Journal for Geometry and Graphics
Volume 10 (2006), No. 1, 1–13.

Counting Escher's m×m Ribbon Patterns

J. Joseph Fowler[2], Ellen Gethner[1]

[1] *Dept. of Computer Science and Engineering, Univ. of Colorado at Denver*
Campus Box 109, PO Box 173364 Denver, CO 80217, USA
email: Ellen.Gethner@cudenver.edu

[2] *Department of Computer Science, University of Arizona USA*
email: jfowler@CS.Arizona.edu

Abstract. Using a construction scheme originally devised by M.C. Escher, one can generate doubly-periodic patterns of the xy-plane with the operations of rotation, reflection and translation acting on an asymmetric square *motif*. Rotating and/or reflecting the original motif yields eight distinct *aspects*. By selecting m^2 (not necessarily distinct) motif aspects and arranging them in an $m \times m$ *Escher tile*, one can then tile the xy-plane by translating the Escher tile by integer multiples of m in the x and/or y direction to create *wallpaper patterns*.

Two wallpaper patterns are considered equivalent if there is some isometry between the two. Previously, the general formula was given by the second author (Gethner) in [6] for the number of inequivalent patterns generated by $m \times m$ Escher tiles composed of the four rotated aspects of a single asymmetric motif by applying Burnside's Lemma. Here we extend that formula to include the four additional reflected aspects when composing $m \times m$ Escher tiles with which to tile the plane.

Key Words: motif, wallpaper pattern, Escher tile, symmetry, group action, geometric structure

MSC 2000: 51F15, 52C20

1. Introduction

By carving an asymmetric motif into a square block of wood and its mirror image into a second block (Fig. 1(a) and (b)), the Dutch graphic artist Maurits Cornelis Escher created wallpaper patterns generated by the two blocks acting as stamps with which to fill the plane. Since each block could be rotated one of four ways (by 0°, 90°, 180° or 270°), the two blocks together yielded a total of eight distinct aspects of the original motif (Fig. 2). To create a repeating wallpaper pattern in the xy-plane with period 2 in each direction, Escher first constructed a 2×2 square tile filled with four copies of (not necessarily distinct) aspects

ISSN 1433-8157/\$ 2.50

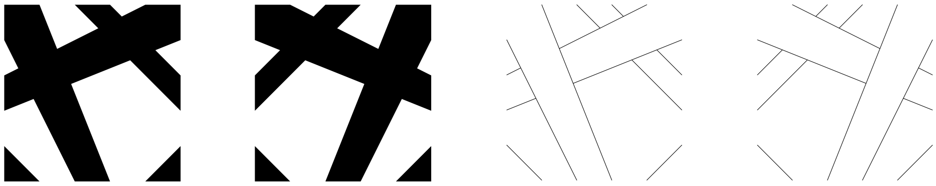

Figure 1: (a) and (b): ESCHER's original motif and its mirror image that he carved into two square blocks. (c) and (d): Escher later carved blocks that showed only the outlines of the bands

of the original motif (see Fig. 3). Translating the tile vertically and horizontally produced doubly-periodic patterns in the xy-plane.

ESCHER found that if he populated a 2×2 tile with one of the four rotated aspects (Fig. 2(a)–(d)), that of the $4^4 = 256$ tiles, only 23 of the tiles generated visually distinct wallpaper patterns. Any of the other $256 - 23 = 233$ patterns produced in this manner could be translated and/or rotated so as to exactly match one of the 23 patterns he had sketched (or stamped) in his workbooks. In [10], Doris SCHATTSCHNEIDER mathematically verified ESCHER's result by applying Burnside's Lemma to show that there are 23 inequivalent patterns generated from this set of 2×2 tiles, where two patterns are considered to be equivalent if there is some isometry between the two. ESCHER performed a similar analysis using all eight aspects from Fig. 2 when composing his 2×2 tiles, but restricted which aspects could be placed in relation to each other as he considered two different cases. Here he included reflection as an operation to transform one wallpaper pattern into another. ESCHER correctly determined the number of inequivalent tiles for the first case, but did not complete the second case. Further details regarding ESCHER's work with illustrations of his sketches can be found in [9] and [10].

In the $m \times m$ case, there are a^{m^2} different tiles that can be created using a distinct aspects of a single asymmetric motif, where it will be understood that if $a = 4$, then the set of allowed aspects are the four rotated aspects in Fig. 2(a)–(d), and if $a = 8$, then the set of aspects include all those in Fig. 2. Let $N_a(m)$ denote the number of inequivalent wallpaper patterns generated by these a^{m^2} distinct tiles. An outgrowth of ESCHER's work is determining $N_a(m)$ for arbitrary m and the appropriate set of a aspects. In [1], Dan DAVIS determined that $N_8(2) = 154$ by using a computer search. The second author (Gethner) gave the formula for $N_4(m)$ for an arbitrary m by applying Burnside's Lemma in [6]. In this paper we extend those results to provide the general formula for $N_8(m)$.

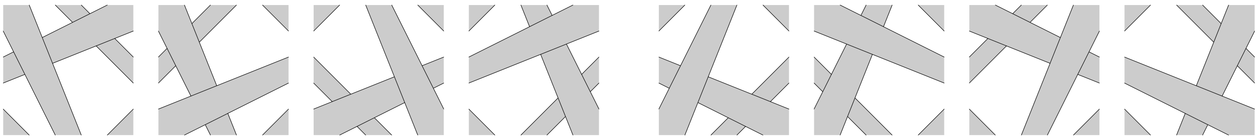

Figure 2: The 8 aspects of ESCHER's ribbon motif. (a)–(d) are counterclockwise rotations of 0°, 90°, 180° or 270° of Fig. 1(c) with the bands partially shaded. (e)–(h) are reflections of each of these in a horizontal line

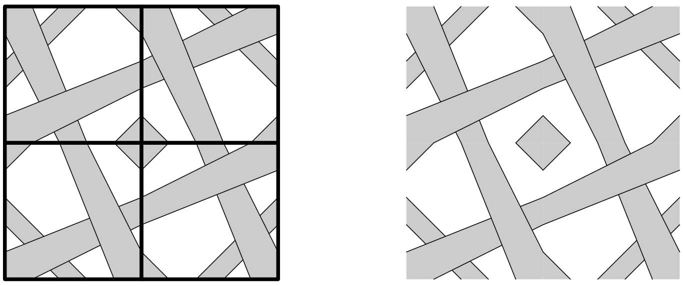

Figure 3: Construction of a 2×2 Escher tile T consisting of the four rotated aspects of the motif shown in Fig. 2(a)–(d). For clarity, (a) outlines the four aspects to illustrate their placement in T, and (b) depicts the final Escher tile T

2. Preliminaries

Let A_4 denote the set of four rotated aspects of an asymmetric motif under 0°, 90°, 180°, and 270° rotations; let $A_8 = A_4 \cup R(A_4)$ where $R(A_4)$ is the reflection of the aspects of A_4 along a horizontal line; and let A denote the set of aspects with which we are working. Unless otherwise stated, $A = A_8$, and $a = |A| = 8$. Let E denote the set of $m \times m$ Escher tiles composed of a (not necessarily distinct) aspects chosen from A; let S denote the set of wallpaper patterns of the xy-plane generated by an Escher tile from E.

We establish a 1-1 correspondence between the $m \times m$ Escher tiles of E and the wallpaper patterns of S by embedding the tiles from E in *standard position* in the xy-plane. We do this by first labeling the centers of the subsquares of the $m \times m$ Escher tile $T \in E$ with the ordered pairs (i, j) for $i, j \in [m]$, where $[m] = \{0, \ldots, m-1\}$, such that the center of the lower-left subsquare is labeled $(0, 0)$ (see Fig. 4). Each of the other subsquares are labeled by their relative distance (in terms of subsquares) from the $(0, 0)$ subsquare where the first coordinate is the number of columns to the right, and the second coordinate is the number of rows above.

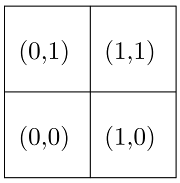

Figure 4: Assigning coordinates for the four subsquares of a 2×2 Escher tile

Using these labels as x, y coordinates, we can then embed the tile T in the xy-plane by placing the centers of its subsquares at the points corresponding to the labels. Each of the m^2 subsquares then becomes a unit square. Translating T repeatedly by m units in the x and y directions generates the corresponding wallpaper pattern $W \in S$. This embedding of T and W in the xy-plane establishes a 1-1 correspondence φ where $W = \varphi(T)$. Fig. 5 illustrates the placement of T from Fig. 3(b) in standard position. For clarity, the subsquares of T are outlined, and the x and y axes are overlaid as dashed lines. Henceforth, we will assume that an Escher tile $T \in E$ is in standard position, and the wallpaper pattern it generates $\varphi(T) \in S$ is embedded in the xy-plane.

Let G signify the group generated by the following transformations of the xy-plane:

1. Translation by 1 unit in either the x or y direction, denoted as $S_{1,0}$ and $S_{0,1}$, respectively.

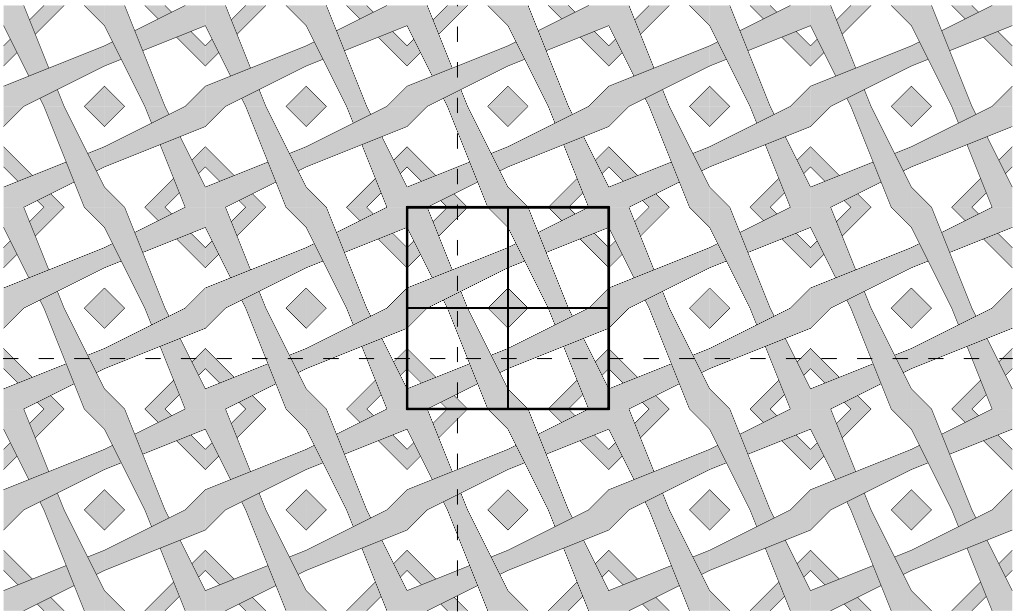

Figure 5: The wallpaper $\varphi(T)$ generated by the embedded 2×2 Escher tile T in Fig. 3

2. Rotation (counterclockwise) by 90° about the origin, denoted by R_{90}.
3. Reflection through the x-axis, denoted by R.

We will say that two wallpaper patterns $W, W' \in S$ are *equivalent with respect to* G, or simply *equivalent*, if there exists some element $g \in G$ that transforms one of the patterns into the other, i.e. $g \circ W = W'$ where $\circ$ denotes the natural action of g on the xy-plane. We define the action of $g \in G$ on the Escher tile $T \in E$ to produce another (not necessarily distinct) tile $T' \in E$ using the 1-1 correspondence φ between E and S, i.e. $g \circ T = T'$ if and only if $g \circ \varphi(T) = \varphi(T')$. Hence, we can say two Escher tiles T and T' are equivalent, denoted as $T \sim T'$, if and only if there exists some $g \in G$ such that $g \circ T = T'$.

2.1. Actions on the coordinates and the aspects of the subsquares

Each element g of G maps the coordinates (x, y) of each subsquare of an Escher tile T to some new set of coordinates modulo m. This rearrangement of coordinates induces a permutation $\hat{g}$ on the set of ordered pairs $C = \{(i, j) : 0 \le i, j \le m - 1\}$ corresponding to the locations of the centers of the subsquares. In fact, the action of G on C forms a natural homomorphism from G onto $\hat{G} = \{\hat{g} | g \in G\}$. As such, $\hat{G}$ forms its own group, which is a subgroup of S_{m^2}, the symmetric group of degree m^2. Hence, by mapping the coordinates modulo m, the action of each element of $g \in G$ on the coordinates of the subsquares of an $m \times m$ Escher tile can be described by a permutation $\hat{g} \in \hat{G}$.

Throughout, we will use the notation $\overline{x}$ to denote the congruence classes of x modulo m. For a given $\hat{g} \in \hat{G}$, we define $\hat{g}((\overline{x}, \overline{y}))$ as the action of the permutation $\hat{g}$ (corresponding to the transformation $g \in G$) on the individual element $(\overline{x}, \overline{y}) \in C$. The following list indicates how the fundamental transformations in G act on C.

1. $\hat{S}_{1,0}((\overline{x}, \overline{y})) = (\overline{x+1}, \overline{y})$ (translation by 1 unit in the x-direction).
   $\hat{S}_{0,1}((\overline{x}, \overline{y})) = (\overline{x}, \overline{y+1})$ (translation by a 1 unit in the y-direction).
2. $\hat{R}_{90}((\overline{x}, \overline{y})) = (\overline{-y}, \overline{x})$ (90° counterclockwise rotation about $(0, 0)$).
   $\hat{R}_{180}((\overline{x}, \overline{y})) = (\overline{-x}, \overline{-y})$ (180° counterclockwise rotation about $(0, 0)$).

$\hat{R}_{270}((\overline{x}, \overline{y})) = (\overline{y}, \overline{-x})$ (270° counterclockwise rotation about $(0, 0)$).

3. $\hat{R}((\overline{x}, \overline{y})) = (\overline{-x}, \overline{y})$ (reflection through the x-axis).

Let H denote the group of translations generated by $S_{1,0}$ and $S_{0,1}$. Since $H \cong \mathbb{Z}_m \times \mathbb{Z}_m$, the direct product of two cyclic groups of order m, we can write an arbitrary element of H in the form $S_{i,j}$, which is equivalent to $(S_{1,0})^i(S_{0,1})^j$. A translation in H acts on the coordinates of the subsquares as follows:

$$((\hat{S}_{1,0})^i(\hat{S}_{0,1})^j)((\overline{x}, \overline{y})) = \hat{S}_{i,j}((\overline{x}, \overline{y})) = (\overline{x+i}, \overline{y+j}) \tag{1}$$

Let K be a subgroup of G not containing the translations in H, i.e. $H \cap K = \{e\}$. The largest such group K is generated by R_{90} and R, in which $K \cong \mathbb{D}_4$, the dihedral group of order 8. The elements of K then act on the coordinates of the subsquares as follows:

$$((\hat{R})^s(\hat{R}_{90})^k)((\overline{x}, \overline{y})) = \begin{cases} ((-1)^s \ \overline{x}, \overline{y}) & \text{if } k \equiv 0 \bmod 4 \\ ((-1)^{s+1} \ \overline{y}, \overline{x}) & \text{if } k \equiv 1 \bmod 4 \\ ((-1)^{s+1} \ \overline{x}, \overline{-y}) & \text{if } k \equiv 2 \bmod 4 \\ ((-1)^s \ \overline{y}, \overline{-x}) & \text{if } k \equiv 3 \bmod 4 \end{cases} \tag{2}$$

In order to combine (1) and (2) we need the following Lemma.

Lemma 1 *Let G be the group generated by $S_{1,0}$, $S_{0,1}$ and a subgroup K of the dihedral group generated by R_{90} and R. Then $G = HK$ where each element $k \in K$ forms a distinct coset Hk in G/H. In particular, any element of G can be written uniquely as a composition of a translation of H with an element of K, and $|G| = m^2|K|$.*

Proof: We first consider the action of G on C, the coordinates of the subsquares. Let $\hat{H}$ and $\hat{K}$ denote the homomorphic image of H and K in $\hat{G}$. We need to show that the generators of $\hat{K}$ normalize the generators of $\hat{H}$, i.e. for generators $\hat{k} \in \hat{K}$ and $\hat{h} \in \hat{H}$, $\hat{k}^{-1}\hat{h}\hat{k} \in \hat{H}$, or equivalently, $\hat{h}\hat{k} = \hat{k}\hat{h}'$ for some permutation $\hat{h}' \in H$. Since $\hat{H}$ has two generators $\hat{S}_{1,0}$ and $\hat{S}_{0,1}$, and $\hat{K}$ can have up to two generators $\hat{R}_{90}$ and $\hat{R}$, there are four combinations to consider with regard to the permutations in $\hat{G}$:

$$\begin{array}{rclcl}
(\hat{S}_{1,0}R_{90})((\overline{x}, \overline{y})) & = & \hat{S}_{1,0}((\overline{-y}, \overline{x})) & = & (\overline{-y+1}, \overline{x}) \\
 & = & \hat{R}_{90}((\overline{x}, \overline{y-1})) & = & (\hat{R}_{90}\hat{S}_{0,-1})((\overline{x}, \overline{y})) \\
(\hat{S}_{0,1}\hat{R}_{90})((\overline{x}, \overline{y})) & = & \hat{S}_{0,1}((\overline{-y}, \overline{x})) & = & (\overline{-y}, \overline{x+1}) \\
 & = & \hat{R}_{90}((\overline{x+1}, \overline{y})) & = & (\hat{R}_{90}\hat{S}_{1,0})((\overline{x}, \overline{y})) \\
(\hat{S}_{1,0}\hat{R})((\overline{x}, \overline{y})) & = & \hat{S}_{1,0}((\overline{-x}, \overline{y})) & = & (\overline{-x+1}, \overline{y}) \\
 & = & \hat{R}((\overline{x-1}, \overline{y})) & = & (\hat{R}\hat{S}_{-1,0})((\overline{x}, \overline{y})) \\
(\hat{S}_{0,1}\hat{R})((\overline{x}, \overline{y})) & = & \hat{S}_{0,1}((\overline{-x}, \overline{y})) & = & (\overline{-x}, \overline{y+1}) \\
 & = & \hat{R}((\overline{x}, \overline{y+1})) & = & (\hat{R}\hat{S}_{0,1})((\overline{x}, \overline{y}))
\end{array}$$

Furthermore, since the translations of H do not affect the aspects of the motif, our argument above extends to show that for any generators h of H and k of K, $hk = kh'$ for some translation $h' \in H$. As such, any element of $g \in G$ can be written as an alternating sequence of the generators and identity elements of H and K, i.e. $g = h_1k_1 \cdots h_rk_r$, for some $h_1, \ldots, h_r \in \{e, S_{1,0}, S_{0,1}\}$ and $k_1, \ldots, k_r \in \{e, R_{90}, R\}$. Reordering by swapping adjacent elements gives $g = h_1' \cdots h_r'k_1 \cdots k_r$ where $h_1', \ldots, h_r' \in H$. Hence, $g = hk$ for some unique $h \in H$ and $k \in K$. Since $|H| = m^2$, $|G| = m^2|K|$. □

Hence, the action of any element of G on the coordinates of the subsquares is as follows:

$$(\hat{S}_{i,j}(\hat{R})^s(\hat{R}_{90})^k)((\overline{x},\overline{y})) = \begin{cases} (\overline{(-1)^s x + i},\ \overline{y+j}) & \text{if } k \equiv 0 \bmod 4 \\ (\overline{(-1)^{s+1} y + i},\ \overline{x+j}) & \text{if } k \equiv 1 \bmod 4 \\ (\overline{(-1)^{s+1} x + i},\ \overline{-y+j}) & \text{if } k \equiv 2 \bmod 4 \\ (\overline{(-1)^s y + i},\ \overline{-x+j}) & \text{if } k \equiv 3 \bmod 4 \end{cases} \tag{3}$$

When necessary, we denote $K_{\{R_{90}\}}$ as the subgroup generated by R_{90}, and $K_{\{R_{90},R\}}$ as the full dihedral subgroup to disambiguate to which K we are referring.

Additionally, elements of K act naturally on the set A of the aspects of the motif. We denote the natural action of $k \in K$ on an aspect $\mathcal{A} \in A$ as $k \circ \mathcal{A}$. For a set of aspects A to be complete, it must be the case that for any aspect $\mathcal{A} \in A$, the transformed aspect $k \circ \mathcal{A} \in A$ for all $k \in K$. Clearly this is the case for $K_{\{R_{90}\}}$ acting on either A_4 and A_8, and for $K_{\{R_{90},R\}}$ acting on A_8. We will refer to such a set of aspects A as a *complete set of aspects under the action of K*.

2.2. Burnside's Lemma

Burnside's Lemma[1] can be used to count the number of inequivalent tiles under the transformations of G. Proofs of Burnside's Lemma can be found in [7] and [11].

Burnside's Lemma *For a finite set S of objects, let G be the group of operations acting on S and* $\mathrm{fix}(x)$ *be the set of objects in S fixed by a given operation $x \in G$. Then s, the number of inequivalent classes of S under G, is given by*

$$s = \frac{1}{|G|} \sum_{x \in G} |\mathrm{fix}(x)| \tag{4}$$

Here S is the set of wallpaper patterns for which we have established a 1-1 correspondence with the set of $m \times m$ Escher tiles. If $K = K_{\{R_{90}\}}$, then the set of aspects $A = A_4$ or A_8. Otherwise, if $K = K_{\{R_{90},R\}}$, then $A = A_8$. Regardless, $|G| = |K|m^2 = |A|m^2 = am^2$.

2.3. Key observation

Lemma 1 allows us to make a crucial observation that will be used to eliminate several cases when applying Burnside's Lemma in the next section.

Lemma 2 *Let $g = hk$ where $h \in H$ and $k \in K \setminus \{e\}$. Let A be a complete set of aspects under the action of K. Then g fixes no Escher tile composed of motifs with aspects from A if and only if $\hat{g}((\overline{x},\overline{y})) = (\overline{x},\overline{y})$ for some subsquare coordinate $(x,y) \in C$. Furthermore, if $\hat{g}((\overline{x},\overline{y})) \neq (\overline{x},\overline{y})$ for any $(x,y) \in C$, then g fixes exactly $a^{m^2/|g|}$ tiles where $a = |A|$ and $m^2/|g|$ is the number of orbits of $\hat{g}$ acting on C.*

Proof: We show sufficiency by simply noting that since $k \neq e$, $g = hk$ will change the aspects of motifs in every subsquare. However, if the location of a subsquare remains fixed but its

[1] Also known as the Cauchy-Frobenius Lemma [2] was originally proved by FROBENIUS (1887).

aspect changes, it is impossible for g to fix the whole Escher tile. This is immediate from the fact that the motif is assumed to be totally asymmetric.

For necessity, we assume the contrapositive that $\hat{g}((\overline{x}, \overline{y})) \neq (\overline{x}, \overline{y})$ for every $(x, y) \in C$. We then show how to construct an Escher tile T fixed by g. For each orbit of $\hat{g}$, pick a subsquare (x, y), freely choosing its aspect $\mathcal{A} \in A$. Then set the aspect of $\hat{g}^i((\overline{x}, \overline{y}))$ to the aspect $k^i \circ \mathcal{A}$ for $i = 1, \ldots, |g| - 1$. Clearly, any Escher tile T fixed by g must adhere to this method of construction. However, for this construction to work it must be the case that the orbit of $\hat{g}$ acting on an ordered pair $(x, y) \in C$ yields $|g|$ distinct elements. If this is not the case, then there exists an $\alpha \in \{1, \ldots, |g| - 1\}$ where $\hat{g}^\alpha((\overline{x}, \overline{y})) = (\overline{x}, \overline{y})$ for some $(x, y) \in C$. By the Orbit-Stabilizer theorem, all the orbits of $\hat{g}$ acting on C have the same cardinality. Hence, $\hat{g}^\alpha((\overline{x}, \overline{y})) = (\overline{x}, \overline{y})$ for all $(x, y) \in C$, i.e. $\hat{g}^\alpha = \hat{e}$. However, for $g^\alpha \neq e$, g^α must alter the aspect of *every* subsquare without changing its location. If $m > 2$, then clearly no such element of G exists by examination of (3). If $m = 2$, we can apply the technique used in Lemma 1, to see that $(hk)^\beta = (h')^\beta k^\beta$ for some translation $h' \in H$, so that $|g| = \mathrm{lcm}(|h'|, |k|)$. Hence, $|g| = 2$ or 4. If $|g| = 2$, then $\alpha = 1 < |g|$ is dispatched by our assumption on $\hat{g}$. If $|g| = 4$, then $|k| = 4$, implying $k = R_{90}$ or $k = R_{270}$. However, (3) cannot be solved for an $\hat{S}_{i,j}\hat{R}_{90} = \hat{e}$ or an $\hat{S}_{i',j'}\hat{R}_{270} = \hat{e}$.

Knowing that $|\hat{g}| = |g|$ and $|C| = m^2$, we can apply the Orbit-Stabilizer Theorem to determine that $\hat{g}$ has a total of $m^2/|g|$ orbits when acting on C. Since each orbit had a freely chosen aspect, this gives the $a^{m^2/|g|}$ tiles that are fixed by g. □

3. Counting Fixed Tiles

In this section we determine the number of $m \times m$ Escher tiles fixed by each element of our group of transformations G that acts on S. Summing over these and dividing by the order of G, gives the number of inequivalent tilings as dictated by Burnside's Lemma. We partition the elements of G into right cosets of H. By Lemma 1, the elements of K form a complete set of the coset representatives. We denote the elements of $K\{R_{90}\}$ as $\{e, R_{90}, R_{180}, R_{270}\}$, and the coset $RK\{R_{90}\}$ as $\{\mathcal{H}, \mathcal{D}, \mathcal{V}, \mathcal{D}'\}$, where $\mathcal{H} = Re$, $\mathcal{D} = RR_{90}$, etc. If for a $k \in K$ the total number of tiles fixed by the elements of the coset Hk is n, then we will say that Hk *fixes* n *tiles*. Similarly, if the total number of tiles fixed by HL is n where $L \subseteq K$, we will say that HL *fixes* n *tiles*.

In [6], the first author (Gethner) determined the number of tiles fixed by $HK\{90\}$ in which $A = A_4$. If instead we use the set of aspects $A = A_8$, we can extend the formulas provided in [6] by simply replacing the fixed number of 4 aspects with the variable $a = |A|$. We see that this holds by noting that A_8 is composed of two distinct orbits from the action of $K\{R_{90}\}$ on A_8, namely A_4 and $R(A_4)$. Combining Propositions 4.1 and 5.1 of [6] gives us our first proposition.

Proposition 3 *The subgroup $HK\{R_{90}\}$ fixes*

$$\sum_{d|m} \left(2d\phi(d) - \phi(d)^2\right) a^{m^2/d} + \sum_{d|m} \left(2^{r_d} - 2\right) a^{m^2/d} + \sigma(m)\left(m^2 \left(a^{m^2/4} + \frac{3}{4} a^{m^2/2}\right)\right)$$

tiles where r_d is the number of (not necessarily distinct) prime divisors of d and

$$\sigma(m) = \begin{cases} 1 \text{ if } m \text{ is even} \\ 0 \text{ if } m \text{ is odd.} \end{cases}$$

Next we determine the number of elements fixed by the cosets $H\mathcal{H}$ and $H\mathcal{V}$.

Proposition 4 *If m is even, then the cosets $H\mathcal{H}$ and $H\mathcal{V}$ together fix*

$$ma^{m^2/2} + 2m \sum_{\substack{d|m \\ 2|d}} \phi(d) a^{m^2/d}$$

tiles. Otherwise, for odd m they fix no tiles.

Proof: Let us begin by determining for which, if any, coordinates $\hat{g}((\overline{x}, \overline{y})) = (\overline{x}, \overline{y})$ for $\hat{g} \in \{\hat{\mathcal{H}}, \hat{\mathcal{V}}\}$. From (3) we extract the actions of $H\hat{\mathcal{H}}$ and $H\hat{\mathcal{V}}$ (i.e. $k = 0$ and $k = 2$ for $s = 1$) on the set of coordinates, C, to give us the following two sets of congruences:

$$\begin{array}{rclrcll} \overline{x} & \equiv & \overline{-x+i}, & \overline{y} & \equiv & \overline{y+j} & \text{for } k = \mathcal{H} \\ \overline{x} & \equiv & \overline{x+i}, & \overline{y} & \equiv & \overline{-y+j} & \text{for } k = \mathcal{V}. \end{array}$$

We can reduce these sets of simultaneous congruences to obtain

$$\begin{array}{rclrcll} \overline{2x} & \equiv & \overline{i}, & \overline{0} & \equiv & \overline{j} & \text{for } k = \mathcal{H} \\ \overline{0} & \equiv & \overline{i}, & \overline{2y} & \equiv & \overline{j} & \text{for } k = \mathcal{V}. \end{array} \tag{5}$$

Let $g = hk$ where $h \in H$ and $k \in \{\langle, \mathcal{V}\}$. For $h = S_{i,j}$, we consider two cases based on whether $\overline{j} = \overline{0}$ for $k = \mathcal{H}$ and $\overline{i} = \overline{0}$ for $k = \mathcal{V}$.

Case 1: Assume that $g \in S_{i,0}\mathcal{H} \cup S_{0,j}\mathcal{V}$, i.e. $\overline{j} = \overline{0}$ for $k = \mathcal{H}$ and $\overline{i} = \overline{0}$ for $k = \mathcal{V}$. There are the two cases of m being odd or even. If m is odd, then 2^{-1} exists modulo m, namely $\frac{m+1}{2}$, since $2(\frac{m+1}{2}) \equiv m + 1 \equiv 1 \bmod m$. Hence, the congruences of (5) can be solved for odd m, which means some subsquare is fixed, so none of these elements fix any Escher tiles by Lemma 2. However, if m is even, then we get the two sets of congruences

$$\begin{array}{rcllrcll} x & \equiv & \frac{i}{2} & \bmod \frac{m}{2}, & \overline{j} & \equiv & \overline{0} & \text{for } k = \mathcal{H} \\ y & \equiv & \frac{j}{2} & \bmod \frac{m}{2}, & \overline{i} & \equiv & \overline{0} & \text{for } k = \mathcal{V}. \end{array} \tag{6}$$

These two congruences can only be solved for an x if j is even, or for a y or i is even, respectively. Specifically the solutions will be $\overline{x} \equiv \overline{\frac{i}{2}}$ and $\overline{x} \equiv \overline{\frac{i}{2} + \frac{m}{2}}$ for $k = \mathcal{H}$, and $\overline{y} \equiv \overline{\frac{j}{2}}$ and $\overline{y} \equiv \overline{\frac{j}{2} + \frac{m}{2}}$ for $k = \mathcal{V}$.

Thus, for $k = \mathcal{H}$, no subsquares are fixed for odd i and $j = 0$, and for $k = \mathcal{V}$, no subsquares are fixed for $i = 0$ and odd j. Between these two k there are a total of m such pairs of i and j. For those g, we can directly calculate the order of the orbits of $\hat{g}$ acting on C.

$$\begin{array}{rclclcl} (\hat{S}_{i,0}\hat{\mathcal{H}})^2((\overline{x}, \overline{y})) & = & (\hat{S}_{i,0}\hat{\mathcal{H}})((\overline{-x+i}, \overline{y})) & = & \hat{S}_{i,0}((\overline{x-i}, \overline{y})) & = & (\overline{x}, \overline{y}) \\ (\hat{S}_{0,j}\hat{\mathcal{V}})^2((\overline{x}, \overline{y})) & = & (\hat{S}_{0,j}\hat{\mathcal{V}})((\overline{x}, \overline{-y+j})) & = & \hat{S}_{0,j}((\overline{x}, \overline{y-j})) & = & (\overline{x}, \overline{y}) \end{array} \tag{7}$$

Hence, $|\hat{g}| = 2$ (where $|g| = |\hat{g}|$ by Lemma 2). Given that there are m elements of $\hat{S}_{i,0}\hat{\mathcal{H}} \cup \hat{S}_{0,j}\hat{\mathcal{V}}$ that fix no subsquare coordinate, we apply Lemma 2 to determine that these elements fix $ma^{m^2/|g|} = ma^{m^2/2}$ tiles in all.

Case 2: Assume that for $g = hk$ for $h = S_{i,j}$ where $\overline{j} \neq \overline{0}$ for $k = \mathcal{H}$, and $\overline{i} \neq \overline{0}$ for $k = \mathcal{V}$. We begin by determining the order of the orbits corresponding to g.

$$\begin{array}{rcl}
(\hat{S}_{i,j}\hat{\mathcal{H}})^1((\overline{x},\overline{y})) & = & (\overline{-x+i},\overline{y+j}) \\
(\hat{S}_{i,j}\hat{\mathcal{H}})^2((\overline{x},\overline{y})) & = & (\overline{x-i+i},\overline{y+j+j}) = (\overline{x},\overline{y+2j}) \\
 & \vdots & \\
(\hat{S}_{i,j}\hat{\mathcal{H}})^d((\overline{x},\overline{y})) & = & \left\{ \begin{array}{rl} (\overline{x},\, \overline{y+dj}) & \text{if } d \text{ is even} \\ (\overline{-x+i},\, \overline{y+dj}) & \text{if } d \text{ is odd} \end{array} \right. \\
(S_{i,j}\mathcal{V})^1((\overline{x},\overline{y})) & = & (\overline{x+i},\overline{-y+j}) \\
(S_{i,j}\mathcal{V})^2((\overline{x},\overline{y})) & = & (\overline{x+i+i},\overline{y-j+j}) = (\overline{x+2i},\overline{y}) \\
 & \vdots & \\
(S_{i,j}\mathcal{V})^d((\overline{x},\overline{y})) & = & \left\{ \begin{array}{ll} (\overline{x+di},\, \overline{y}) & \text{if } d \text{ is even} \\ (\overline{x+di},\, \overline{-y+j}) & \text{if } d \text{ is odd} \end{array} \right.
\end{array} \tag{8}$$

Using the technique in the proof of Lemma 1, the $g^d = (hk)^d$ can be written as $g^d = h'k^d$ for some $h' \in H$. Since k is a reflection, $k^d = k$ if d is odd. As such, the order of g cannot be d for odd d, since $g^d = h'k \neq e$ for any $h' \in H$ and $k \in \{\hat{\mathcal{H}}, \mathcal{V}\}$. Hence, d must be even in (8), which gives the following two congruences to solve:

$$\begin{array}{rcll}
\overline{y} & \equiv & \overline{y+dj} & \text{for } k = \mathcal{H} \\
\overline{x} & \equiv & \overline{x+di} & \text{for } k = \mathcal{V}
\end{array}$$

which simplify to

$$\begin{array}{rcccccl}
\overline{0} & \equiv & \overline{dj} & \Longrightarrow & m|dj & \Longrightarrow & d = \frac{m}{(m,j)} \quad \text{for } k = \mathcal{H} \\
\overline{0} & \equiv & \overline{di} & \Longrightarrow & m|di & \Longrightarrow & d = \frac{m}{(m,i)} \quad \text{for } k = \mathcal{V}
\end{array}$$

There are $\phi(d)$ values of $j \in [m]$ such that $d = \frac{m}{(m,j)}$ and $\phi(d)$ values of $i \in [m]$ such that $d = \frac{m}{(m,i)}$, where $\phi(d)$ is Euler's totient function. Please note that neither $\overline{(m,j)} \not\equiv \overline{0}$ or $\overline{(m,i)} \not\equiv \overline{0}$ since $i \nmid m$ for $k = \mathcal{H}$ and $j \nmid m$ for $k = \mathcal{V}$. Since d is even, we are only considering even divisors of m, which forces m to be even.

Applying Lemma 2 we know that for each such orbit of even order d there are m^2/d such orbits. Further, since each value of i can range independently over $[m]$ for $S_{i,j}\mathcal{H}$ and each value of j can range independently over $[m]$ for $S_{i,j}\mathcal{V}$, there are a total of $2m\phi(d)$ possible pairs of i and j with orbits of length d between the two sets of transformations. Summing over these terms gives us

$$2m \sum_{\substack{d|m \\ 2|d}} \phi(d) a^{m^2/d}$$

tiles fixed by these g, which together with the $ma^{m^2/2}$ tiles fixed by case 1, gives us the final total we need. $\square$

Finally, we determine the number of elements fixed by the cosets $H\mathcal{D}$ and $H\mathcal{D}'$.

Proposition 5 *If m is even, then the cosets $H\mathcal{D}$ and $H\mathcal{D}'$ together fix*

$$2m \sum_{\substack{d|m \\ d \neq 1}} \phi(d) a^{m^2/2d}$$

tiles. Otherwise, for odd m they fix no tiles.

Proof: First, we need to determine for which, if any, coordinates $\hat{g}((\overline{x},\overline{y})) = (\overline{x},\overline{y})$ for $\hat{g} \in \{\hat{\mathcal{D}}, \hat{\mathcal{D}}'\}$. From (3) we extract the actions of $H\hat{\mathcal{D}}'$ and $H\hat{\mathcal{D}}'$ (i.e. $k = 1$ and $k = 3$ for $s = 1$) on the set of coordinates C to gives us the following two sets of congruences:

$$\begin{array}{rclrcll} \overline{x} & \equiv & \overline{y+i}, & \overline{y} & \equiv & \overline{x+j} & \text{for } k = \mathcal{D} \\ \overline{x} & \equiv & \overline{-y+i}, & \overline{y} & \equiv & \overline{-x+j} & \text{for } k = \mathcal{D}', \end{array} \tag{9}$$

which reduce to

$$\begin{array}{rclrcll} \overline{i} & \equiv & \overline{-j}, & \overline{2(x-y)} & \equiv & \overline{i-j} & \text{for } k = \mathcal{D} \\ \overline{i} & \equiv & \overline{j}, & \overline{2(x+y)} & \equiv & \overline{i+j} & \text{for } k = \mathcal{D}'. \end{array} \tag{10}$$

Substituting $\overline{-j}$ for $\overline{i}$ for $k = \mathcal{D}$, and $\overline{i}$ for $\overline{j}$ for $k = \mathcal{D}$, gives

$$\begin{array}{rclcrclll} \overline{2(x-y)} & \equiv & \overline{2i} & \Longrightarrow & x-y & \equiv & i & \bmod \frac{m}{2} & \text{for } k = \mathcal{D} \\ \overline{2(x+y)} & \equiv & \overline{2j} & \Longrightarrow & x-y & \equiv & j & \bmod \frac{m}{2} & \text{for } k = \mathcal{D}'. \end{array} \tag{11}$$

Clearly for one of m given pairs of i and j that satisfy (10) then there are multiple solutions for x and y. Moreover, given that there are m pairs of x and y that satisfy (11) modulo $\frac{m}{2}$, there are $2m$ such pairs of x and y that satisfy the original congruence modulo m.

Thus by Lemma 2, given that $2m$ locations of subsquares are fixed for each of these m pairs of i and j, the elements $\hat{S}_{i,-i}\mathcal{D}$ and $S_{i,i}\mathcal{D}'$ fix no Escher tiles for $i \in [m]$.

Hence, there are only the remaining cases of $\overline{-i} \neq \overline{j}$ for $k = \mathcal{D}$, and $\overline{i} \neq \overline{j}$ for $k = \mathcal{D}'$ to consider. Given that (10) cannot be solved for these cases, we need to determine the order of $g = hk$ where $h \neq S_{-i,j}$ and $h \neq S_{i,j}$ for $k = \mathcal{D}$ and $k = \mathcal{D}$, respectively.

$$\begin{array}{rcl} (\hat{S}_{i,j}\hat{\mathcal{D}})^1((\overline{x},\overline{y})) & = & (\overline{y+i}, \overline{x+j}) \\ (\hat{S}_{i,j}\hat{\mathcal{D}})^2((\overline{x},\overline{y})) & = & (\overline{x+i+j}, \overline{y+i+j}) \\ & \vdots & \\ (\hat{S}_{i,j}\hat{\mathcal{D}})^{2d}((\overline{x},\overline{y})) & = & (\overline{x+d(i+j)}, \overline{y+d(i+j)}) \\ (\hat{S}_{i,j}\hat{\mathcal{D}})^{2d+1}((\overline{x},\overline{y})) & = & (\overline{y+d(i+j)+i}, \overline{x+d(i+j)+j}) \\ (\hat{S}_{i,j}\hat{\mathcal{D}}')^1((\overline{x},\overline{y})) & = & (\overline{-y+i}, \overline{-x+j}) \\ (\hat{S}_{i,j}\hat{\mathcal{D}}')^2((\overline{x},\overline{y})) & = & (\overline{x+i-j}, \overline{y-i+j}) \\ & \vdots & \\ (\hat{S}_{i,j}\hat{\mathcal{D}}')^{2d}((\overline{x},\overline{y})) & = & (\overline{x+d(i-j)}, \overline{y+d(j-i)}) \\ (\hat{S}_{i,j}\hat{\mathcal{D}}')^{2d+1}((\overline{x},\overline{y})) & = & (\overline{-y+d(i-j)+i}, \overline{-x+d(j-i)+j}) \end{array} \tag{12}$$

By applying the technique used in the proof of Lemma 1, we again see that g cannot have odd order $2d+1$, i.e. for some $h' \in H$ and $k \in \{\mathcal{D}, \mathcal{D}'\}$, $g^{2d+1} = (hk)^{2d+1} = h'k^{2d+1} = h'k \neq e$. Hence, we only consider orbits of order $2d$.

To solve for an ordered pair (x, y) fixed by these g^{2d}, we need to solve the congruences

$$\begin{array}{l} \left.\begin{array}{rcl} \overline{x} & \equiv & \overline{x+d(i+j)} \\ \overline{y} & \equiv & \overline{y+d(i+j)} \end{array}\right\} \text{ for } k = \mathcal{D} \\ \left.\begin{array}{rcl} \overline{x} & \equiv & \overline{x+d(i-j)} \\ \overline{y} & \equiv & \overline{y+d(j-i)} \end{array}\right\} \text{ for } k = \mathcal{D}', \end{array} \tag{13}$$

which simplify to

$$\begin{array}{lllllll} \overline{0} & \equiv & \overline{d(i+j)} & \Longrightarrow & m|d(i+j) & \Longrightarrow & d = \frac{m}{(m,(i+j))} \quad \text{for } k = \mathcal{D} \\ \overline{0} & \equiv & \overline{d(i-j)} & \Longrightarrow & m|d(i-j) & \Longrightarrow & d = \frac{m}{(m,(i-j))} \quad \text{for } k = \mathcal{D}'. \end{array} \tag{14}$$

Let $\overline{u} = \overline{i+j}$ and $\overline{v} = \overline{i-j}$ such that $0 \leq u, v < m$. There are $\phi(d)$ distinct values of $u \in [m]$ such that $d = \frac{m}{(m,u)}$, and $\phi(d)$ distinct values of $v \in [m]$ such that $d = \frac{m}{(m,v)}$, where again $\phi(d)$ is Euler's totient function.

Note that $0 < (m, u) < m$ and $0 < (m, v) < m$ since $\overline{u} = \overline{i+j} \neq \overline{0}$ for $k = \mathcal{D}$, and $\overline{v} = \overline{i-j} \neq \overline{0}$ for $k = \mathcal{D}'$. Since orbits of length $2d$ are even, this again forces m to be even. However, we are no longer restricted to even divisors of m as before with $k = \mathcal{H}$ and $k = \mathcal{V}$.

By again applying Lemma 2, we know that for each such orbit of order $2d$ there are $m^2/2d$ such orbits. Further, each distinct congruent value of $\overline{u} = \overline{i+j}$ and $\overline{v} = \overline{i-j}$ can each range independently over $[m]$ for $S_{i,j}\mathcal{D}$ and $S_{i,j}\mathcal{D}'$, respectively. Thus, there are a total of $2m\phi(d)$ possible pairs of i and j that have orbits of length $2d$ between the two sets of elements. Summing over these terms gives us

$$2m \sum_{\substack{d|m \\ d \neq 1}} \phi(d) a^{m^2/2d}$$

tiles fixed by these g, which gives us the total we need. □

Propositions 3, 4 and 5 together gives us our final proposition.

Proposition 6 *The group $HK_{\{R_{90},R\}}$ fixes*

$$\sum_{d|m} \Big(2d\phi(d) - \phi(d)^2\Big) a^{m^2/d} + \sum_{d|m} \Big(2^{r_d} - 2\Big) a^{m^2/d} +$$

$$\sigma(m)\left(m^2\left(a^{m^2/4} + \tfrac{3}{4}\, a^{m^2/2}\right) + m a^{m^2/2} + 2m\Big(\sum_{\substack{d|m \\ 2|d}} \phi(d) a^{m^2/d} + \sum_{\substack{d|m \\ d \neq 1}} \phi(d) a^{m^2/2d}\Big)\right)$$

tiles where r_d and $\sigma(m)$ are as defined in Prop. 3.

4. Applying Burnside's Lemma

We know the order of G from Lemma 1 and the number of Escher tiles fixed by each element in g from Prop. 6, so that we can apply formula (4) of Burnside's Lemma to give the main theorem of this paper.

Theorem 7 *Let $K = K_{\{R_{90},R\}}$ and let the group $G = HK$ (where $|G| = 8m^2$) act on the set of $m \times m$ with aspects chosen from a complete set of aspects A under the action of K with $a = |A|$. Let $N_a(m)$ denote the number of inequivalent tiles under the action of G. Burnside's Lemma gives the following:*

$$N_8(m) = \tfrac{1}{8m^2}\Bigg[a^{m^2} + \sum_{\substack{d|m \\ d \neq 1}} \Big(2d\phi(d) - \phi(d)^2) a^{m^2/d}\Big) + \sum_{d|m} \left(2^{r_d} - 2\right) a^{m^2/d}$$

$$+ \sigma(m)\left(m^2\left(a^{m^2/4} + \tfrac{3}{4}\, a^{m^2/2}\right) + m a^{m^2/2} + 2m\Big(\sum_{\substack{d|m \\ 2|d}} \phi(d) a^{m^2/d} + \sum_{\substack{d|m \\ d \neq 1}} \phi(d) a^{m^2/2d}\Big)\right)\Bigg]$$

where r_d is the number of (not necessarily distinct) prime divisors of d and

$$\sigma(m) = \begin{cases} 1 \text{ if } m \text{ is even} \\ 0 \text{ if } m \text{ is odd.} \end{cases}$$

In particular for $m = 2$ and $a = 8$,

$$\begin{aligned} N_8(2) &= 2^{-5}(2^{12} + 3 \cdot 2^6 + 0 + 2^7 + 2^5 + 3 \cdot 2^6 + 2^2(2^6 + 2^3)) \\ &= 128 + 6 + 4 + 1 + 6 + 8 + 1 = 154. \end{aligned}$$

This confirms the result of Dan Davis in [1] that $N_8(2) = 154$.

5. Future Work

There are many open problems including:

1. Generalize this formula for the case for a motif having rotation or reflection symmetry. Lemma 2 would not hold in this case. Additional symmetry makes it more difficult to determine which operations fix tiles.
2. Generalize the formula here to higher dimensions; in particular, consider the $m \times m \times m$ case.
3. Generalize the results here to regular triangular and hexagonal tilings of the xy-plane, or investigate similar problems in the hyperbolic plane, which admits tiling by regular n-gons for any $n \geq 3$.

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Received June 2, 2005; final form April 21, 2006