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# On the Simson-Wallace Theorem and its Generalizations

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**Abstract.** In this contribution we show generalizations of the well known Simson-Wallace Theorem into the space. We use methods of commutative algebra which are based on Gröbner basis computations (see [3]). This method enables to find and to prove such statements which are often very difficult to prove by techniques of synthetic geometry. In order to display geometric objects we use the dynamic geometry software Cabri and the mathematical software Maple. All computations were done by the computer algebra system CoCoA.

*Key Words:* Simson-Wallace Theorem, commutative algebra, normal form, Gröbner basis, automatic theorem proving, cubic surface

*MSC 2000:* 51N20, 51M04, 13P10

## 1. Introduction

There is a nice property of the circumcircle of a triangle, which is often ascribed to R. Simson (1687-1768), but it was really discovered by W. Wallace in 1799 (see [4]). Therefore it is quite common to call the following statement the Simson-Wallace Theorem:

*Let $ABC$ be a triangle and $P$ a point of the circumcircle of $ABC$. Then the feet of perpendiculars from $P$ onto the sides of $ABC$ lie on a straight line (see Fig. 1).*

The properties of the Simson-Wallace line and generalizations of this theorem have been investigated very often. A survey of results is given in [4, 9, 14]. For the latest references see [5, 13].

In this paper we will present two generalizations of the Simson-Wallace Theorem into the Euclidean space $E^3$. First we generalize the theorem on an *arbitrary* tetrahedron $ABCD$ and investigate points $P$ whose orthogonal projections $K, L, M, N$ on the faces of $ABCD$ form a tetrahedron of fixed volume (cf. [11]), where the same problem is solved for a special class of tetrahedra. We will show that these points $P$ lie on a cubic surface. Some properties of this cubic surface are given. In the second generalization we take a skew quadrilateral instead of a

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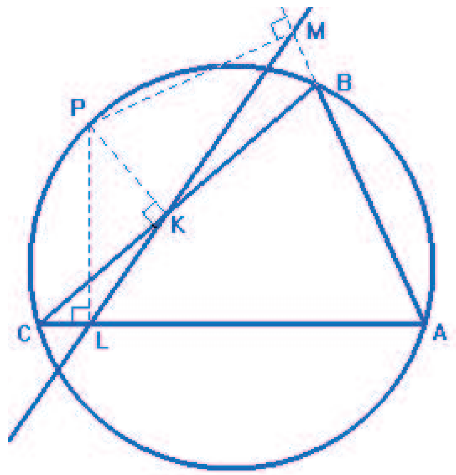


Figure 1: The "classical" Simson-Wallace Theorem

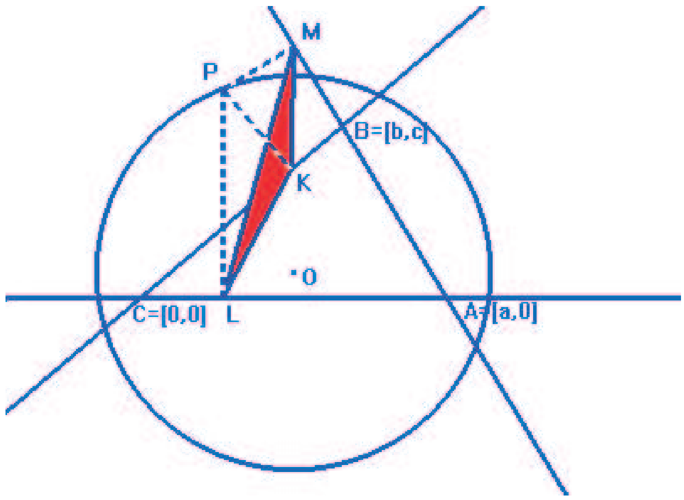


Figure 2: GERGONNE's generalization — the triangle $KLM$ has fixed area

tetrahedron and explore points, whose orthogonal projections on the sides of a quadrilateral are coplanar. In both cases we use the methods of commutative algebra based on Gröbner basis computations and automatic theorem proving (see [3]). Also a synthetic way of solving the above problems is shown.

The paper is organized as follows. At the beginning the basic algebraic tools of the theory of automatic theorem proving are introduced. Then some known generalizations of the Simson-Wallace Theorem in the plane are given ([6, 5]. Finally the two generalizations of the Simson-Wallace Theorem into the space are described.

All the computations were done on Intel Pentium 2.00GHz/1572MB RAM using the computer algebra system CoCoA 4.3[1] and Maple 8 and 9.5. The figures were produced with the aid of Maple and the dynamic geometry software Cabri II.

## 2. Basic algebraic tools in automatic theorem proving

One of the most useful applications of Gröbner bases is automated theorem proving. Many non trivial theorems have been proved and even discovered by this theory. In the last 20 years of the last century efficient methods were developed for automatic theorem proving of theorems from elementary geometry. In this section we will give a brief overview of the theory of automatic theorem proving (see [2, 16, 8, 3, 15, 12]).

*Automated theorem proving* treats statements of the kind $\mathbf{H} \Rightarrow \mathbf{T}$, where $\mathbf{H}$ is the set hypotheses and $\mathbf{T}$ the set of theses or conclusions. We are to decide whether the statement is true or not.

- In the first step of automatic proving theorems we *algebraize* the geometric problem. To do this we have to specify a coordinate system and by means of variables to express the relations between geometric objects and geometric magnitudes like areas, squares of distances (to avoid radicals) etc. This stage is characterized by establishing the set of hypotheses in the form of polynomial equations

$$h_1(x_1, x_2, \ldots, x_n) = 0,\ h_2(x_1, x_2, \ldots, x_n) = 0,\ \ldots,\ h_r(x_1, x_2, \ldots, x_n) = 0$$

[1]The software CoCoA is freely distributed at http://cocoa.dima.unige.it .

and the thesis, which is expressed by the polynomial equation

$$t(x_1, x_2, \dots, x_n) = 0.$$

After the first step the statement has the form

$$\forall x \in \mathbb{C}^n, \quad h_1(x) = 0,\ h_2(x) = 0,\ \dots,\ h_r(x) = 0 \quad \Longrightarrow \quad t(x) = 0, \tag{1}$$

where $h_1, h_2, \dots, h_r \in \mathbb{Q}[x_1, x_2, \dots, x_n]$, $\mathbb{C}$ is the field of complex numbers, and $\mathbb{Q}[x_1, x_2, \dots, x_n]$ is the ring of polynomials with rational coefficients. If the thesis consists of more polynomial equations we will solve each of them separately.

It can happen that such a statement is not true because of the absence of so called *non degeneracy conditions*.

- The second step is characterized by finding conditions under which the statement becomes meaningless, e.g., a triangle collapses to the segment, the segment to the point etc., i.e.,

$$g_1(x_1, x_2, \dots, x_n) \neq 0,\ g_2(x_1, x_2, \dots, x_n) \neq 0,\ \dots,\ g_s(x_1, x_2, \dots, x_n) \neq 0.$$

Then the geometric statement can be translated into the form

$$\forall x \in \mathbb{C}^{\mathsf{K}}, \quad h_1(x) = 0, \dots, h_r(x) = 0,\ g_1(x) \neq 0, \dots,\ g_s(x) \neq 0 \quad \Longrightarrow \quad t(x) = 0. \tag{2}$$

- The third step involves the *verification* of (2). The hypothesis variety $H$ is the set of all solutions of the system

$$h_1 = 0,\ \dots,\ h_r = 0,\ g_1 \neq 0,\ \dots,\ g_s \neq 0. \tag{3}$$

The thesis variety $T$ is the set of all solutions of $t = 0$. The statement (2) is true if $H$ is contained in $T$. By Hilbert's Nullstellensatz the statement (2) is true iff 1 belongs to the ideal

$$J(h_1, \dots, h_r, g_1 t_1 - 1, \dots, g_s t_s - 1, ct - 1),$$

where $t_1, t_2, \dots, t_s, t$ are slack variables. In practice it usually suffices to show that $t$ belongs to the ideal

$$I(h_1, \dots, h_r, g_1 t_1 - 1, \dots, g_s t_s - 1).$$

With *automatic deriving* we mean finding geometric formulas holding among prescribed geometric magnitudes which follow from given assumptions.

On the other hand, *automatic discovery* stands for searching complementary assumptions which are necessary to add to the geometric statement (which is in general not valid), so that it becomes true. The discovery of loci belongs to automatic discovery; here we search for the "unknown" locus of points. This method will be demonstrated at Guzman's generalization of the Simson-Wallace Theorem in the plane and then by generalizations in the space.

## 3. Generalizations of the Simson-Wallace Theorem in the plane

Let $K, L, M$ be the feet of perpendiculars dropped from a point $P$ to the sides $AB, BC, CA$ of the triangle $ABC$, respectively. Instead of demanding $K, L, M$ being collinear we look for points $P$ leading to a pedal triangle $KLM$ of fixed area. The locus of such points $P$ is a circle, due to J. D. Gergonne (see [2]). His theorem reads as follows:

*Let $ABC$ be a triangle and $P$ a point of a circle which is concentric with the circumcircle of $ABC$. Then the feet of perpendiculars from $P$ onto the sides of $ABC$ form a triangle of the constant area $f$ (Fig. 2).*

In the previous cases we projected a point $P$ orthogonally to each side of a triangle $ABC$ to obtain the points $K, L, M$. Now we will project a point $P$ onto the sides $BC$, $AC$, $AB$ of a triangle $ABC$ in three *arbitrary* directions $u, v, w$ given by vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ to obtain the points $K, L, M$, respectively. We exclude the case when all the three directions $u, v, w$ are parallel (in this case points $P$ fill the whole plane) and the case that the directions $u, v, w$ are parallel to the sides $BC, AC, AB$, respectively. We will investigate the locus of points $P$ such that the triangle $KLM$ has a fixed area $s$ (see [6], where the problem is solved in a synthetic way).

Let us choose the Cartesian system of coordinates so that

$$A = [a, 0], \quad B = [b, c], \quad C = [0, 0], \quad P = [p, q],$$
$$K = [k_1, k_2], \quad L = [l_1, l_2], \quad M = [m_1, m_2],$$
$$\mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2), \quad \mathbf{w} = (w_1, w_2).$$

From $K = P + t_1\mathbf{u}$, $L = P + t_2\mathbf{v}$, $M = P + t_3\mathbf{w}$, $K = C + s_1(B - C)$, $L = C + s_2(A - C)$, $M = A + s_3(B - A)$, where $t_1, t_2, t_3, s_1, s_2, s_3$ are real parameters, we get the system of equations

$$\begin{array}{llll} h_1\colon\ k_1 = p + t_1u_1, & h_2\colon\ k_2 = q + t_1u_2, & h_3\colon\ l_1 = p + t_2v_1, & h_4\colon\ l_2 = q + t_2v_2, \\ h_5\colon\ m_1 = p + t_3w_1, & h_6\colon\ m_2 = q + t_3w_2, & h_7\colon\ k_1 = s_1b, & h_8\colon\ k_2 = s_1c, \\ h_9\colon\ l_1 = s_2a, & h_{10}\colon\ l_2 = 0, & h_{11}\colon\ m_1 = a + s_3(b - a), & h_{12}\colon\ m_2 = s_3c. \end{array}$$

The conclusion $h_{13}$ is given by

$$\text{area of } KLM = s \iff h_{13}\colon\ 2s = k_1l_2 + l_1m_2 + m_1k_2 - m_1l_2 - k_1m_2 - l_1k_2.$$

It is obvious that in general $h_{13}$ doesn't follow from the assumptions $h_1, h_2, \dots, h_{12}$ .Hence we will add the conclusion polynomial $h_{13}$ to $h_1, h_2, \dots, h_{12}$ and eliminate the dependent variables $k_1, k_2$, $l_1, l_2$, $m_1, m_2$, $t_1, t_2, t_3$, $s_1, s_2, s_3$ from the ideal $I = (h_1, h_2, \dots, h_{13})$. We enter (in CoCoA)

```
UseR::=Q[abcpqfu[1..2]v[1..2]w[1..2]k[1..2]l[1..2]m[1..2]t[1..3]s[1..3]];
I:=Ideal(k[1]-p-t[1]u[1],k[2]-q-t[1]u[2],l[1]-p-t[2]v[1],l[2]-q-t[2]v[2],m[1]
-p-t[3]w[1],m[2]-q-t[3]w[2],k[1]-s[1]b,k[2]-s[1]c,l[1]-s[2]a,l[2],m[1]-a-s[3]
(b-a),m[2]-s[3]c,k[1]l[2]+l[1]m[2]+m[1]k[2]-m[1]l[2]-k[1]m[2]-l[1]k[2]-2s);
Elim(k[1]..s[3],I);
```

and get a single algebraic equation of second degree in $p, q$

$$C(s) = 0, \tag{4}$$

where

$$\begin{aligned} C(s) \;=\;& c^2v_2p^2(u_1w_2 - u_2w_1) + cpq(cu_2v_1w_1 - au_2v_2w_1 + bu_2v_2w_1 - cu_1v_1w_2 + \\ &+ au_2v_1w_2 - bu_1v_2w_2) + cq^2(-bu_2v_1w_1 + au_1v_2w_1 - au_1v_1w_2 + bu_1v_1w_2) + \\ &+ ac^2v_2p(u_2w_1 - u_1w_2) + acq(-cu_1v_2w_1 + cu_1v_1w_2 - bu_2v_1w_2 + bu_1v_2w_2) + \\ &+ 2v_2s(cu_1 - bu_2)(cw_1 + aw_2 - bw_2). \end{aligned}$$

As the constant $s$ occurs only in the last term of (4) we can write

$$C(s) = C(0) + s \cdot Q, \tag{5}$$

where $Q = 2v_2(cu_1 - bu_2)(cw_1 + w_2(a - b))$. We have proved that (4) is a necessary condition for the area of $KLM$ being $s$.

Now we shall prove that the condition (4) is also sufficient:
By the HILBERT's Nullstellensatz we are to prove that the polynomial $h_{13}$ belongs to the radical ideal of $(h_1, h_2, \dots, h_{12}, C(s))$ or — which is equivalent — that 1 belongs to the ideal $J = (h_1, h_2, \dots, h_{12}, C(s), h_{13}t - 1)$ where $t$ is a slack variable. We compute the normal form $\mathtt{NF}$ of 1 with respect to the ideal $J$ and get $\mathtt{NF}(1, J) = 1$, so it not known whether the condition (4) is sufficient for the area of $KLM$ being $s$ or not. The reason is that most geometric theorems are generically true.

We shall search for non-degeneracy conditions: Eliminating all dependent variables $k_1, k_2$, $l_1, l_2$, $m_1, m_2$, $t_1, t_2, t_3$, $s_1, s_2, s_3$ plus a slack variable $t$ in the ideal $J$ we get the condition

$$d\colon\; v_2(cu_1 - bu_2)(cw_1 + w_2(a - b)) = 0,$$

which means that at least one of directions $u, v, w$ is parallel to the sides $BC, AC, AB$, respectively. In order to avoid this, we add the polynomial $dr - 1$ to the ideal $J$ and compute the normal form of 1 w.r.t. the ideal $J' = J \cup \{dr - 1\}$, where $r$ is another slack variable. We obtain $\mathtt{NF}(1, J') = 0$, i.e., the condition (4) is sufficient for the area of $KLM$ being $s$.

We thus arrived at the theorem which is due to M. DE GUZMÁN [6]

*Project $P$ onto the sides $BC, AC, AB$ of a triangle $ABC$ in given directions $u, v, w$, which are not parallel to the sides $BC, AC, AB$, onto the points $K, L, M$, respectively. Then the locus of points $P$ such that the area of the triangle $KLM$ equals $s$ is a conic $C(s)$ given by (4).*

We see that this generalization confines the previous cases. A demonstration of this generalization is carried out with the dynamic geometry software Cabri II (see Fig. 3).

Suppose that $c \neq 0$ and $a \neq 0$, i.e., $A, B, C$ are not collinear and $A \neq B$. The family of conics $C(s)$ for a given $s$ and arbitrary directions $u, v, w$ has interesting properties, which follow from (5). Let us recall some of them [6, 5]:

a) $C(0)$ *passes through the vertices $A, B, C$, i.e., $C(0)$ is a circumconic of $ABC$.*

b) *Varying the area $s$ the curves $C(s)$ to given directions $u, v, w$ form a pencil of homothetic central conics (ellipses or hyperbolas with common axes) or a pencil of congruent parabolas with common axis.*

c) $C(0)$ *is singular if and only if two of the directions $u, v, w$ are parallel.*

d) *If all the directions $u, v, w$ are pairwise different then $C(s)$ is a regular conic.*

e) $C(0)$ *passes through the points $A', B', C'$, where $A' = A'B \cap A'C$, $B' = B'A \cap B'C$ and $C' = C'A \cap C'B$, where $A'B \parallel B'A \parallel w$, $A'C \parallel C'A \parallel v$ and $B'C \parallel C'B \parallel u$.*

Fig. 3 shows an ellipse and a hyperbola as the locus of points $P$ such that $\triangle KLM$ has the area $s$ for different choices of directions $u, v, w$.

<u>*Remarks:*</u> 1) In [5] affine and projective generalization of the Simson-Wallace Theorem have been introduced. If the affine feet $K, L, M$ lie on the affine Wallace line of $P$ with respect to a center $Z$ or if the projective feet $K, L, M$ lie on the projective Wallace line of $P$ with respect to a center $Z$ and an axis $z$ then $P$ lies on a conic, which depends on two parameters given be coordinates of the center $Z$.

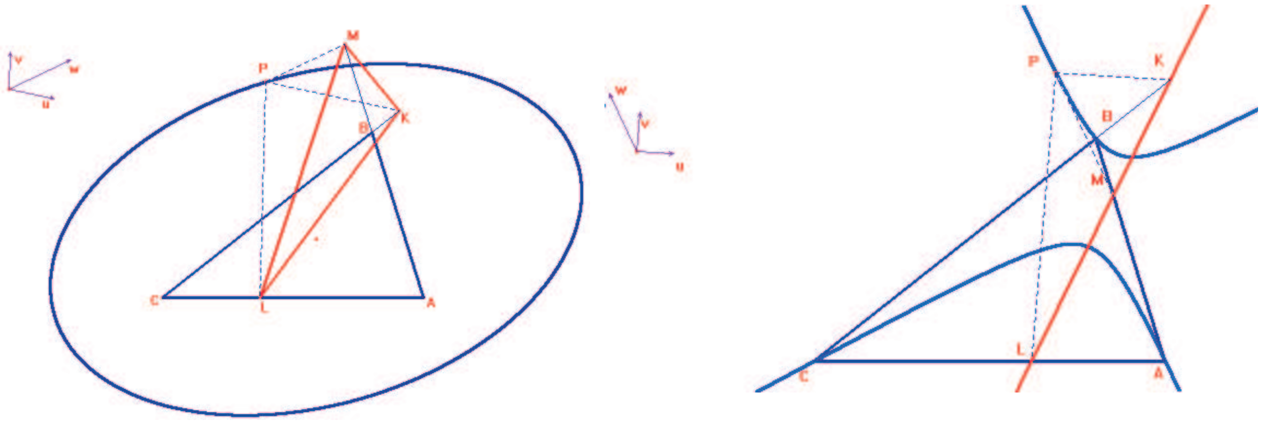


Figure 3: Ellipse and hyperbola for various kinds of $u, v, w$

2) By a given triangle $ABC$ the conic (4) is defined by seven parameters $u_1, u_2, v_1, v_2, w_1, w_2$ and by $s$, but in fact four parameters are enough setting $u_2 = v_2 = w_2 = 1$. Briefly we could write $C(u, v, w, s)$. 3) In the classical Simson-Wallace Theorem the directions $u, v, w$ and the directions of the sides of the given triangle form three pairs of an involutoric projectivity, which includes the so called 'absolute involution' $\iota$ on the ideal line of the plane. Like any circle in plane, the circles of Gergonne's extension of the Simson-Wallace Theorem pass through the imaginary fixed points of $\iota$. Choosing $u, v, w$ arbitrarily, the above mentioned pairs of directions define an elliptic or hyperbolic or parabolic projectivity $\pi$ on the ideal line of the plane and the solution conics $C(s)$ of Guzmán's generalization pass through the fixed points points of this projectivity $\pi$. As a consequence, triplets $(u_i, v_i, w_i)$ defining projectivities $\pi_i$ with common fixed points lead to identical sets of conics $\{C_i(s)\}$.

## 4. Generalization to three dimensions

In this part we generalize the Simson-Wallace Theorem to the space. We will show two generalizations. First we extend the Simson-Wallace Theorem to the space considering a tetrahedron $ABCD$ instead of a triangle and arbitrary projections $K, L, M, N$ of a point $P$ onto the faces of $ABCD$ such that $\text{vol}(KLMN) = s$. Then we will replace a tetrahedron by a skew quadrilateral and investigate the same problem.

### 4.1. Generalization of the Simson-Wallace Theorem on a tetrahedron

Consider a tetrahedron $ABCD$ in a Euclidean space $E^3$. Let $P$ be an arbitrary point and $K, L, M, N$ the feet of perpendiculars dropped from $P$ onto the faces $BCD$, $ACD$, $ABD$, and $ABC$ of the tetrahedron $ABCD$, respectively. We are looking for the locus of points $P$ such that $\text{vol}(KLMN) = s$ (cf. [11]), where the same problem is solved for a special class of tetrahedra (in our notation) $A = [0, 0, 0]$, $B = [1, 0, 0]$, $C = [0, c, 0]$, $D = [1, e, f]$.

Choose the Cartesian system of coordinates such that

$$A = [0, 0, 0], \quad B = [a, 0, 0], \quad C = [b, c, 0], \quad D = [d, e, f], \quad P = [p, q, r],$$
$$K = [k_1, k_2, k_3], \quad L = [l_1, l_2, l_3], \quad M = [m_1, m_2, m_3], \quad N = [n_1, n_2, n_3].$$

Then the following relations hold:

$$\begin{array}{lcl}
PK \perp BCD & \iff & h_1\colon\ (b-a)(p-k_1)+c(q-k_2)=0 \ \wedge \\
 & & h_2\colon\ (d-a)(p-k_1)+e(q-k_2)+f(r-k_3)=0, \\
k \in BCD & \iff & h_3\colon\ -acf-aek_3+afk_2+ack_3+cfk_1+bek_3+cdk_3-bfk_2=0, \\
PL \perp ACD & \iff & h_4\colon\ b(p-l_1)+c(q-l_2)=0 \ \wedge \ h_5\colon\ d(p-l_1)+e(q-l_2)+f(r-l_3)=0, \\
L \in ACD & \iff & h_6\colon\ cfl_1+bel_3-cdl_3-bfl_2=0, \\
PM \perp ABD & \iff & h_7\colon\ a(p-m_1)=0 \ \wedge \ h_8\colon\ d(p-m_1)+e(q-m_2)+f(r-m_3)=0, \\
M \in ABD & \iff & h_9\colon\ aem_3-afm_2=0, \\
PH \perp ABC & \iff & h_{10}\colon\ a(p-n_1)=0 \ \wedge \ h_{11}\colon\ b(p-n_1)+c(q-n_2)=0, \\
N \in ABC & \iff & h_{12}\colon\ acn_3=0.
\end{array}$$

The conclusion $h_{13}\colon$ vol$(KLMN)=s \iff$

$$h_{13}\colon \begin{vmatrix} k_1 & k_2 & k_3 & 1 \\ l_1 & l_2 & l_3 & 1 \\ m_1 & m_2 & m_3 & 1 \\ n_1 & n_2 & n_3 & 1 \end{vmatrix} = 6s. \tag{6}$$

A direct elimination of the dependent variables $k_1$, $k_2$, $k_3$, $l_1$, $l_2$, $l_3$, $m_1$, $m_2$, $m_3$, $n_1$, $n_2$, $n_3$ from the ideal $I=(h_1,h_2,\dots,h_{12},h_{13})$ fails. Hence we use the following *successive* elimination. First eliminate $m_1,m_2,m_3,n_1,n_2,n_3$ in the ideal $(h_7,\dots,h_{13})$ to obtain the elimination ideal generated by the only polynomial $p_1$. Then eliminate $l_1,l_2,l_3$ in the ideal $(h_4,h_5,h_6,p_1)$. We get the elimination ideal with one generator $p_2$. In the end we eliminate $k_1,k_2,k_3$ in the ideal $(h_1,h_2,h_3,p_2)$ to obtain the single condition

$$F(s)=ac^2f^3G+s\cdot Q, \tag{7}$$

where

$$\begin{aligned}
G \;=\; & bf^2q^3(b-a)+fr^3(abe-acd+cd^2-b^2e-c^2e+ce^2)+c^2f^2p^2q+ \\
& +cfp^2r(e^2-ce+f^2)+cf^2q^2p(a-2b)+fq^2r(abe-acd+cd^2-b^2e+cf^2)+ \\
& +cf^2r^2p(a-2d)+f^2r^2q(b^2-ab+c^2-2ce)+2cefpqr(b-d)+abcf^2q^2+ \\
& +r^2(abce^2-ac^2de+c^2d^2e+acde^2-2bcde^2-abe^3+b^2e^3+acdf^2-abef^2+ \\
& +b^2ef^2+c^2ef^2)-ac^2f^2pq+acfpr(ce-e^2-f^2)+ \\
& +fqr(ac^2d-2abce-c^2d^2+2bcde-b^2e^2+abe^2+abf^2-b^2f^2-c^2f^2)
\end{aligned}$$

and

$$\begin{aligned}
Q \;=\; & -6(e^2+f^2)\left((cd-be)^2+b^2f^2+c^2f^2\right)\left(a^2c^2-2ac^2d+c^2d^2-2a^2ce+2abce+\right. \\
& \left.+2acde-2bcde+a^2e^2-2abe^2+b^2e^2+a^2f^2-2abf^2+b^2f^2+c^2f^2\right),
\end{aligned}$$

which is a constant which doesn't depend on $p,q,r,s$.

We established the following

**Theorem 1** *Let $P$ be an arbitrary point and $K,L,M,N$ the feet of perpendiculars dropped from $P$ onto the faces $BCD$, $ACD$, $ABD$, $ABC$ of a tetrahedron $ABCD$, respectively. Then the locus of points $P$ such that the tetrahedron $KLMN$ has constant volume $s$ belong to the surface $F(s)=0$ from (7).*

Remarks: 1) $Q$ can be written in the form

$$Q = -6 \cdot \frac{1}{a^2} \cdot |(B-A)\times(D-A)|^2 \cdot |(D-A)\times(C-A)|^2 \cdot |(D-B)\times(C-B)|^2. \quad (8)$$

2) Note that (7) is the necessary condition for the tetrahedron $KLMN$ having constant volume. We didn't succeed to show that (7) is (in this general form) also sufficient for the moment. Hence we do not know whether every point of the surface $F(s) = 0$ obeys the conditions of the theorem. In [11] a similar problem for a non general tetrahedron is solved by Wu's method [16], which is based on pseudodivision. In this way a necessary and sufficient condition was found in accordance with our results. We will show that for concrete values $a, b, c, d, e, f$ we are able to verify sufficiency as well.

Now we list some properties of a surface $F(s) = 0$ for $s$ being zero, i.e., when $K, L, M, N$ are coplanar. From (7) follows $F(0) = 0 \iff G = 0$.

**Theorem 2** *The surface $G$ has the following properties:*

- a) *$G$ contains the edges $AB, AC, AD, BC, BD, CD$ of $ABCD$, i.e., $G$ is a circumsurface of $ABCD$.*
- b) *$G$ is a cubic surface.*
- c) *$G$ has 4 singular points — the vertices $A, B, C, D$ of the tetrahedron.*
- d) *The point of intersection between three planes which contain, e.g., the edges $AB$, $BD$, $DA$ and which are perpendicular to the planes $ABC$, $BDC$, $DAC$, respectively, belongs to the surface $G$. Similarly we will proceed for other triples of edges.*
- e) *The lines $AB, AC, AD, BC, BD, CD$ are torsal lines of the cubic $G$, i.e., the tangent plane at an arbitrary point of the line contains the whole line. The tangent planes at three pairs of opposite edges intersect at three other straight lines which are coplanar. Each of these three lines intersects the pair of corresponding skew torsal lines.*
- f) *There exists a simple rational parametrization of $G$.*

*Proof:* a) Note that an arbitrary point $P$ of an edge coincides with *two* points from the feet of perpendiculars $K, L, M, N$, which are then coplanar. The other way to verify this is a direct computation.
b) This statement follows from the fact, that the cubic surface contains all the six edges of a tetrahedron $ABCD$ (see [1]).
c) If all the cubic terms in the equation $G = 0$ vanish, then the surface is a quadric, which is not possible, because the surface contains all six edges of $ABCD$.
d) This follows from the fact that in this case all the feet $K, L, M, N$ lie in the plane $ABD$.
e) For the proof see [1, pp. 567–568].
f) Let $X = A + t\mathbf{u}$ be a straight line with $\mathbf{u} = (u, v, 1)$. Then, because of $A$ being a double point, it intersects the cubic surface $G$ at most at one point $X(u, v, 1)$ (see Example 1). □

Remarks: 1) It is well known that every cubic surface (in the complex projective space) contains 27 lines. In this case we have $6 \times 4 + 3 = 27$ lines, because each edge is counted four times (see [1, 7, 10].
2) To prove Theorem 1 we could also proceed in the more synthetic way as Guzmán did in plane [6]. The feet $K, L, M, N$ form a tetrahedron of fixed volume $s$. Hence $K, L, M, N$ fulfil

the formula (6). The coordinates of points $K, L, M, N$ being intersections of perpendiculars from $P = [p, q, r]$ with the faces of $ABCD$ are *linear* in $p, q, r$. Thus (6) is a cubic algebraic equation in $p, q, r$, which has in general 20 real coefficients. To determine these coefficients we need at least 19 points of the surface. We know that each edge of $ABCD$ contains two double points, which makes together $4 + 6 \times 2 = 16$ points. It remains to determine the last 3 points. A construction of such points follows from d) in Theorem 2.

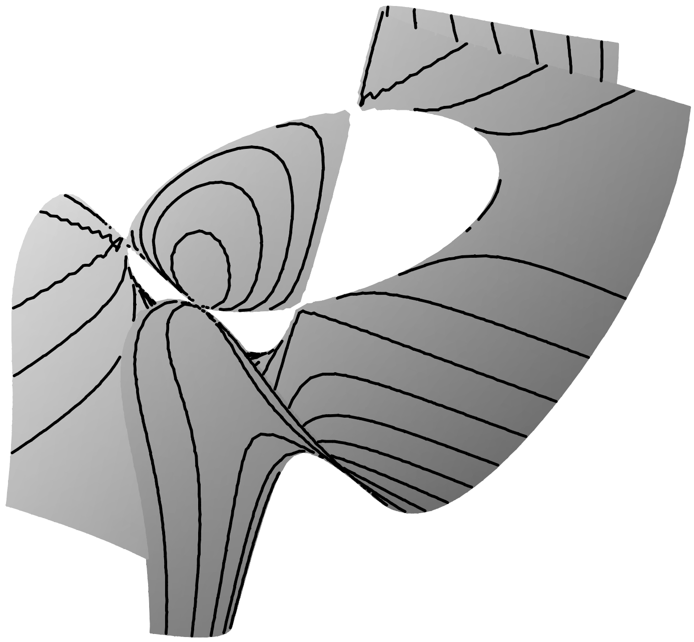

Figure 4: Cubic surface $p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr = 0$ as the locus of points $P$ with coplanar feet in planes of a (special) tetrahedron

Example 1: For special values $a = 1,\ b = 0,\ c = 1,\ d = 0,\ e = 0,\ f = 1,\ s = 0$ we get from (7)

$$p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr = 0, \tag{9}$$

see Fig. 4.
First we will prove that (9) is also a sufficient condition for $K, L, M, N$ being coplanar (see Example 3), where a detailed computation is carried out.
This surface can be easily parametrized taking into account that the surface has 4 double points. Putting $p = ur, q = vr, r = r$ and setting this into (9) we get

$$\begin{gathered} p = \frac{u(u + uv + v)}{u^2v + u^2 + uv^2 + v^2 + u + v}, \qquad q = \frac{v(u + uv + v)}{u^2v + u^2 + uv^2 + v^2 + u + v} \\ r = \frac{u + uv + v}{u^2v + u^2 + uv^2 + v^2 + u + v} \end{gathered} \tag{10}$$

for real $u, v$.

Example 2: The choice $a = 2,\ b = 1,\ c = \sqrt{3},\ d = 1,\ e = 1/\sqrt{3},\ f = \sqrt{8/3}$ with the centroid of $ABCD$ in the origin gives for an arbitrary $s$ a one-parametric system of surfaces which are associated with a regular tertrahedron (writing $x, y, z$ instead of $p, q, r$):

$$\begin{gathered} 24\sqrt{6}x^2y + 24\sqrt{3}x^2z + 24\sqrt{3}y^2z - 8\sqrt{6}y^3 - 16\sqrt{3}z^3 + \\ + 36\sqrt{2}x^2 + 36\sqrt{2}y^2 + 36\sqrt{2}z^2 - 18\sqrt{2} - 729s = 0. \end{gathered} \tag{11}$$

For $s=0$ we obtain the locus of points $P$ such that the feet $K,L,M,N$ are coplanar. This cubic surface has the vertices $A,B,C,D$ as the only singular points.
For $s\neq 0$ all the cubic surfaces associated with a regular tetrahedron $ABCD$ don't contain singular points unless the value $s=-18\sqrt{2}/729$. This leads to the cubic surface

$$6\sqrt{6}x^2y+6\sqrt{3}x^2z+6\sqrt{3}y^2z-2\sqrt{6}y^3-4\sqrt{3}z^3+9\sqrt{2}x^2+9\sqrt{2}y^2+9\sqrt{2}z^2=0, \qquad (12)$$

with one singular point — an isolated point placed in the centroid of $ABCD$.
In Fig. 5 we see the cubic surface (11) associated with a regular tetrahedron $ABCD$ for $s=10\sqrt{2}/729$.

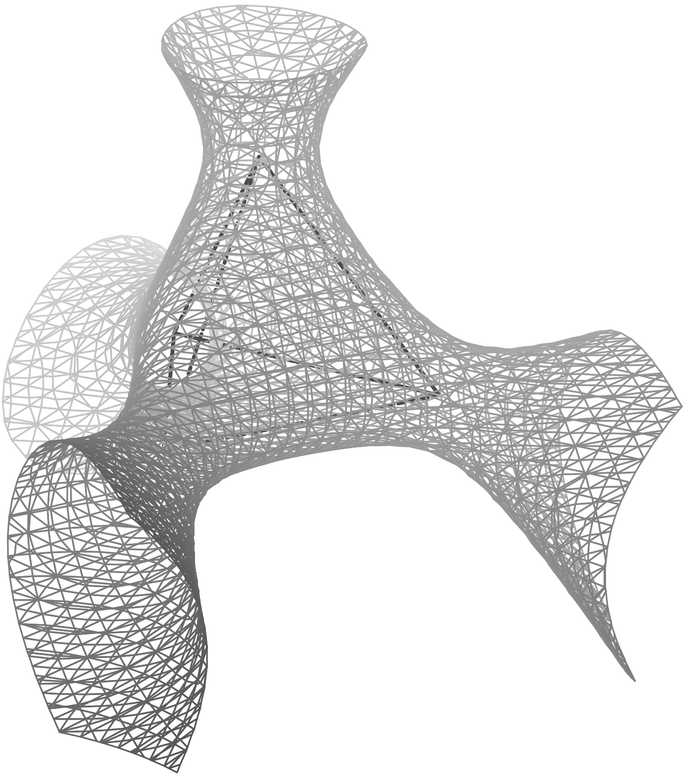

Figure 5: Cubic surface associated with a regular tetrahedron as the locus of points $P$ with constant volume $s=10\sqrt{2}/729$ of the tetrahedron $KLMN$

### 4.2. Generalization of the Simson-Wallace Theorem on skew quadrilaterals

Another generalization of the Simson-Wallace Theorem consists in a consideration of a skew quadrilateral $ABCD$ in $E^3$ instead of a tetrahedron. Denote by $K,L,M,N$ the feet of perpendiculars which are dropped from a point $P$ to the sides $AB$, $BC$, $CD$ $DA$ of a quadrilateral $ABCD$, respectively. We are to find the locus of points $P$ such that feet $K,L,M,N$ are coplanar.

Choose the Cartesian system of coordinates such that

$$A=[0,0,0],\quad B=[a,0,0],\quad C=[b,c,0],\quad D=[d,e,f],$$
$$P=[p,q,r],\quad K=[k_1,0,0],\quad L=[l_1,l_2,0],\quad M=[m_1,m_2,m_3],\quad N=[n_1,n_2,n_3].$$

The conditions are as follows:

$$\begin{array}{rcl}
PK\perp AB & \Longleftrightarrow & h_1:\ p-k_1=0,\\
L\in BC & \Longleftrightarrow & h_2:\ l_2(b-a)-c(l_1-a)=0,\\
PL\perp BC & \Longleftrightarrow & h_3:\ (p-l_1)(b-a)+c(q-l_2)=0,\\
M\in CD & \Longleftrightarrow & h_4\dots h_6:\ d-b)(m_2-c)-(e-c)(m_1-b)=0\ \wedge\\
 & & (e-c)m_3-(m_2-c)f=0\ \wedge\ (m_1-b)f-m_3(d-b)=0,\\
PM\perp CD & \Longleftrightarrow & h_7:\ (p-m_1)(d-b)+(q-m_2)(e-c)+(r-m_3)f=0,\\
N\in DA & \Longleftrightarrow & h_8\dots h_{10}:\ dn_2-en_1=0\ \wedge\ dn_3-fn_1=0\\
 & & \wedge\ fn_2-en_3=0,\\
PN\perp DA & \Longleftrightarrow & h_{11}:\ (p-n_1)d+(q-n_2)e+(r-n_3)f=0,\\
\text{coplanar } K,L,M,N & \Longleftrightarrow & h_{12}:\ k_1l_2m_3-l_2m_3n_1-k_1m_3n_2+l_1m_3n_2-\\
 & & -k_1l_2n_3+l_2m_1n_3+k_1m_2n_3-l_1m_2n_3=0.
\end{array}$$

The successive elimination (eliminate $n_1, n_2, n_3$ first and then $m_1, m_2, m_3$ etc.) of the 9 variables $k_1, \dots, n_3$ from the ideal $(h_1, h_2, \dots, h_{11}, h_{12})$ gives the equation $H = 0$ of a cubic surface $H$ which involves 176 terms:

$$\begin{gathered}
H := p^3(cd(c(-a+d)+2e(a-e))-b(e^2+f^2)(a-b))+cp^2q(ae(-c+e)-f^2(a-2b))+\\
+cfp^2r(-c(a-2d)+2e(a-b))+p^2(a^2c^2d-c^2d^3-2a^2cde+abcde+b^2cde+c^3de-acd^2e+\\
+bcd^2e+a^2be^2-b^3e^2+ac^2e^2-bc^2e^2-c^2de^2-ace^3+bce^3+a^2bf^2-b^3f^2+ac^2f^2-\\
-bc^2f^2-c^2df^2-acef^2+bcef^2)+pq^2(-ac^2d+c^2d^2+2acde-2bcde-abe^2+b^2e^2+\\
+c^2f^2)+pq(c(d^2+e^2+f^2)(ab-b^2-c^2)+e(d^2+e^2+f^2-bd-ce)(ab-b^2-c^2)+\\
+ef^2(-ab+b^2+c^2)-a^2bcd+ab^2cd+ac^3d+a^2cd^2-abcd^2-acd^3+bcd^3+a^2c^2e-\\
-ac^2de+c^2d^2e-a^2ce^2-acde^2+bcde^2+c^2e^3+a^2cf^2+abef^2-2abcf^2-acdf^2+cf^2bd-\\
-ef^2b^2)+2fpqr(cd-be)(a-b)+p(-a^2c^2d^2+ac^2d^3+a^2bcde-ab^2cde-ac^3de+a^2cd^2e-\\
-abcd^2e-a^2b^2e^2+ab^3e^2-a^2c^2e^2+abc^2e^2+ac^2de^2+a^2ce^3-abce^3-a^2b^2f^2+ab^3f^2-\\
-a^2c^2f^2+abc^2f^2+ac^2df^2+a^2cef^2-abcef^2)+f^2pr^2(-ab+b^2+c^2)+\\
+fpr(ac(c(a-2d)-2e(a-b))-(bd-d^2+ec-e^2-f^2)(ab-b^2-c^2)))+aceq^3(-c+e)+\\
+acfq^2r(-c+2e)+aeq^2((-c+e)(ab-b^2-c^2)+c(bd-d^2+ce-e^2-f^2))+\\
+qr^2(acf^2)+afqr((-c+2e)(ab-b^2-c^2)+c(bd-d^2+ce-e^2-f^2))+\\
+aeq(bd-d^2+ce-e^2-f^2)(ab-b^2-c^2)+afr^2(ab-b^2-c^2)+\\
+afr(bd-d^2+ce-e^2-f^2)(ab-b^2-c^2)=0
\end{gathered}$$

The equation $H = 0$ gives a necessary condition for $P = [p, q, r]$ such that feet $K, L, M, N$ are coplanar. We can state

**Theorem 3** *Let $P$ be an arbitrary point and $K, L, M, N$ the feet of perpendiculars dropped from $P$ onto the sides $AB$, $BC$, $CD$, $DA$ of a skew quadrilateral $ABCD$, respectively. Then points $P = [p, q, r]$ such that $K, L, M, N$ are coplanar obey the equation $H = 0$.*

*Remark:* The author failed for the moment in proving that the condition $H = 0$ (in this general form) is also sufficient for $K, L, M, N$ being coplanar. In concrete cases (as in the next example) this verification has been done.

*Example 3:* For the choice $a = 1$, $b = 0$, $c = 1$, $d = 0$, $e = 0$, $f = 1$ we get the cubic surface

$$-p^2q+pq^2-p^2r-q^2r+pr^2+qr^2+p^2-r^2-p+r=0, \tag{13}$$

or after factorization

$$(p-r)(pq-q^2+pr+qr-p-r+1)=0.$$

Thus the cubic surface decomposes into the plane and one sheet hyperboloid (see Fig. 6).

The verification that (13) is also sufficent for $K, L, M, N$ being coplanar is as follows:

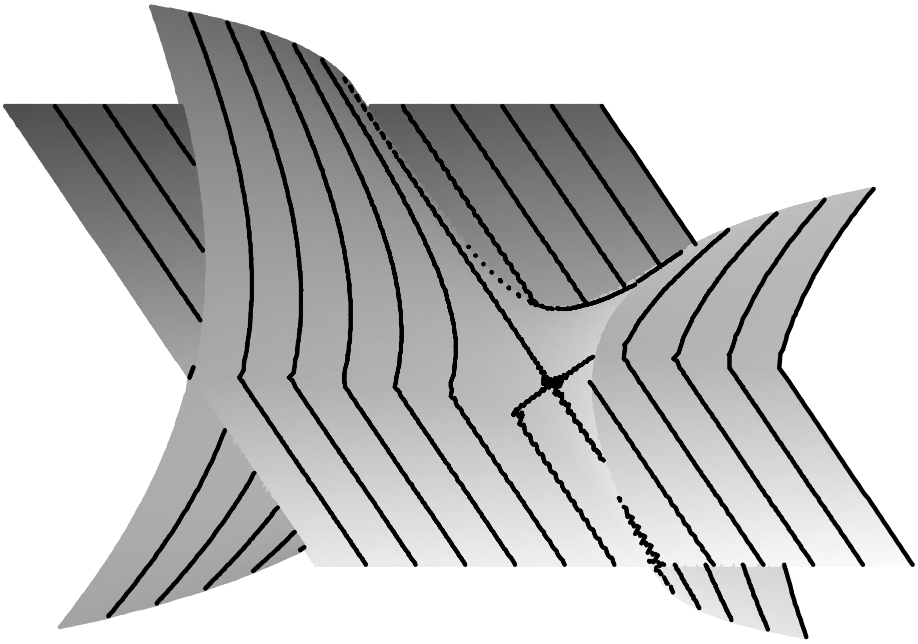

Figure 6: Plane and quadric $(p-r)(pq-q^2+pr+qr-p-r+1)=0$ as the locus of points $P$ with coplanar feet on the edges of a skew quadrilateral

```
Use R::=Q[abcdefpqrk[1..3]l[1..3]m[1..3]n[1..3]tuv];
I:=Ideal(p-k[1],l[2](b-a)-c(l[1]-a),(p-l[1])(b-a)+c(q-l[2]),(d-b)(m[2]-c)-(e-c)
(m[1]-b),(e-c)m[3]-(m[2]-c)f,(m[1]-b)f-m[3](d-b),(p-m[1])(d-b)+(q-m[2])(e-c)+
(r-m[3])f,dn[2]-en[1],dn[3]-fn[1],fn[2]-en[3],(p-n[1])d+(q-n[2])e+(r-n[3])f,a-1,
b,c-1,d,e,f-1,-p^2q+pq^2-p^2r-q^2r+pr^2+qr^2+p^2-r^2-p+r,(k[1]l[2]m[3]-l[2]m[3]
n[1]-k[1]m[3]n[2]+l[1]m[3]n[2]-k[1]l[2]n[3]+l[2]m[1]n[3]+k[1]m[2]n[3]-
l[1]m[2]n[3])t-1);
NF(1,I);
```

The answer NF$=0$ follows immediately.

*Remarks:* 1) The generalization above stimulates immediately the following question: The 6 edges of a tetrahedron allow three possibilities of skew edge quadrilaterals. How are the three solution surfaces (Fig. 6) of these three possibilities related?

2) For the construction of the generalization above it is not essential that the four edges form a skew quadrilateral. One could equally treat the case of four skew given lines $a, b, c, d$ and ask for coplanar pedal points $K, L, M, N$ of a point $P$. And now it would be interesting to know if it makes a difference whether the given lines are generators of a regulus or not.

3) In Fig. 6 the respective cubic surface decomposes into a quadric and a plane. In another case, e.g., $a=1$, $b=1$, $c=1$, $d=0$, $e=0$, $f=1$ we get an irreducible cubic. Why?

## 5. Final remarks

The Wallace-Simson Theorem has been generalized several times in the history. The two generalizations to three dimensions presented in this paper are based on results of commutative algebra in the last third of the last century. There are many questions arising from this. Some of them were indicated in remarks. There are problems with computational complexity. The equations of surfaces are too long in its general form, which prevents us from another generalizations (arbitrary directions $u, v, w$, investigation of vol($KLMN$) in the second generalization etc.) Projective extension as well as a synthetic attitude would also be possible.

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## References

[1] B. Bydžovský: *Úvod do algebraické geometrie.* Jednota českých matematikåu a fyzikåu, Praha 1948.

[2] Shang-Ching Chou: *Mechanical Geometry Theorem Proving.* D. Reidel Publishing Company, Dordrecht 1987.

[3] D. Cox, J. Little, D. O'Shea: *Ideals, Varieties, and Algorithms.* Second Edition, Springer 1997.

[4] H.S.M. Coxeter, S.L. Greitzer: *Geometry revisited.* Toronto - New York 1967.

[5] O. Giering: *Affine and Projective Generalization of Wallace Lines.* J. Geometry Graphics **1**, 119–133 (1997).

[6] M. de Guzmán: *An Extension of the Wallace-Simson Theorem: Projecting in Arbitrary Directions.* Amer. Math. Monthly **106**, 574–580 (1999).

[7] D. Hilbert, S. Cohn-Vossen: *Anschauliche Geometrie.* Berlin 1932.

[8] D. Kapur: *A refutational approach to geometry theorem proving.* Artificial Intelligence **37**, 61–94.

[9] H. Martini: *Neuere Ergebnisse der Elementargeometrie.* In: O. Giering, J. Hoschek (eds): *Geometrie und ihre Anwendungen.* C. Hanser Verlag, München 1994.

[10] D. Pedoe: *Geometry.* Dover Publ., New York 1988.

[11] E. Roanes M., M. Roanes-Lozano: *Automatic Determination of Geometric Loci. 3D-Extension of Simson-Steiner Theorem.* In J.A. Campbell and E. Roanes-Lozano (eds.): AISC 2000, LNAI 1930, Springer, 157–173 (2001).

[12] T. Recio, M.P. Vélez: *Automatic Discovery of Theorems in Elementary Geometry.* J. Automat. Reason. **12**, 1–22 (1998).

[13] R. Riesinger: *On Wallace Loci from the Projective Point of View.* J. Geometry Graphics **8**, 201–213 (2004).

[14] M. Simon: *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert.* Teubner, Leipzig 1906.

[15] D. Wang: *Gröbner Bases Applied to Geometric Theorem Proving and Discovering.* In B. Buchberger, F. Winkler (eds): *Gröbner bases and applications*, Lecture Notes of Computer Algebra, Cambridge Univ. Press, Cambridge 1998, pp. 281–301.

[16] W.T. Wu: *Mechanical Theorem Proving in Geometries.* Springer, Vienna 1994.

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