Definition 1, the functions $N_l^k(u, \, u_i)$ can be computed even for values u_i outside of $[u_{i-1}, \, u_{i+1}]$, however this is originally not allowed when points of B-spline curves are computed. This mathematical extension yields a geometrical extension of the path $\mathbf{s}(u, \, u_i)$. In this way one can find limit points of the paths as well, computing the mathematical limit of the functions $N_l^k(u, \, u_i)$, $u_i \to \pm\infty$. We have to emphasize, that the points of these extended arcs of the path do not belong to any B-spline curve defined by the original data, but help us to describe the characteristics of the restricted part of the path.

For these extended paths the following statement has been verified in [3]:

Theorem 1 *Modifying the single multiplicity knot u_i of the B-spline curve $\mathbf{s}(u)$, points of the extended paths of the arcs $\mathbf{s}_{i-1}(u)$ and $\mathbf{s}_i(u)$ tend to the control points $\mathbf{d}_i$ and $\mathbf{d}_{i-k}$ as u_i tends to $-\infty$ and ∞, respectively, i.e.,*

$$\lim_{u_i \to -\infty} \mathbf{s}(u, u_i) = \mathbf{d}_i, \quad \lim_{u_i \to \infty} \mathbf{s}(u, u_i) = \mathbf{d}_{i-k}, \quad u \in [u_{i-1}, u_{i+1}).$$

The purpose of the present paper is to prove a generalization of Theorem 1 for the description of the effects of the modification of two knots.

2. Symmetric alteration of two knots

Here we prove a general theorem about the extended paths obtained by the symmetric modification of two knots.

Theorem 2 *When symmetrically altering the knots u_i and u_{i+z} ($z \in \{1, 2, \ldots, k\}$, where k is the order of the original B-spline curve), the extended paths of points of the arcs $\mathbf{s}_j$, ($j = i, i+1, \ldots, i+z-1$), converge to the midpoint of the segment bounded by the control points $\mathbf{d}_i$ and $\mathbf{d}_{i+z-k}$ when $\lambda \to -\infty$, i.e.,*

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, \lambda) = \frac{1}{2}(\mathbf{d}_i + \mathbf{d}_{i+z-k}), \quad u \in [u_i, u_{i+z}). \tag{1}$$

Proof: We will prove that for $z = 1, 2, \ldots, k-1$

$$\lim_{\lambda \to -\infty} N_{i+z-k}^{k}(u, \lambda) = \lim_{\lambda \to -\infty} N_i^k(u, \lambda) = 1/2$$

and

$$\lim_{\lambda \to -\infty} N_j^k(u, \lambda) = 0, \quad (j \neq i, i+z-k), \quad u \in [u_i, u_{i+z}),$$

while for $z = k$

$$\lim_{\lambda \to -\infty} N_i^k(u, \lambda) = 1$$

and

$$\lim_{\lambda \to -\infty} N_j^k(u, \lambda) = 0, \quad (j \neq i), \quad u \in [u_i, u_{i+z}).$$

We prove these equalities by induction on k.

i) $k = 3$: We have to prove that modifying u_i and u_{i+z} for $z = 1, 2$

$$\lim_{\lambda \to -\infty} N_{i+z-3}^{3}(u, \lambda) = \lim_{\lambda \to -\infty} N_i^3(u, \lambda) = 1/2$$

and

$$\lim_{\lambda \to -\infty} N_j^3(u, \lambda) = 0, \quad (j \neq i, i+z-3), \quad u \in [u_i, u_{i+z})$$

while for $z = 3$

$$\lim_{\lambda \to -\infty} N_i^3(u, \lambda) = 1$$

and

$$\lim_{\lambda \to -\infty} N_j^3(u, \lambda) = 0, \quad (j \neq i), \quad u \in [u_i, u_{i+3}).$$

On the interval $[u_i,\, u_{i+1})$ the original basis function $N_i^3(u)$ is of the form

$$N_i^3(u) = \frac{u-u_i}{u_{i+2}-u_i}\,\frac{u-u_i}{u_{i+1}-u_i}\,.$$

Substituting $\hat{u}_i = u_i + \lambda$, $\hat{u}_{i+z} = u_{i+z} - \lambda$ and $\hat{u}_j = u_j$, $j \neq i, i+z$, we obtain three different forms of $N_i^3(u,\,\lambda)$ for $z = 1, 2, 3$, but for all cases the numerator as well as the denominator are second degree polynomials in λ. In the numerator the coefficient of λ^2 is 1 for all z, while in the denominator this coefficient is equal to 2 for $z = 1, 2$ (due to $\hat{u}_{i+1} - \hat{u}_i = u_{i+1} - u_i - 2\lambda$ for $z = 1$ and $\hat{u}_{i+2} - \hat{u}_i = u_{i+2} - u_i - 2\lambda$ for $z = 2$), while for $z = 3$ the coefficient equals 1. This yields

$$\lim_{\lambda\to-\infty} N_i^3(u,\,\lambda) = \begin{cases} \frac{1}{2}\,, \text{ if } z = 1, 2 \\ 1\,, \text{ if } z = 3, \end{cases} \qquad u \in [u_i,\, u_{i+1}).$$

On the interval $[u_i,\, u_{i+1})$ the remaining nonzero basis functions are of the following original forms

$$\begin{aligned} N_{i-2}^3(u) &= \frac{u_{i+1}-u}{u_{i+1}-u_{i-1}}\,\frac{u_{i+1}-u}{u_{i+1}-u_i}\,, \\ N_{i-1}^3(u) &= \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\,\frac{u_{i+1}-u}{u_{i+1}-u_i} + \frac{u_{i+2}-u}{u_{i+2}-u_i}\,\frac{u-u_i}{u_{i+1}-u_i}. \end{aligned}$$

Similarly to the case $N_i^3(u)$ it is easy to calculate that after the substitution $\hat{u}_i = u_i + \lambda$, $\hat{u}_{i+z} = u_{i+z} - \lambda$, and $\hat{u}_j = u_j$, $j \neq i, i+z$, the limit of these functions for $\lambda \to -\infty$ is $1/2$ and 0 for $z = 1$, while 0 and $1/2$ for $z = 2$, respectively. For $z = 3$ the limit of both functions is 0. For the rest of the indices (i.e., $j \neq i-2, i-1, i$) $N_j^3(u) \equiv 0$ holds for any $u \in [u_i,\, u_{i+1})$.

On the intervals $[u_{i+1},\, u_{i+2})$ and $[u_{i+2},\, u_{i+3})$ a similar calculation proves the statement for $k = 3$.

ii) Now we assume, that for $u \in [u_i,\, u_{i+z})$ the following equalities hold:

$$\lim_{\lambda\to-\infty} N_i^{k-1}(u,\,\lambda) = \begin{cases} \frac{1}{2}\,, \text{ if } z \in \{1, \ldots, k-2\} \\ 1\,, \text{ if } z = k-1 \end{cases} \tag{2}$$

$$\lim_{\lambda\to-\infty} N_{i+z-k+1}^{k-1}(u,\,\lambda) = \begin{cases} \frac{1}{2}\,, \text{ if } z \in \{1, \ldots, k-2\} \\ 1\,, \text{ if } z = k-1 \end{cases} \tag{3}$$

$$\lim_{\lambda\to-\infty} N_j^{k-1}(u,\,\lambda) = 0, \quad (j \neq i, i+z-k+1), \quad z \in \{1, \ldots, k-1\}. \tag{4}$$

At first we will prove that the assumptions (2)–(4) yield

$$\lim_{\lambda\to-\infty} N_i^k\,(u,\lambda) = \begin{cases} \frac{1}{2}\,, \text{ if } z \in \{1, \ldots, k-1\} \\ 1\,, \text{ if } z = k \end{cases} \qquad u \in [u_i,\, u_{i+z}). \tag{5}$$

By definition

$$N_i^k(u) = \frac{u-u_i}{u_{i+k-1}-u_i}\,N_i^{k-1}(u) + \frac{u_{i+k}-u}{u_{i+k}-u_{i+1}}\,N_{i+1}^{k-1}(u).$$

Substitute again $\hat{u}_i = u_i + \lambda$, $\hat{u}_{i+z} = u_{i+z} - \lambda$ and $\hat{u}_j = u_j$, $j \neq i, i+z$. For $z \in \{1, \ldots, k-2\}$ the limit of the first term is $1/2$ due to eq. (2). For $z = 1$ $N_{i+1}^{k-1}(u) \equiv 0$, thus the limit of the second term equals 0, while for $z \in \{1, \ldots, k-2\}$ it is equal to 0 due to (4). For $z = k-1$ the limit of the fraction in the first term is $1/2$, while (2) yields $\lim\limits_{\lambda\to-\infty} N_i^{k-1}(u,\,\lambda) = 1$, thus the

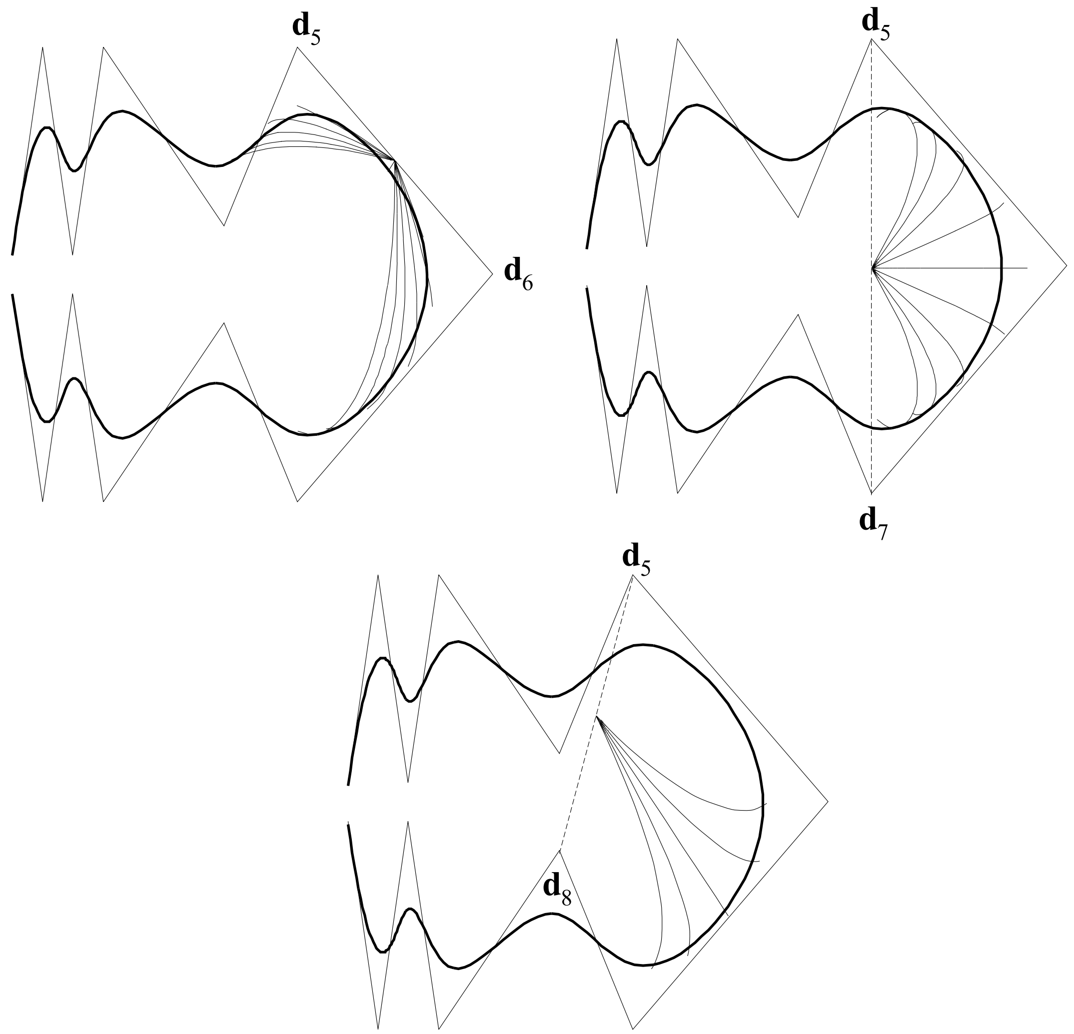

Figure 1: Extended paths of the points of a cubic B-spline curve obtained by a symmetric modification of u_6 and u_9 (above left), u_7 and u_9 (above right) and u_8 and u_9 (below), respectively

limit of the first term equals 1/2. The limit of the second term is equal to 0 due to equalities (4). For $z = k$ the statement is equivalent to that of Theorem 1.

Now we prove that equalities (2)–(4) also imply

$$\lim_{\lambda \to -\infty} N^k_{i+z-k}(u, \lambda) = \begin{cases} \frac{1}{2}, \text{ if } z \in \{1, \dots, k-1\} \\ 1, \text{ if } z = k \end{cases} \qquad u \in [u_i, u_{i+z}). \tag{6}$$

By definition

$$N^k_{i+z-k}(u) = \frac{u - u_{i+z-k}}{u_{i+z-1} - u_{i+z-k}} N^{k-1}_{i+z-k}(u) + \frac{u_{i+z} - u}{u_{i+z} - u_{i+z-k+1}} N^{k-1}_{i+z-k+1}(u).$$

After the usual substitution one can see, that for $z \in \{1, \dots, k-1\}$ the limit of the first term equals 0 due to (4). For $z \in \{1, \dots, k-2\}$ the limit of the fraction at the second term equals 1, while the limit of the basis function at the second term equals 1/2 due to (3). For $z = k-1$ an easy calculation shows that the limit of the fraction is 1/2, while the limit of the basis function equals 1 due to (4) again. Thus the limit of the second term is always equal to 1/2. Finally for $z = k$ cf. Theorem 1.

At the *final step* of the proof we verify that (2)–(4) imply

$$\lim_{\lambda \to -\infty} N_j^k(u, \lambda) = 0, \quad (j \neq i, i+z-k), \quad u \in [u_i, u_{i+z}) \tag{7}$$

for all z. By definition

$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u).$$

If $j = i + z - k + 1$, then $j + k - 1 = i + z$, thus the limit of the fraction in the first term is 0, while for any other cases the limit of the basis function equals 0 due to (4). Hence the limit of the first term is always equal to 0. If $j + 1 = i$ then the limit of the fraction at the second term equals 0, while for any other cases the limit of the basis function equals 0 due to (4) again. Thus the limit of the second term equals 0 as well.

Hence (2)–(4) yield (5), (6) and (7) which completes the proof. □

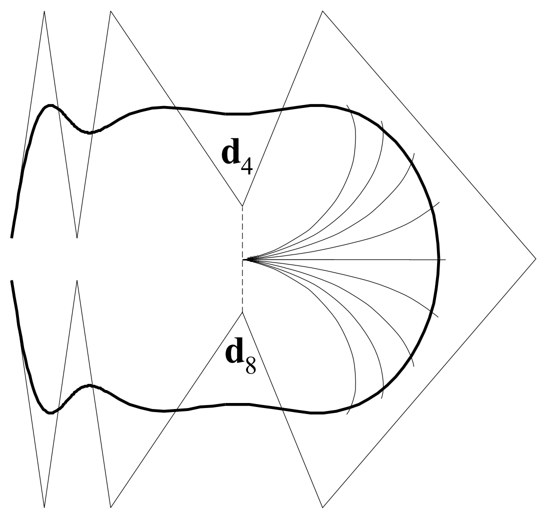

Figure 2: Extended paths of the points of a 6^{th} order B-spline curve obtained by a symmetric modification of u_8 and u_{10}.

3. Conclusion

The symmetric modification of two knots of B-spline curves has been discussed in the paper. We proved, that the points of some arcs of the curve tend to the midpoint of the segment of two control points. This can be seen in the figures for different orders and knot pairs: for a cubic curve in Fig. 2 and for a 6^{th} order B-spline curve in Fig. 2. Future research includes some further aspects of the coordinated motion of two or more knot values and the generalization of these results to surfaces.

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