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A Transformation Based on the Cubic Parabola $y = x^3$

Eugeniusz Korczak

ul. św. Rocha 6B m. 5, PL 61-142 Poznań, Poland
email: ekorczak@math.put.poznan.pl

Abstract. In this paper a particular geometric transformation is investigated, the Λ-transformation. It is defined on the set T of tangent lines of the cubic parabola $C^3:\ y = x^3$ in the Euclidean plane $\mathbb{R}^2$. Let t be any line from the set T. The point $X \in t$ is called the image of a certain point $M \in t$ under the Λ-transformation, if the condition $(PQMX) = \lambda$ ($\lambda \in \mathbb{R}$ and $\lambda \neq 0, 1$) holds, where $(PQMX)$ denotes the cross-ratio of the four points; P is the point of contact, and Q is the remaining point of intersection between the tangent line t and the basic curve C^3. Varying the line t in the set T and the point M along the line t we obtain a transformation of the plane $\mathbb{R}^2$ into $\mathbb{R}^2$. The image of any straight line $p \in \mathbb{R}^2$ is discussed too.

Key Words: Λ-transformation, quadratic transformation

MSC 2000: 51N15, 51N35

1. Introduction

In most of my papers, among other in [1, 2], I discuss the properties of different versions of the Λ-transformation, in the plane as well as in the three-dimensional space. Each time the transformation is based on the requirement that the cross-ratio of four distinct points has to equal a given real number λ. In the following paper we again refer to a similar problem. This time we define the Λ-transformation on the set T of all tangent lines of the cubic parabola $C^3 \subset \mathbb{R}^2$. To any point $M \in t$ (where $t \in T$) there corresponds a single point $X \in t$ such that for a given $\lambda \in \mathbb{R}$, $\lambda \neq 0, 1$, the cross-ratio $(PQMX) = \lambda$. Here P is the point of tangency and Q is the remaining point of intersection between $t \in T$ and the basic curve C^3 (see Fig. 1). Varying the tangent line t in the set T and the point M along the tangent line, one can say that the Λ-transformation defines a transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

It is well known that through any arbitrary point $M \notin C^3$ there pass three in general distinct tangent lines t_1, t_2, t_3 (see Fig. 2). One of these tangent lines is always real, the other two can be real or conjugate complex. On each of these tangent lines the transformation Λ can be defined so that in fact the plane $\mathbb{R}^2$ is covered by the images $X = \Lambda(M)$. For the investigations let us start on one of the real tangent lines.

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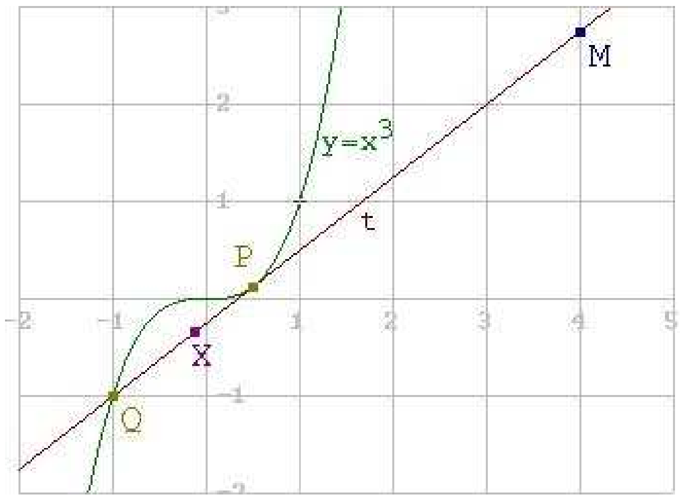

Figure 1: Corresponding points $M \mapsto X$ on the tangent line t

2. Equations of the Λ-transformation

In the Euclidean plane $\mathbb{R}^2$ let us choose the cubic parabola C^3 with the equation

$$y = x^3. \tag{1}$$

The equation of the tangent line $t \in T$ at a specified point $P(x_0,\, x_0^3)$ of the basic curve C^3 is

$$y - y_0^3 = y'(x_0)(x - x_0), \quad \text{hence} \;\; y = 3x_0^2 x - 2x_0^3\,. \tag{2}$$

In order to get the coordinates of the third common point Q between t and C^3, we must solve the system of equations

$$\begin{aligned} x^3 - y &= 0, \\ 3x_0^2 x - 2x_0^3 - y &= 0. \end{aligned}$$

So we get

$$x^3 - 3x_0^2 x + 2x_0^3 = 0. \tag{3}$$

We know that x_0 is a root of multiplicity 2 of the equation (3). Let us find the third one. It's easy to see that

$$\left(x^3 - 3x_0^2 x + 2x_0^3\right) : (x - x_0)^2 = x + 2x_0.$$

Hence, the third root is equal to $-2x_0$, and the coordinates of the point Q are $(-2x_0,\, -8x_0^3)$.

The coordinates of a certain third point $M(x, y)$ lying on the straight line joining two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$, can be expressed as an affine combination. So we have

$$x = x_1 + (x_2 - x_1)s, \qquad y = y_1 + (y_2 - y_1)s, \tag{4}$$

where $s = PM/QP$ is an inhomogeneous parameter with $s \in (-\infty, +\infty)$. For $s = 0$ we get the point $P(x_1, y_1)$, for $s = 1$ the point $Q(x_2, y_2)$. Substituting the corresponding values of P and Q we get finally the system of equations

$$x = x_0(1 - 3s), \qquad y = x_0^3(1 - 9s). \tag{5}$$

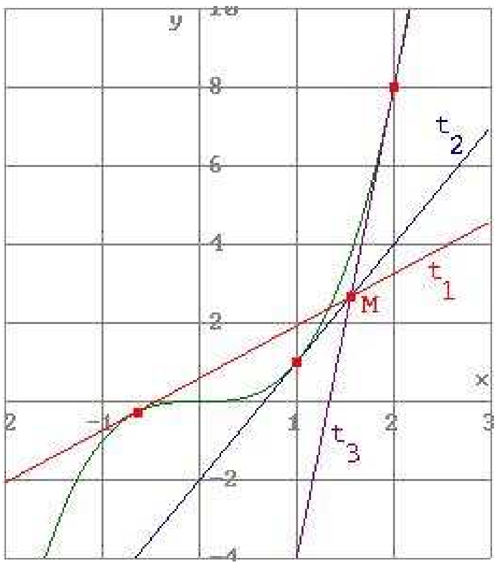

Figure 2: The three tangent lines passing through M

Eqs. (5) define the coordinates (x, y) of the point $M \in t$, when t touches the curve C^3 at the point $P(x_0,\, x_0^3)$.

Let us define the fourth point $X(\xi, \eta) \in t$ according to the definition of the Λ-transformation, i.e., such that the cross-ratio $(PQMX) = \lambda$ for a given $\lambda \in \mathbb{R}$, $\lambda \neq 0, 1$. It is known (e.g., [3]) that the cross-ratio of these four points can be expressed in terms of the coordinates by

$$\frac{x - x_1}{x - x_2} : \frac{\xi - x_1}{\xi - x_2} = \lambda \quad \text{or} \quad \frac{y - y_1}{y - y_2} : \frac{\eta - y_1}{\eta - y_2} = \lambda\,.$$

After substituting the corresponding values of the coordinates we get the formulas of the Λ-transformation in the form

$$\xi = \frac{x_0[\lambda - s(2 + \lambda)]}{s(1 - \lambda) + \lambda}\,, \qquad \eta = \frac{x_0^3[\lambda - s(8 + \lambda)]}{s(1 - \lambda) + \lambda}\,.$$

As we see, the coordinates ξ, η depend on the two parameters $x_0, s \in (-\infty, +\infty)$ and a real number λ. It will be appropriate to denote x_0 by u. Hence, the above given system of equations can be written finally as

$$\xi = \frac{u[\lambda - s(2 + \lambda)]}{s(1 - \lambda) + \lambda}\,, \qquad \eta = \frac{u^3[\lambda - s(8 + \lambda)]}{s(1 - \lambda) + \lambda}\,. \tag{6}$$

Any value of the parameter u corresponds to a certain tangent line t from the set T (see the Eq. (2)). After fixing u, to any s corresponds a certain point M on the tangent line t. Hence, any pair of values (u, s) determines a single point $M(x, y)$ (see Eqs. (5)). The equations (6) determine the point $X(\xi, \eta) \in t$, the image of $M(x, y)$ under the Λ-transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Since the equations (6) are rational, the Λ-transformation defines a special rational transformation in the Euclidean plane.

3. Properties of the Λ-transformation

It follows immediately from (6) that for a fixed value u_0 of the parameter u the formulas describe a certain straight line, one of the tangents lines of C^3. Hence, we get the following

Lemma 1 *The elements of the set T remain fixed under the Λ-transformation, but they are not pointwise fixed.*

Let us consider the cross-ratio $(PQMX) = \lambda$ and let us assume that the points P and M coincide, i.e., $P = M$. So, we get $\frac{MM}{QM} = \lambda \frac{MX}{QX}$, in other words, $0 = \lambda \frac{MX}{QX}$. This equation is satisfied only when $M = X$ (due to $\lambda \neq 0$). One gets a similar result for $M = Q$.

It is known that through any point M of the parabolic cubic C^3 there pass two, in general distinct tangent lines t_1, t_2 (see Fig. 3). Hence, in this case we get $M = X_1 = X_2$ and therefore

Lemma 2 *The basic curve C^3 is pointwise fixed under the Λ-tranformation and its points are twofold singular points of the transformation.*

The origin $O(0,0)$ is an inflection point of the basic curve C^3 obeying (1) with the x-axis as tangent line. Any point M on the x-axis, $M \neq O$, corresponds to parameter values $u = 0$ and $s \neq 0$. We substitute these values into (6) and get $M = O$as the image.

In the case $u = v = 0$ we have $O = P = Q = M$. Then the cross-ratio loses any sense and the image of this point can be any point on the x-axis, the tangent line at the inflection point. Notice that this is similar to the case of constructing the polar of a point lying on a conic. The polar is tangent to the conic in this very point, and the construction of the polar based on the cross-ratio fails. So, we can formulate the following

Lemma 3 *The image of points M lying on the tangent line in the inflection point coincides with the inflection point. The image of the point of inflection is undetermined; any point on its tangent line can be seen as its image.*

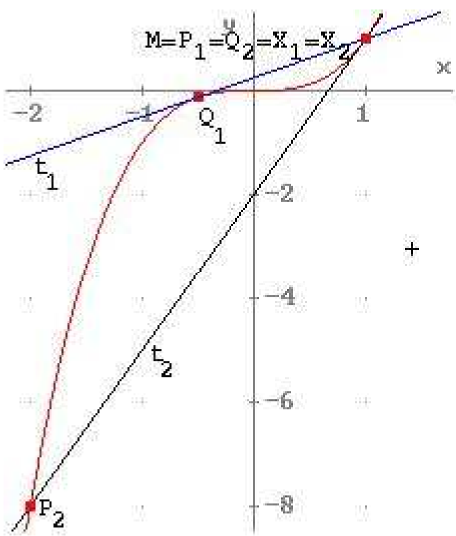

Figure 3: There are two tangent lines passing through the point $M \in C^3$

From the equations

$$x = u(1 - 3s), \qquad y = u^3(1 - 9s)$$

one can see that for fixed $u = u_0$ the system describes a single tangent line t. When $s = s_0$ is fixed too, then we get a point M on this tangent line. When fixing s_0 and varying parameter u, we obtain a certain cubic parabola, the *trajectory* of the point M (see Fig. 4).

Substituting $u = u_0$ in the formulas (6) yields the same tangent line t. Then $s = s_0$ defines the point X, the image of point M. Next, varying the parameter u we obtain again a cubic parabola, the trajectory of the point X and at the same time the image of the trajectory of the point M (see Fig. 4).

Finally, let us consider the problem, what happens with the points at infinity: Now we need to introduce homogenous coordinates. This can be done by the substitution

$$\xi = \frac{x_1}{x_3}, \qquad \eta = \frac{x_2}{x_3}.$$

Hence, from (6) we get the system

$$x_1 = u[\lambda - s(2+\lambda)], \quad x_2 = u^3[\lambda - s(8+\lambda)], \quad x_3 = s(1-\lambda) + \lambda.$$

Points at infinity are characterized by $x_3 = 0$. So we get the condition $s(1-\lambda) + \lambda = 0$, or $s = \dfrac{\lambda}{\lambda - 1}$. Substituting the last expression in (5), we obtain the coordinates of all those points $M(x, y)$, for which the images $\Lambda(M) = X$ are at infinity:

$$x = \frac{3\lambda u}{1-\lambda}, \qquad y = \frac{9\lambda u^3}{1-\lambda}. \tag{7}$$

As we see, the trajectory of such points M is the cubic parabola defined by the system (7).

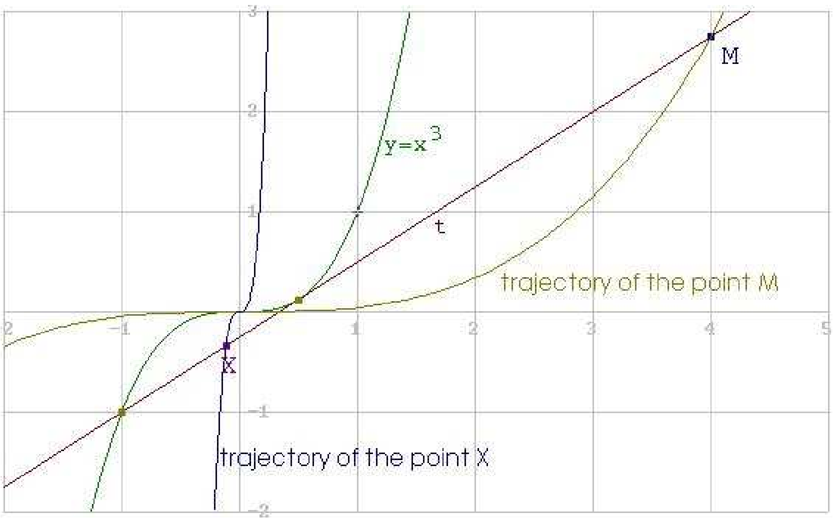

Figure 4: Locus of point M for variable u and fixed s together with it image X

In the case $\lambda = -1$ the Λ-transformation is an involution, i.e., $\Lambda(M) = X$ implies $\Lambda(X) = M$.

Image of a straight line

Let the straight line with equation

$$ax + by + c = 0 \tag{8}$$

be given in the Euclidean plane $\mathbb{R}^2$. What is the image of this line under the Λ-transformation?

We have to look for (5) under the condition that the coordinates (x, y) of M obey (8). This implies for the parameter s

$$s = \frac{u(a + bu^2) + c}{3u(a + 3bu^2)}. \tag{9}$$

Substituting this in (6), we obtain the equations describing the image of the line (8):

$$\begin{aligned} \xi &= \frac{u[2\lambda u(a+4bu^2) - 2u(a+bu^2) - c(2+\lambda)]}{2\lambda u(a+4bu^2) + u(a+bu^2) + c(1-\lambda)}, \\ \eta &= \frac{u^3[2\lambda u(a+4bu^2) - 8u(a+bu^2) - c(2+\lambda)]}{2\lambda u(a+4bu^2) + u(a+bu^2) + c(1-\lambda)}. \end{aligned} \tag{10}$$

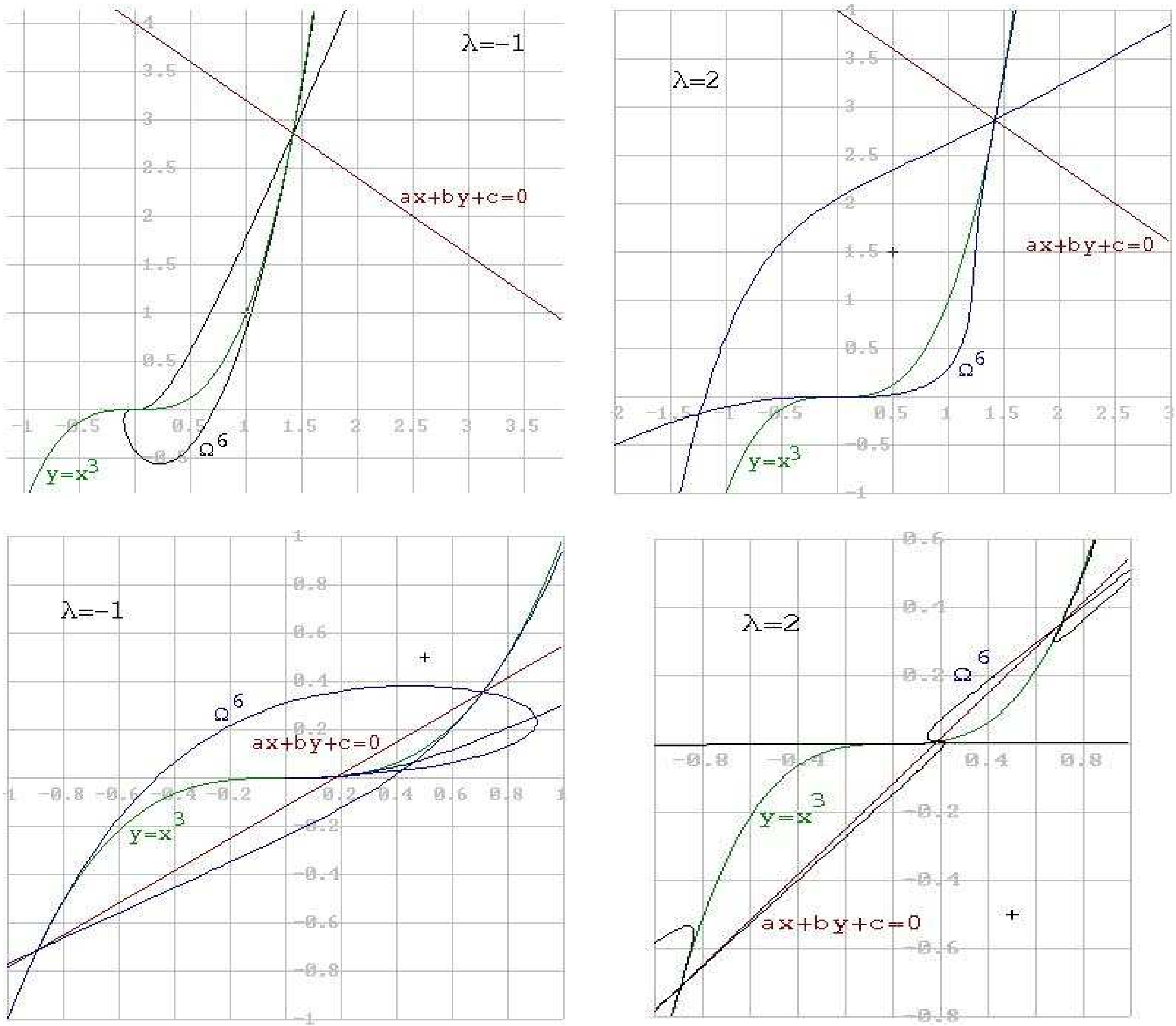

Figure 5: Image of a straight line under the Λ-transformation for different λ

The formulas in (10) contain rational functions. Hence the image curves are rational with singularities at the points of intersection between the line (8) and the basic curve (see Lemma 2 and Fig. 5).

As we have proved on page 19, the locus of points M which are mapped onto points at infinity, is a cubic curve with the parametric representation (7). The straight line (8) has three common points with the curve (7), so the curve (10) has three points in infinity. Hence we can formulate the following

Theorem 1 *The image of a straight line under the* Λ*-transformation is a rational curve* Ω^6 *of degree six with three twofold points, three points at infinity and the parametric representation given by (10) (see Fig. 5).*

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