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Geometry of Regular Heptagons

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Abstract. This paper explores some new geometric properties of regular heptagons. We add to the list of results from the Bankoff-Garfunkel famous paper on regular heptagons 30 years ago enlisting the help from computers. Our idea is to look at the central points (like incenters, centroids, circumcenters and orthocenters) of certain triangles in the regular heptagon to find new related regular heptagons which have simple constructions with ruler and compass from the original heptagon. In the proofs we use complex numbers and the software Maple V. The eleven figures are made with the Geometer's Sketchpad.

Key Words: regular heptagon, heptagonal triangle

MSC 2000: 51N20, 51M04, 14A25, 14Q05

1. Introduction

Leon Bankoff and Jack Garfunkel in the reference [1] thirty years ago gave the review of several results on regular heptagons and on the associated heptagonal triangle (see Fig. 1). In [3], [4] and [5] some of these initial theorems in [1] have been improved. In this paper our goal is to continue these investigations. We use again complex numbers to discover new relationships in these geometric configurations.

In order to simplify our statements we use the following notation. The midpoint of points X and Y is $[X;\,Y]$ while $X \parallel \ell$ and $X \perp \ell$ are the parallel and the perpendicular to the line ℓ through the point X.

Let $\Theta = ABCDEFG$ be a regular heptagon inscribed into the circle k with the center O and the radius R. We now define fourteen regular heptagons associated to Θ. It suffices to describe only their first vertex because the other vertices are obtained by rotations about the point O. The first vertices are shown in Fig. 1 and are defined as follows:

$$A_m = [O;\,A], \quad A = [O;\,A_2], \quad A' = [A;\,B], \quad A^d = AC \cap BG,$$
$$A^s = BC \cap AG, \quad A'_m = [A_m;\,B_m], \quad A'_2 = [A_2;\,B_2], \quad A^d_m = A_mC_m \cap B_mG_m,$$
$$A^d_2 = A_2C_2 \cap B_2G_2, \quad A^s_m = B_mC_m \cap A_mG_m, \quad A^s_2 = B_2C_2 \cap A_2G_2,$$

and let A^*, A^*_m, A^*_2 be the midpoints of the shorter arcs AB, A_mB_m, A_2B_2.

For different points X and Y and a real number $r > 0$ let $\gamma(X;\,Y)$ and $\gamma(X;\,r)$ denote circles with the center at X which goes through Y and with the radius r, respectively, while $\varepsilon(XY)$ is the complement of the segment XY in the line XY.

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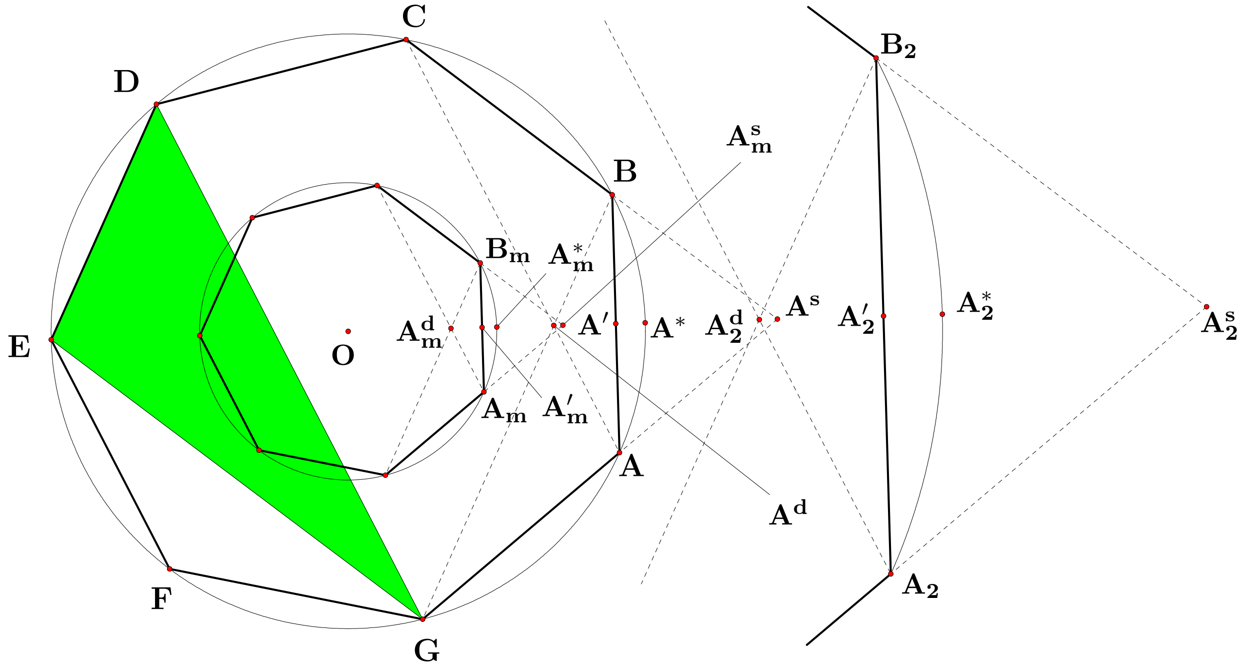

Figure 1: First vertices of 14 regular heptagons associated to the regular heptagon $ABCDEFG$ and one of its heptagonal triangles DEG

2. New theorems

We begin with an improvement of the observations attributed to THÉBAULT on the pages 10 and 11 of [1]. The parts (1)–(4) are from there while (5)–(10) are new. When investigating regular heptagons it seems natural to look for various regular heptagons associated to it. In our first theorem we discovered four such heptagons.

Theorem 1

(1) The segment $A'F_m$ is the diagonal of the square build on the inradius of Θ.

(2) Extend $A'B$ over B to the point W so that $|A'W| = |A'B^*|$. The segment WO is the diagonal of the square constructed on half the side of the equilateral triangle inscribed to the circle k.

(3) The circle with the center at W which is orthogonal to k has $|A'B_m^*|$ as radius.

(4) Let $T = AB \cap (F_m \perp OF)$, $S = [O; T]$, and $m = \gamma(S; O)$. The points A' and F_m are on the circle m and the line $A'B_m^*$ is its tangent (Fig. 2).

(5) Let $L = (m \cap OB) \setminus \{O\}$, $H = m \cap \varepsilon(OA)$, $I = m \cap \varepsilon(OD)$, $J = m \cap \varepsilon(OG)$, $K = m \cap \varepsilon(OC)$. Then $F_m LA'HIJK$ is a regular heptagon with the length of sides $|A'B_m^*|$.

(6) Let $H' = m \cap \varepsilon(A'B')$, $U' = m \cap \varepsilon(A'C')$, $L' = (m \cap A'D') \setminus \{A'\}$, $Q' = (m \cap A'E') \setminus \{A'\}$, $K' = m \cap \varepsilon(A'F')$, $J' = m \cap \varepsilon(A'G')$. Then $TH'U'L'Q'K'J'$ is a regular heptagon.

(7) The midpoints I'', H'', U'', L'', Q'', K'', J'' of shorter arcs TI, $H'H$, $U'A'$, $L'L$, $Q'F_m$, $K'K$, $J'J$ are vertices of a regular heptagon whose sides are parallel with sides of $DEFGABC$.

(8) Let $n = \gamma(A'; B_m^*)$. The circles m and n intersect in the points H and L.

(9) Let $M = n \cap A'F_m$, $N = n \cap A'J$, $P = n \cap \varepsilon(A'L)$, $Q = n \cap \varepsilon(A'K)$, $U = n \cap \varepsilon(A'I)$. Then $B_m^* MNHPQU$ is a regular heptagon.

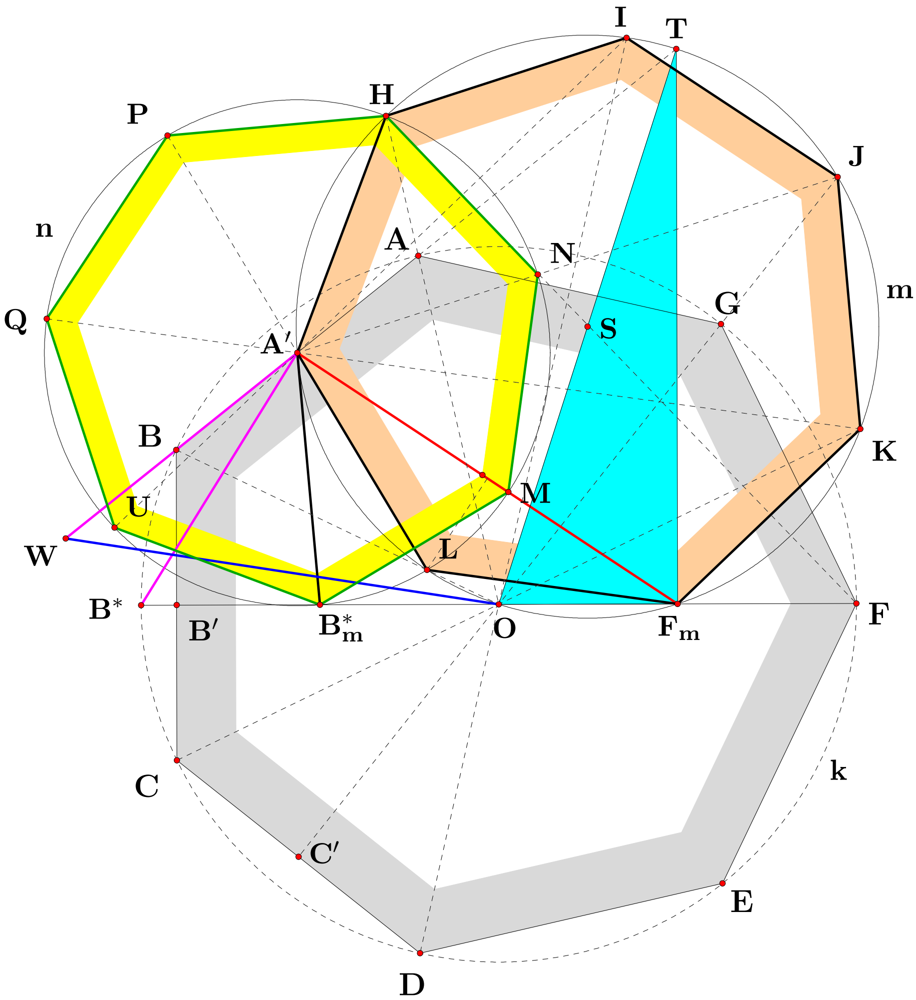

Figure 2: The circles m and n with the regular heptagons inscribed into them

(10) The circle with the center at W which is orthogonal to n has $|A'A|$ as radius.

Proof: (1) In this proof we shall assume that the complex coordinates (or the affixes) of the vertices of the heptagon $ABCDEFG$ are

$$F = 1, \quad G = f^2, \quad A = f^4, \quad B = f^6, \quad C = f^8, \quad D = f^{10}, \quad E = f^{12},$$

where f is the 14th root of unity.

From $A' = \frac{f^6 + f^4}{2}$ and $F_m = \frac{1}{2}$ follows that $2\,|A'O|^2 - |A'F_m|^2$ is equal to $\frac{(f^4 + f^2 - 1)p_+ \, p_-}{4}$, where

$$p_- = f^6 - f^5 + f^4 - f^3 + f^2 - f + 1 \quad \text{and} \quad p_+ = f^6 + f^5 + f^4 + f^3 + f^2 + f + 1.$$

But, $f^{14} - 1$ is $(f^2 - 1)\, p_-\, p_+$ and $p_+ = 1 + 2\,i\,(1 + 2\,\cos\frac{\pi}{7})\sin\frac{2\pi}{7} \neq 0$. We see that $p_- = 0$ so that $|A'F_m| = (R\cos\frac{\pi}{7})\,\sqrt{2}$.

(2) The point W is on AB and $\gamma(A';\, B^*)$ if $f^4\, W + \overline{W} - f^{10} - f^8 = 0$ and

$$2W\overline{W} - (f^{10} + f^8)W - (f^6 + f^4)\overline{W} + f^{13} + f^{11} + f^3 + f - 2 = 0.$$

By solving this system we get

$$W = \frac{f^6 + f^4 - f^3\sqrt{f(f-1)^2(f^2+f+2)(2\,f^2+f+1)}}{2}.$$

It is now easy to check that $\frac{3}{2} - |WO|^2 = \dfrac{(f^{11} - 3f^8 - 3f^7 + 3f + 3)\, p_-}{2} = 0$.

(3) One of the intersections of k and the circle with diameter WO is the point Z with affix

$$\frac{2 + \sqrt{2(f^{13} - f^{12} + f^{11} - f^2 + f - 2)}}{f^7(f^3 + f + \sqrt{f(f-1)^2(f^2+f+2)(2f^2+f+1)})}.$$

The difference $|WZ|^2 - |A'B_m^*|^2$ contains p_- as a factor.

(4) The equations of AB and $F_m \perp OF$ are $f^6 z + f^2 \overline{z} = 1 - f^{10}$ and $z + \overline{z} = 1$. Hence, $S = \dfrac{f^{10} + f^2 - 1}{2f^2(1 - f^4)}$. Since both $|SO|^2 - |A'S|^2$ and $|SO|^2 - |F_m S|^2$ contain p_- as a factor we infer that A' and F_m are on m. Also, the lines $A'B_m^*$ and $A'S$ are perpendicular.

(5) Note that $f^{2k}\,(F_m - S) + S$ for $k = 1, 2, 3, 4, 6$ lie on lines OC, OG, OD, OA, OB while for $k = 5$ it agrees with A'. Hence, these are the vertices of $F_m KJIHA'L$. Moreover, $|A'L| = |A'B_m^*|$.

(6) Now $f^{2k}\,(T - S) + S$ for $k = 1, \ldots, 6$ lie on lines $A'B'$, $A'C'$, $A'D'$, $A'E'$, $A'F'$, $A'G'$ so that these are the last six vertices of the regular heptagon $TH'U'L'Q'K'J'$.

(7) This part is more complicated even on a computer so that we only outline main steps. First find the equation of m and of the line ℓ joining S with the midpoint of the segment IT. One of the points in the intersection $m \cap \ell$ is I''. Then we rotate six times through the angle $\frac{2\pi}{7}$ to get points H'', U'', L'', Q'', K'', and J''. Finally, we check that (only one) corresponding sides of $I''H''U''L''Q''K''J''$ and $DEFGABC$ are parallel.

(8) The equations of the circles m and n (Fig. 2) are

$$2(f^4 - 1)\, z\,\overline{z} + f^4(f^6 + f^4 - 1)\, z - (f^{10} + f^8 - 1)\,\overline{z} = 0,$$

and

$$4\, z\,\overline{z} - 2f^4(f^2 + 1)(f^4\, z + \overline{z}) + f^{13} + f^{11} + f^3 + f - 1 = 0.$$

Their intersections have rather complicated affixes but after some clever manipulation with square roots one can show that they represent points H and L.

(9) We rotate the point B_m^* about the point A' six times through the angle $\frac{2\pi}{7}$. The third point concides with the point H while the others are on lines $A'F_m$, $A'J$, $A'L$, $A'K$, and $A'I$, so that $B_m^* MNHPQU$ is indeed a regular heptagon.

(10) Similar to the proof of (3). $\square$

Theorem 2 *Let the lines BE and BG intersect the line AD in points M and N. Let U, V, and W denote circumcenters of the triangles BDM, BMN, and ABN (Fig. 3).*

(1) $W = A^*$ and U is the reflection of O at the line BD.

(2) V is the reflection of W at the line BG and the midpoint of the shorter arc BM on the circumcircle of BDM.

(3) $|UV| = R$ and $|UW| = |UH| = R\sqrt{2}$, where H is the intersection of the lines AO and GV. Also, $|VW| = |OV| = |AH|$.

(4) If $X = OA \cap (B \perp DC)$, $Y = OD \cap (X \perp CB)$, and $Z = OC \cap (G \perp GA)$, then $OVWXYGZ$ is a regular heptagon whose sides are perpendicular to the corresponding sides of $FEDCBAG$.

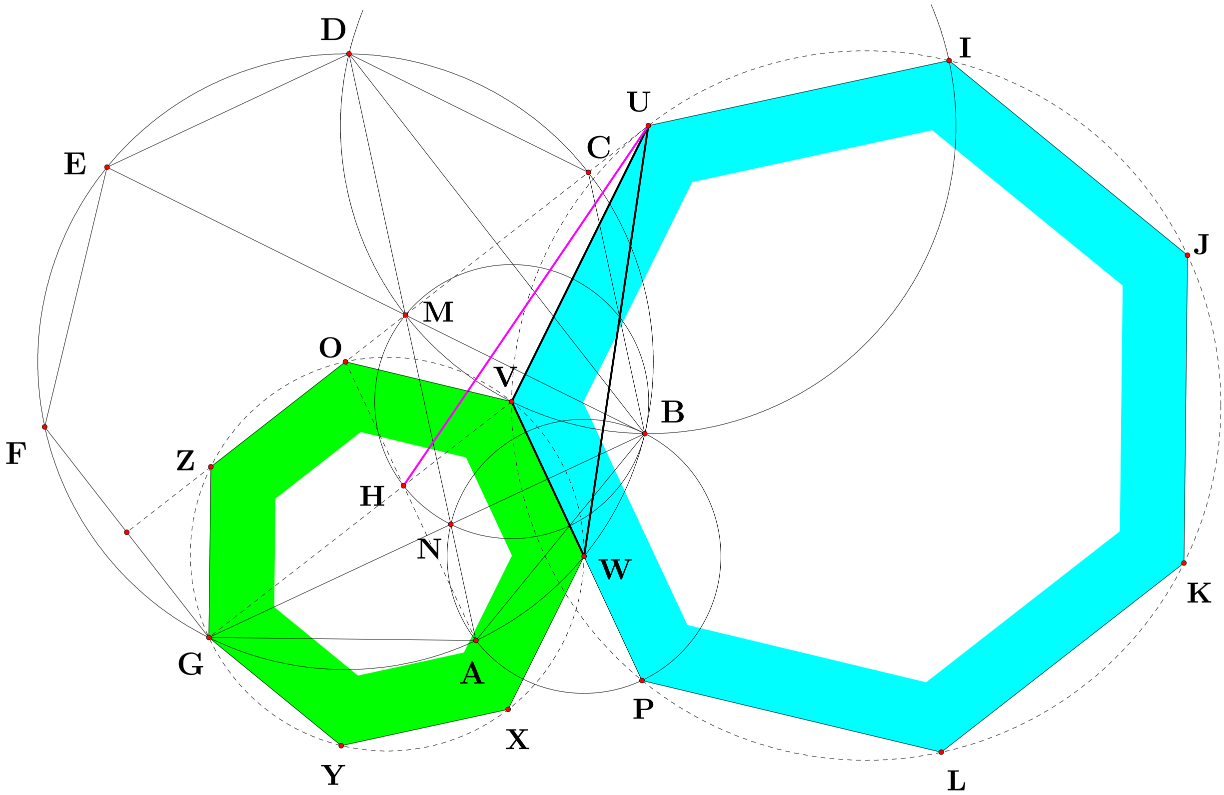

Figure 3: The triangle on circumcenters U, V, W of the triangles BDM, BMN, and ABN and two regular heptagons with sides perpendicular to sides of $ABCDEFG$

(5) If $I = GV \cap (U \perp CB)$, $J = DU \cap (I \perp BA)$, $K = OB \cap (J \perp GA)$, $L = AF^* \cap (K \perp FG)$, and $P = AF^* \cap (D \perp BG)$, then $VUIJKLP$ is a regular heptagon whose sides are perpendicular to the corresponding sides of $DCBAGFE$.

Proof: (1) The affixes of the points M, N, U, V, W, and H are

$$f^{10} - f^8 + f^6, \quad \frac{f^{10} + f^6 + f^4}{f^4 + f^2 + 1}, \quad f^{10} + f^6, \quad \frac{f^{10} + f^6}{f^4 + f^2 + 1}, \quad \frac{f^{10} + f^8 + f^6 + f^4}{f^4 + f^2 + 1}, \quad \frac{f^2(f^4 + 1)^2}{2f^4 + f^2 + 2},$$

respectively. Since $|WA^*|^2 = \left(\frac{f^6\, p_-}{f^4 + f^2 + 1}\right)^2$, we infer that $W = A^*$. It is easy to find the reflection of O in the line BD and check that it coincides with the point U.

(2) Since the reflection of a complex number x in the line determined by different complex numbers y and z is $\frac{y(\overline{x} - \overline{z}) + z(\overline{y} - \overline{x})}{\overline{y} - \overline{z}}$, by direct substitution of affixes, we see that V is a reflection of W in the line BG.

(3) Since $|UV|^2 = |OW|^2$, we get $|UV| = 1 = R$. Also, since $|UW|^2 = \frac{2(f^2+2f^4+f^6-f^{10}-f^{12})}{(f^4 + f^2 + 1)^2}$ and $|UW|^2 - 2$ factors as $\frac{(2f^{10} + f^8 + 2f^6 - 2f^4 - 3f^2 - 2)\, p_+ p_-}{(f^4 + f^2 + 1)^2}$, it follows that $|UW| = \sqrt{2} = R\sqrt{2}$. In a similar fashion we can also prove that $|UH| = \sqrt{2} = R\sqrt{2}$ and that $|VW| = |OV| = |AH|$.

(4) Let $T = \frac{f^{12} + f^8}{f^6 - 1}$ denote the circumcenter of the triangle OVW. Then $f^{2k}(O - T) + T$ for $k = 1, \ldots, 6$ is Z, G, Y, X, W, V. Hence, $OVWXYGZ$ is a regular heptagon. Since OV is perpendicular to FE, its sides are perpendicular to the corresponding sides of $FEDCBAG$.

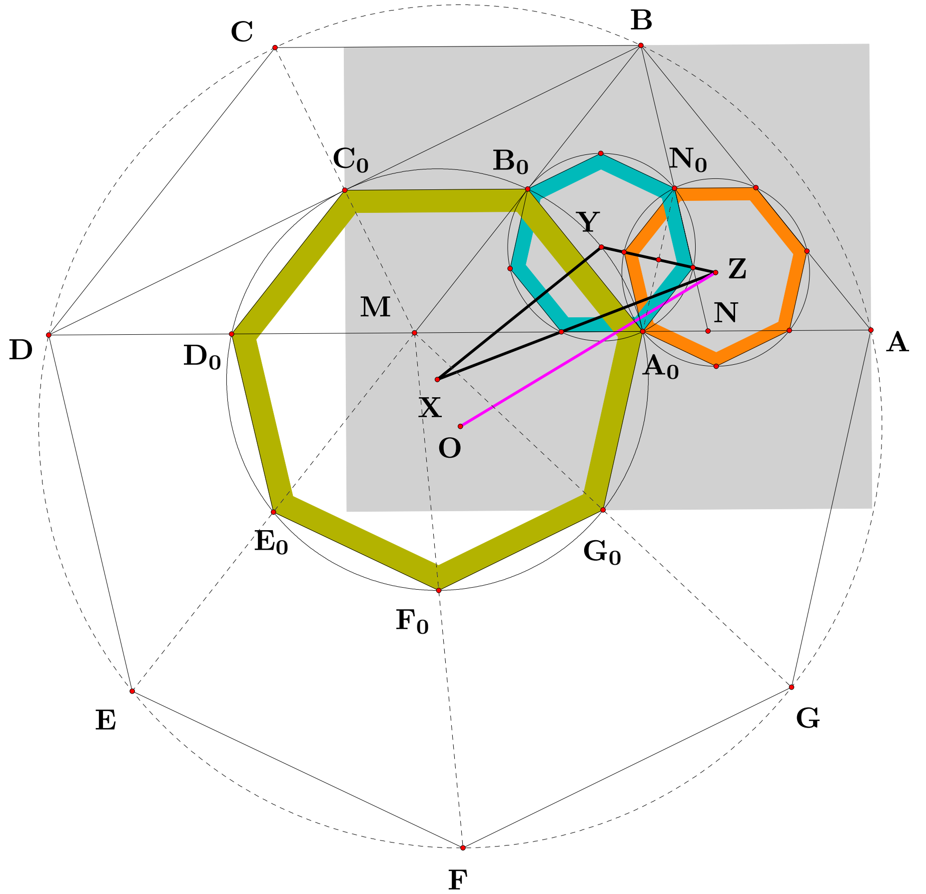

Figure 4: The triangle on centers X, Y, Z of the nine-point circles of triangles BDM, BMN, and ABN with $|XZ| = \frac{R\sqrt{2}}{2}$

(5) Similar to the proof of (4). □

There is a similar result for centers of the nine-point circles instead of circumcenters.

Theorem 3 *Let the lines BE and BG intersect the line AD in points M and N. Let A_0, B_0, C_0, D_0, E_0, F_0, G_0 and N_0 be midpoints of the segments AM, BM, CM, DM, EM, FM, GM and BN. Let X, Y and Z be centers of the nine-point circles d_9, m_9 and a_9 of the triangles BDM, BMN and ABN (Fig. 4).*

(1) The point X is the midpoint of the segment MO and the point Y is the midpoint of the shorter arc A_0B_0 of the nine-point circle d_9 of the triangle BDM.

(2) The point Z is the reflection of Y at the line A_0N_0. Also, $|XY| = \frac{R}{2}$ and $|XZ| = |OZ| = \frac{R\sqrt{2}}{2}$.

(3) The polyhedron $A_0B_0C_0D_0E_0F_0G_0$ is a regular heptagon inscribed into d_9 which is the image of $ABCDEFG$ in the homothety $h(M, \frac{1}{2})$.

(4) Let N' be the projection of the point N on the side AB, let $K = BN \cap YZ$, let L be the reflection of K in the line A_0N_0, let $P = [A; N]$ and $S = [M; N]$. Let $Q = [B; G]$, $H = N'N_0 \cap G_0Y$ and $T = D_0Y \cap A_0C_0$. Then $A_0STB_0HN_0K$ and $N_0A'N'PQA_0L$ are regular heptagons inscribed into m_9 and a_9 related by the homothety $h([Y; Z], -1)$ and homothetic with the heptagon $CBAGFED$.

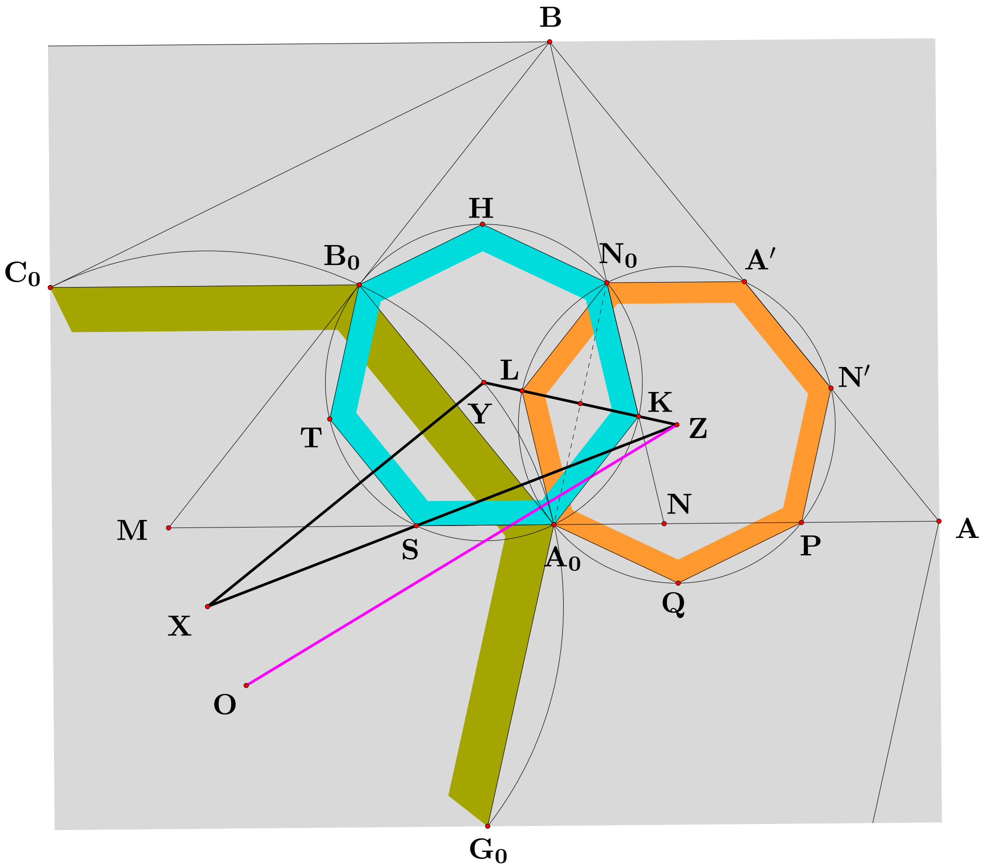

Figure 5: Enlarged rectangular part of Fig. 4 with two regular heptagons homothetic with the heptagon $CBAGFED$

Proof: (1) The affixes of the points M and N have been given in the proof of the previous theorem. It follows that the midpoints A_0, B_0, N_0 are

$$\frac{f^{10}-f^8+f^6+f^4}{2}, \quad \frac{f^{10}-f^8+2f^6}{2}, \quad \frac{2f^{10}+f^8+2f^6+f^4}{2(f^4+f^2+1)}$$

while X, Y, Z are

$$\frac{f^{10}-f^8+f^6}{2}, \quad \frac{2f^{10}+f^8+2f^6+f^4+1}{2(f^4+f^2+1)}, \quad \frac{f^{10}+f^8+2f^6+f^4}{2(f^4+f^2+1)},$$

respectively. By direct inspection we see that X is the midpoint of MO and that Y is the midpoint of the shorter arc A_0B_0 of the nine-point circle of BDM because $|XY| = \frac{1}{2}$ and $|YA_0|^2 = |YB_0|^2$.

(2) Just as easy is to check that Z is the reflection of Y at the line A_0N_0. Finally, since $|XZ|^2 = \dfrac{f^6+2f^4-f^{10}-f^{12}}{2(f^8+2f^6+3f^4+2f^2+1)}$ and both $|XZ|^2-\frac{1}{2}$ and $|OZ|^2-\frac{1}{2}$ contain the polynomial p_- as a factor we get that $|XZ| = |OZ| = \dfrac{\sqrt{2}}{2}$.

(3) This part is obvious.

(4) In a routine fashion we discover that the affixes of the points N', K, L, P, Q, H, S and T are

$$\frac{f^{10}+f^8+3f^6+f^4}{2(f^4+f^2+1)}, \quad \frac{f^{10}+3f^8+3f^6+3f^4+f^2+1}{2(f^{12}+f^4+2f^2+2)}, \quad \frac{f^{12}+2f^{10}+2f^8+3f^6+3f^4+f^2}{2(f^{12}+f^4+2f^2+2)},$$
$$\frac{f^{10}+f^8+2f^6+2f^4}{2(f^4+f^2+1)}, \quad \frac{f^6+f^2}{2}, \quad \frac{5\,f^{12}+f^{10}-3f^8-2f^6+5f^2+6}{2(3\,f^{10}+4f^8+3f^6-2f^2-2)},$$
$$\frac{2f^{10}+2f^6+f^4+1}{2(f^4+f^2+1)}, \quad \frac{-2\,f^{12}-f^{10}+f^4-f^2-1}{2(f^{12}-f^8-2f^6-f^4+1)},$$

respectively. Since $f^{2k}(A_0-Y)+Y$ for $k=1,\ldots,6$ is K, N_0, H, B_0, T, S, we infer that $A_0STB_0HN_0K$ is a regular heptagon inscribed into m_9. Since A_0S is parallel to BC, it is homothetic to $CBAGFED$ (Fig. 5).
Similarly, since $f^{2k}(N_0-Z)+Z$ for $k=1,\ldots,6$ is L, A_0, Q, P, N', A', we get that $N_0A'N'PQA_0L$ is a regular heptagon inscribed into a_9. Since $A'N'$ is parallel to AB, it is homothetic to $CBAGFED$. □

Of course, the triangles UVW and XYZ from the previous two theorems are closely related as the following result clearly shows.

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars at vertices of ABC onto sides YZ, ZX, and XY of XYZ are concurrent. It is well-known that the relation of orthology for triangles is reflexive and symmetric.

Theorem 4 *Let $U'V'W'$ and $X'Y'Z'$ be reflections of UVW and XYZ at the line AD. Let K be the intersection of the lines AF and DG. Then any two among the triangles UVW, XYZ, $U'V'W'$ and $X'Y'Z'$ are orthologic. The triangles UVW and $U'V'W'$ are images under the homotheties $h(K,2)$ and $h(B,2)$ of the triangles $X'Y'Z'$ and XYZ (Fig. 6).*

Proof: Since the points U', V', W', X', Y', Z' have the affixes

$$f^{10}-f^8, \quad \frac{f^{10}+f^6+f^4+1}{f^4+f^2+1}, \quad \frac{f^6+f^4}{f^4+f^2+1}, \quad \frac{2f^{10}-f^8+f^6+f^4}{2},$$
$$\frac{f^{10}+2\,f^6+f^4+1}{2(f^4+f^2+1)}, \quad \text{and} \quad \frac{f^{10}+f^8+2\,f^6+2\,f^4+1}{2(f^4+f^2+1)},$$

respectively, it is easy to check using Theorem 5 in [2] that the triangles UVW and XYZ are orthologic. It is also easy to verify that X', Y', Z' are midpoints of the segments KU, KV, KW and that X, Y, Z are midpoints of the segments BU', BV', BW' which proves the claims about homotheties. □

Theorem 5 *Let $ABCDEFG$ be a regular heptagon inscribed to a circle of radius R. Let the lines BE and BG intersect the line AD in points M and N. If H and K are the de Longchamps points of the triangles BDM and ABN, then $|HK| = R\sqrt{11}$.*

Proof: Since $H = 2\,f^{10}+f^8+2f^6$ and $K = \dfrac{2f^{10}+2f^8+f^6+2f^4}{f^4+f^2+1}$, we get that $|HK|^2-11$ is $\dfrac{-5\,p_+p_-}{f^8+2f^6+3f^4+2f^2+1}$. □

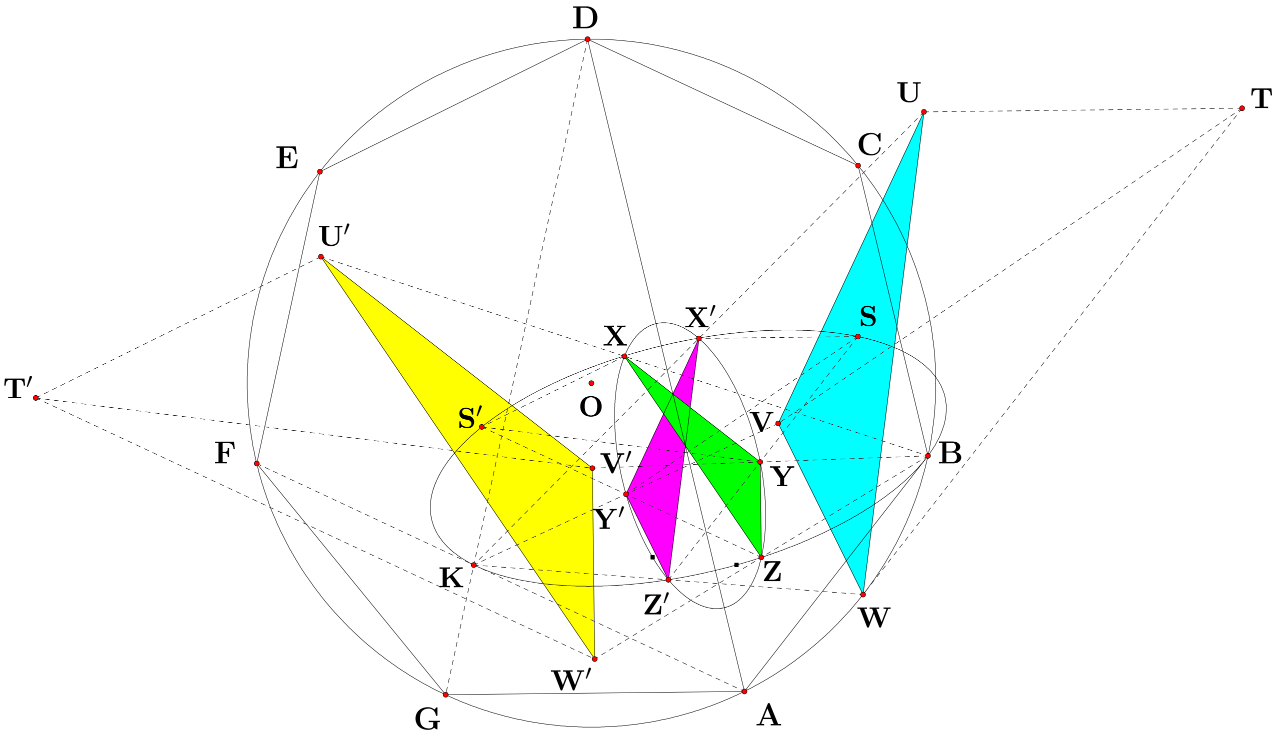

Figure 6: The triangles UVW and XYZ are orthologic and homothetic with their reflections at the line AD

Theorem 6 *Let the line BE in the regular heptagon $ABCDEFG$ intersect the line AD in the point M. Let H, I, J, K be centroids of the triangles BDM, DEM, EGM, BGM. Then $HIJK$ is a rhombus whose side is $\frac{2R}{3}\cos\frac{\pi}{14}$ and whose area is $2/9$ of the area of the quadrangle $BDEG$. The angles $\sphericalangle IHK$ and $\sphericalangle KJI$ are equal to $\frac{4\pi}{7}$ so that the regular heptagons build on IH, HK and on IJ, JK share one side (Fig. 7).*

Proof: Since $M = f^{10} - f^8 + f^6$, we easily find that

$$H = \frac{2f^{10} - f^8 + 2f^6}{3}, \quad I = \frac{f^{12} + 2f^{10} - f^8 + f^6}{3}, \quad J = \frac{f^{12} + f^{10} - f^8 + f^6 + f^2}{3},$$

$$\text{and} \quad K = \frac{f^{10} - f^8 + 2f^6 + f^2}{3}.$$

Now, we compute $|HI|^2 - |IJ|^2$, $|HI|^2 - |JK|^2$, and $|HI|^2 - |KH|^2$ to discover that they are all zero (this is immediate for the second difference while for the first and the third it follows from the fact that both contain p_- as a factor). Hence, the quadrangle $HIJK$ is the rhombus. The claim about the side and the angles is a consequence of easily verified equalities $|IJ|^2 = \frac{4}{9}\cos^2\frac{\pi}{14}$ and $\frac{|HP|^2}{|HI|^2} = \cos^2\frac{2\pi}{7}$, where P is the center of the rhombus. Finally, since the triangles HIJ, BEG and BDE have areas $\frac{i}{36}(f^{12} + 3f^{10} - 3f^4 - f^2)$, $\frac{i}{4}(2f^{10} + f^8 - f^6 - 2f^4)$, and $\frac{i}{4}(f^{12} + f^{10} - f^8 + f^6 - f^4 - f^2)$, respectively, it follows that $\frac{|HIJK|}{|BDEG|} = \frac{2}{9}$. □

Remark (Adrian Oldknow): The statement about $\frac{|HIJK|}{|BDEG|} = \frac{2}{9}$ has actually nothing to do with heptagons — it is just a particular case of the very easily proved result that if M

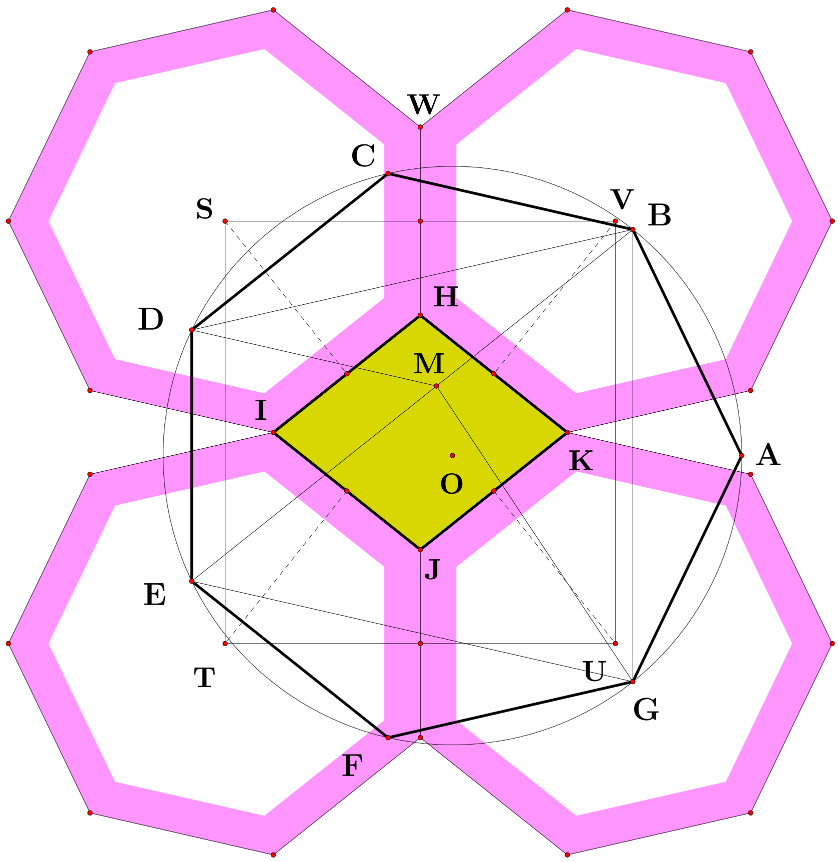

Figure 7: The quadrangle $HIJK$ on centroids
of triangles BDM, DEM, EGM, BGM is a rhombus

is *any* internal point of *any* convex quadrangle $ABCD$, then the centroids of the triangles AMB, BMC, CMD and DMA form a parallelogram with sides $\frac{1}{3}|AC|$ and $\frac{1}{3}|BD|$ and area $\frac{2}{9}|ABCD|$.

Theorem 7 *Let X, Y, Z be the centroids of the triangles BDE, BEG, ABG in the regular heptagon $\Theta = ABCDEFG$. Let S be the intersection of the lines joining E and G with the midpoints H and K of the segments BD and AB. Let $\lambda = \sqrt{u/v}$ where $v = 9 - 18\cos\frac{3\pi}{7}$ and $u = 2\cos\frac{3\pi}{7} - 2\cos\frac{2\pi}{7} + 4\cos\frac{\pi}{7}$. Let P and Q be intersections of the lines EY and FZ with the lines FX and GY.*

1. *If h_1 is the homothety $h(S, \lambda)$, then $XYZ = h_1(EFG)$ (Fig. 8).*
2. *If h_2 and h_3 are the homotheties $h(P, -\lambda)$ and $h(Q, -\lambda)$, then $\Theta_2 = h_2(\Theta)$ and $\Theta_3 = h_3(\Theta)$ are regular heptagons built on segments XY and YZ.*
3. *If U, M and N are the centers of Θ_1, Θ_2 and Θ_3 and Φ and Ψ denote the regular heptagons built on the the segments MN and XZ and containing the vertex B, then the center of Φ is U and Ψ is obtained from Φ by the translation for the vector $\overrightarrow{YU}$.*
4. *The point S lies on the side of Ψ.*

Proof: The centroids are

$$X = \frac{f^{12} + f^{10} + f^6}{3}, \quad Y = \frac{f^{12} + f^6 + f^2}{3}, \quad Z = \frac{f^6 + f^4 + f^2}{3}.$$

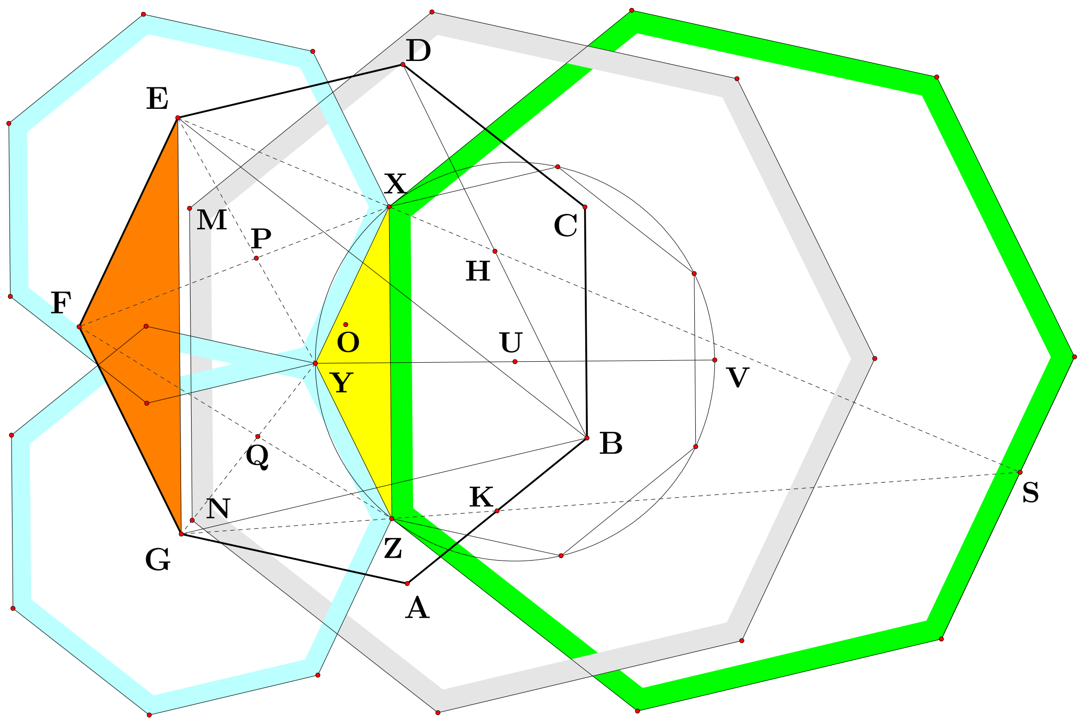

Figure 8: The triangle XYZ on centroids of triangles BDE, BEG, ABG is homothetic with the triangle EFG

The lines EX, FY and GZ concur at the point $S = EH \cap GK$ with the affix $\frac{2f^{12}+2f^{10}+f^8+f^6+1}{2f^8+f^6+f^4+f^2+2}$. The conditions for the corresponding sidelines of the triangles XYZ and EFG to be parallel are satisfied because they are zero (as desired) or are zero because they contain the polynomial p_- as a factor. The quotient $\lambda = \frac{|EF|}{|XY|}$ is equal to the square root of $-\frac{2f^{12}+2f^{10}+f^8+f^6+1}{9(f^8+f^4+1)}$. Let $M = \frac{f^{12}-f^8-f^4}{3}$ be the intersection of the perpendicular bisector of XY with the parallel through X to the line FO. The point $M + f^2(X - M)$ lies on the line GP. Replacing f^2 with other even powers of f we check that $\Theta_2 = h_2(\Theta)$ is the regular heptagon built on the segment XY. Let $U = \frac{f^{12}+f^{10}+f^8+2f^6+f^4+1}{3}$ be the intersection of the perpendicular bisector of XZ with the parallel through M to the line BO. Then U is the center of the regular heptagon built on MN which contains the point B. We can now easily check that MX, NZ and YU are parallel segments of the same length. For the last claim, we translate points $U + f^6(M - U)$ and $U + f^8(M - U)$ for $U - Y$ and check that the point S lies on the segment joining these translated points. □

Theorem 8 *Let U, V, W be the orthocenters of the triangles BDE, BEG, ABG in the regular heptagon $\Theta = ABCDEFG$. Let T be the intersection of the perpendiculars at E and G to the lines BD and AB. Let*

$$\mu = \frac{\sqrt{u}}{\sqrt{w}} \quad \textit{with} \quad u = 2\cos\frac{3\pi}{7} - 2\cos\frac{2\pi}{7} + 4\cos\frac{\pi}{7} \quad \textit{and} \quad w = 1 - 2\cos\frac{3\pi}{7}.$$

Let P be the intersection of the perpendicular bisectors p and q of UV and VW. Let M and N be intersections of p and q with perpendiculars to UW at U and W. Let H and K be the intersections of the lines FU and FW with the lines GO and EV.

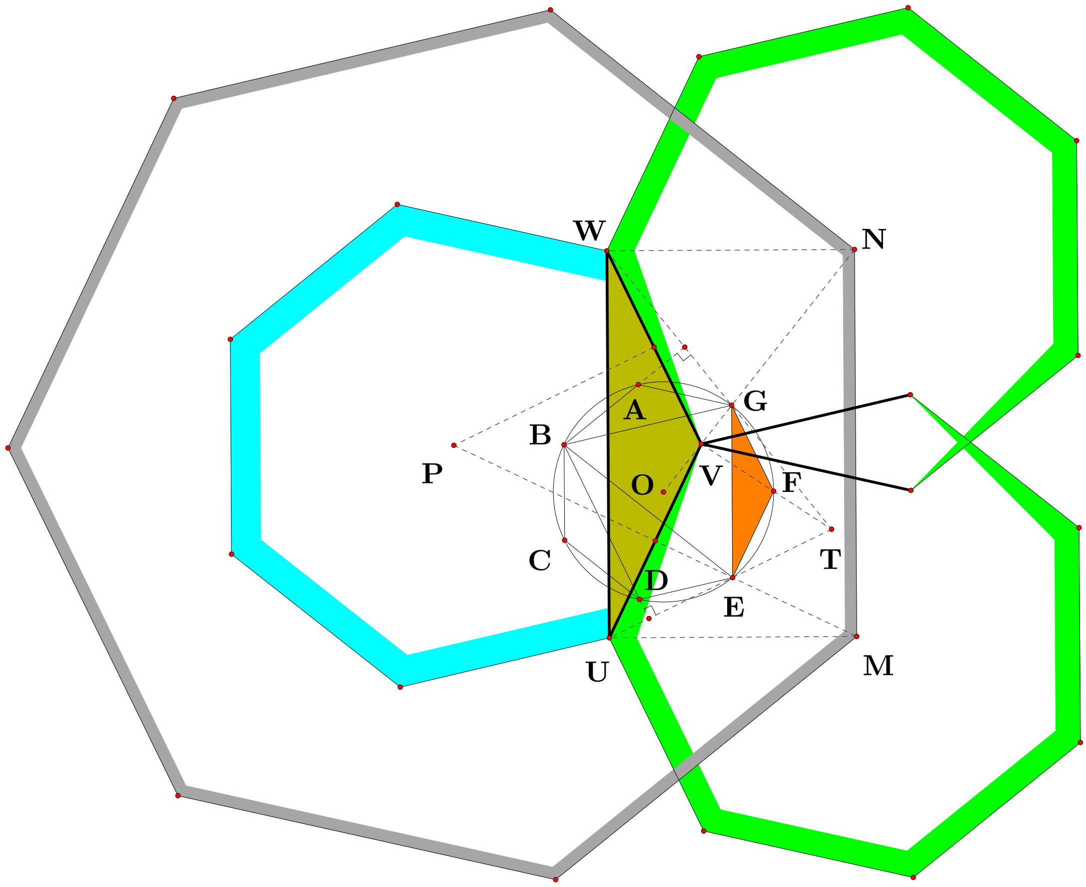

Figure 9: The triangle UVW on orthocenters of triangles BDE, BEG, ABG is homothetic with the triangle EFG

(1) If h_4 is the homothety $h(T, \mu)$, then $UVW = h_4(EFG)$ (Fig. 9).

(2) If h_5 and h_6 are the homotheties $h(H, -\mu)$ and $h(K, -\mu)$, then $\Theta_5 = h_5(\Theta)$ and $\Theta_6 = h_6(\Theta)$ are regular heptagons built on segments UV and VW with centers M and N.

(3) If Φ denotes the regular heptagon built on the the segment MN containing the vertex B and $\Psi = h_4(\Theta)$, then P is a common center of Φ and Ψ.

Proof: The orthocenters are $U = f^{12} + f^{10} + f^6$, $V = f^{12} + f^6 + f^2$, $W = f^6 + f^4 + f^2$. The claims of the theorem are now easily verified in the same way as in the proof of the previous theorem. □

The triangles XYZ and UVW from the previous two theorems are themselves homothetic as the following result shows.

Theorem 9 *The line ST goes through the center O and the triangle UVW on orthocenters is the image under the homothety $h(O, 3)$ of the triangle XYZ on centroids (Fig. 10).*

Proof: Since

$$S = \frac{2f^{12} + f^{10} + f^6 + 2f^2 + 1}{1 - f^2 - f^6 - f^8 - f^{12}} \quad \text{and} \quad T = \frac{2f^{12} + 2f^{10} + f^8 + f^6 + 1}{2f^8 + f^6 + f^4 + f^2 + 2}$$

the free term of the equation of the line ST is zero so that the center O (the origin) lies on ST. The second claim about the triangles UVW and XYZ is clearly true because the complex

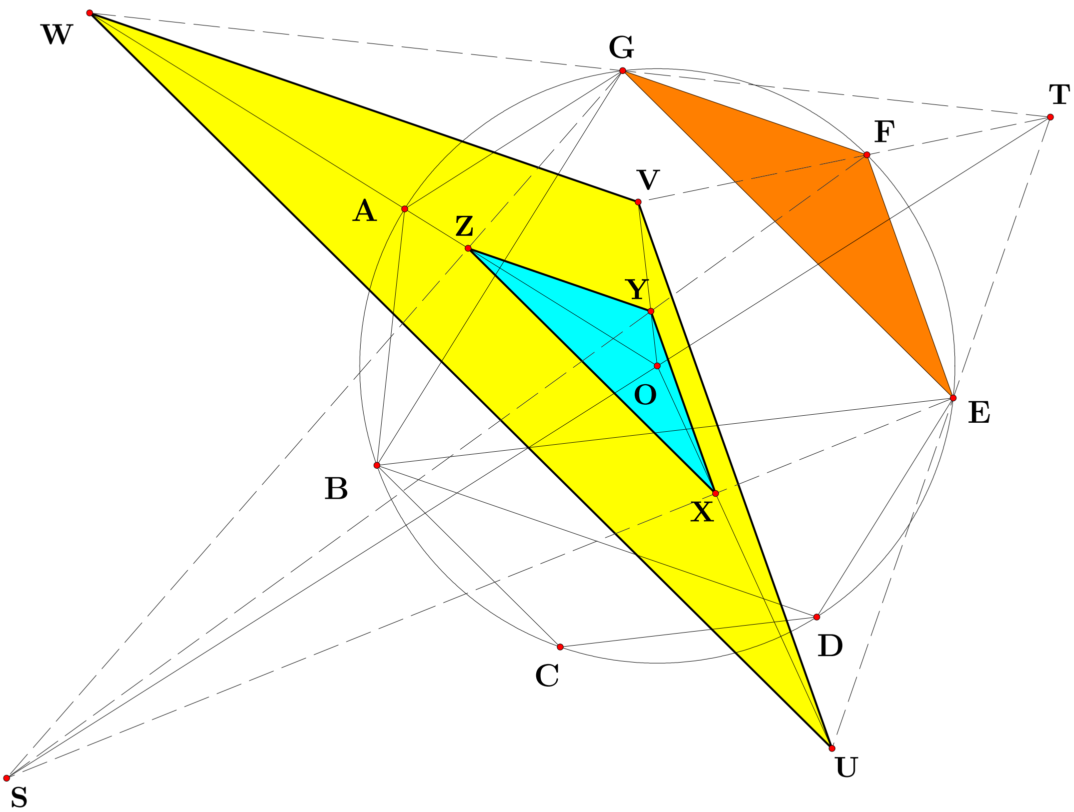

Figure 10: The triangles UVW on orthocenters and XYZ on centroids of triangles BDE, BEG, ABG are homothetic

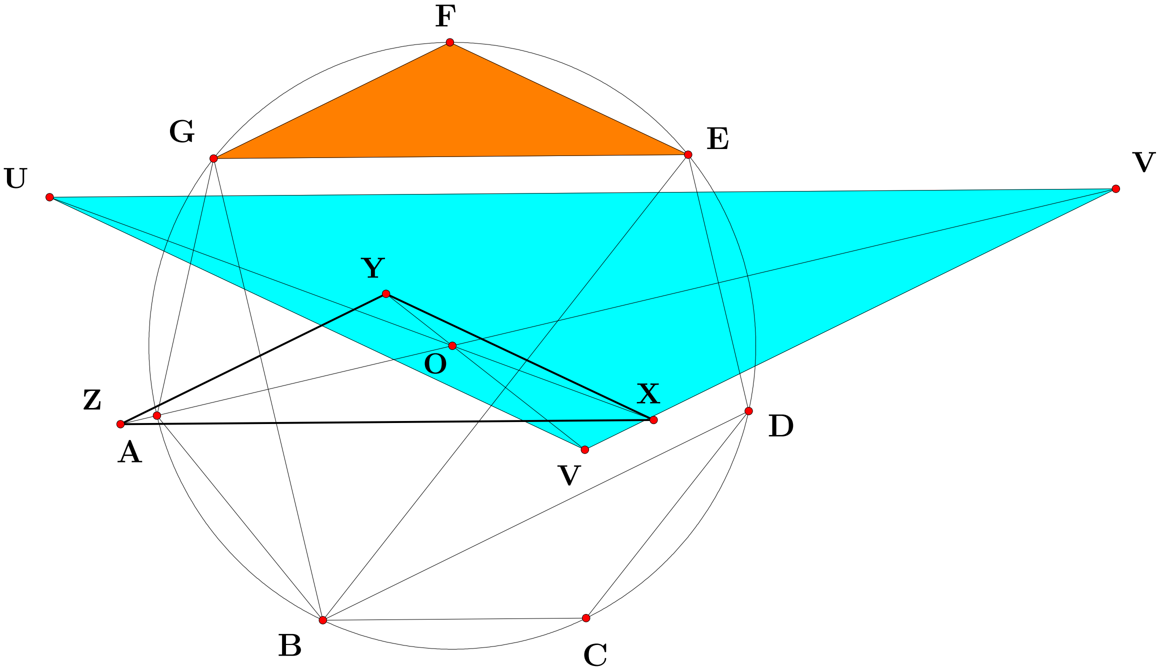

Figure 11: The triangles UVW on the de Longchamps points and XYZ on the centers of the nine-point circles of triangles BDE, BEG, ABG are related in homothety $h(O, -2)$

coordinates of the vertices of the second triangle are one third of the complex coordinates of the vertices of the first triangle. □

There are analogous results for the centers of the nine-point circles and for the de Longchamps points of the triangles BDE, BEG, ABG. We shall not give precise formulations of these theorems leaving this task as an exercise to the reader. In Fig. 11 the isosceles triangles on the de Longchamps points and on the centers of the nine-point circles are shown together.

References

[1] L. Bankoff, J. Garfunkel: *The heptagonal triangle*. Mathematics Magazine **46**, 163–187 (1973).

[2] Z. Čerin: *Hyperbolas, orthology, and antipedal triangles*. Glasnik Mat. **33**, 143–160 (1998).

[3] Z. Čerin: *Geometrija pravilnog sedmerokuta*. Poučak **14**, 5–14 (2003).

[4] Z. Čerin: *Regular heptagon's intersections circles*. Elem. Math. (to appear).

[5] Z. Čerin: *Regular heptagon's midpoints circle*. (preprint).

[6] R. Deaux: *Introduction to the geometry of complex numbers*. Ungar Publishing Co., New York 1956.

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