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Blending Circular Pipes With a Cyclic Surface

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Abstract. This paper deals with circular surfaces joining two cylinders of revolution with axes in a common plane and different radii. Two functions are used in order to define the radii of the desired transition surface. One is polynomial and the other is transcendental. There are two points of view on the problem. One is the theoretical background, and the other relates to its technical use.

Key Words: Transition surface, cyclic surface, cubic

MSC 2000: 68U05, 53A05

1. Geometry of circular pipes

1.1. Transition surface

Let two pipes be given. We show how to connect both cylinders by a transition surface Γ described in detail in Section 1.2. The transition surface is a *cyclic surface* formed by a system of generating circles. The system of generating circles is defined by a set of centers, radii and carrier planes. The choice of the centers is given in Section 1.3, the functions which define the radii and the planes follow in Section 1.4.

Frequently canal surfaces serve as transition surfaces. We address this at the end of the paper.

1.2. Properties of the transition surface

Let k_0 and k_1 denote the circles of contact between the transition surface Γ and the given surfaces γ_0, γ_1, respectively. Henceforth, the following properties of the transition surface Γ are required:

i) Γ is C^1 smooth.

(ii) Its transition to both surfaces γ_0 and γ_1 is also C^1 smooth; thus the surfaces γ_0 and γ_1 and the transition surface Γ are in first-order contact along the circles k_0 and k_1, respectively.

(iii) Γ is free of selfintersections.

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1.3. Centers of the generating circles

Let $\mathcal{S}$ be the locus of centers for the generating circles, i.e., the *spine curve* of the cyclic surface. Then $\mathcal{S}$ is a curve connecting both cylinder axes. Let S_0, S_1 denote the centers of the circles k_0, k_1 and o_0, o_1 the axes of the surfaces γ_0, γ_1, respectively. For a C^1 transition it is necessary that the lines o_0 and o_1 are tangent to $\mathcal{S}$ at the points S_0, S_1.

1.4. Radii of the generating circles and their carrier planes

Let r_0, r_1 denote the radii of the circles k_0, k_1 and of the cylinders γ_0 and γ_1. The radii of the circles sweeping the surface Γ are determined by values of a function R. This function R assigns a positive real number to each point of the curve $\mathcal{S}$. The function R must be continuously differentiable in order to make sure that the surface Γ has the property (i). The function R must have values r_0, r_1 at the points S_0, S_1, resp., and the first derivatives at these terminating points must be zero in order to make sure that the surface Γ has the property (ii).

We choose the normal planes of the curve $\mathcal{S}$ as planes of the generating circles. Then the surface Γ has the property (iii) — at least locally — if the values of the function R are smaller than the corresponding radii of the osculating circles of $\mathcal{S}$.

For generic cyclic surfaces the planes of the circles are not the normal planes of the curve $\mathcal{S}$ (compare, e.g., generic canal surfaces).

2. Mathematical description of the transition surface

The transition surface can be parameterized by the vector function

$$\mathbf{x}(t,u) = \mathbf{y}(t) + R(t)\left(\mathbf{b}_1(t)\cos u + \mathbf{b}_2(t)\sin u\right), \quad t \in I,\ I \subset \mathbb{R},\ u \in [0, 2\pi]. \tag{1}$$

The vector function $\mathbf{y} = \mathbf{y}(t)$ in (1) is a parameterization of $\mathcal{S}$, which is the set of centers of the generating circles. The values of a real function $R = R(t)$ in (1) are the radii of the generating circles. For each $t \in I$ the vectors $\mathbf{b}_1 = \mathbf{b}_1(t)$ and $\mathbf{b}_2 = \mathbf{b}_2(t)$ form an orthonormal basis in the normal plane of $\mathcal{S}$.

In Section 2.1 we will design a curve $\mathcal{S}$. In Section 2.2 we give two functions R with the properties of Section 1.4. In Section 2.3 we will demonstrate how to choose vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ so that the vector function (1) is a parametric representation of the surface Γ. We will finally sum up the results of the previous sections in Section 2.4.

2.1. Parametric representation of the curve $\mathcal{S}$

As the curve $\mathcal{S}$ we choose a cubic with its Hermite representation (see, e.g., [11])

$$\mathbf{y}(t) = \sum_{i=0}^{3} F_i(t)\,\mathbf{s}_i, \quad t \in [0,1], \tag{2}$$

where

$$F_0(t) = 2t^3 - 2t^2 + 1, \quad F_1(t) = -2t^3 + 3t^2, \quad F_2(t) = t^3 - 2t^2 + t, \quad F_3(t) = t^3 - t^2.$$

We set $\mathbf{s}_0 = \overrightarrow{OS_0}$ and $\mathbf{s}_1 = \overrightarrow{OS_1}$ where O denotes the origin of the coordinate system.

The vectors $\mathbf{s}_2$ and $\mathbf{s}_3$ are direction vectors of the cylinder axes o_0 and o_1. The choice of the tangent vector at S_0 defines an orientation of the spine curve of Γ. The tangent vector at the endpoint S_1 of the intermediate spine curve has to be chosen according to this.

Let $[O, x, y, z]$ be a right-handed Cartesian coordinate system such that the parameterization of $\mathcal{S}$ becomes most simple. Hence we specify $O = S_0$, thus $\mathbf{s}_0 = \mathbf{o}$; the x-axis is the line o_0 and the vector $\mathbf{s}_2$ is a positive multiple of the vector $\mathbf{a}_2 = (1, 0, 0)$. The xy-plane is placed such that it includes the point S_1, which gives $\mathbf{s}_1 = (s_{11}, s_{12}, 0)$.

The use of (2) in the stated coordinate system implies that the curve $\mathcal{S}$ is parameterized by the vector function

$$\mathbf{y}(t) = \sum_{i=1}^{3} F_i(t)\,\mathbf{s}_i, \quad t \in [0, 1]. \tag{3}$$

We set

$$\mathbf{s}_2 = k\,\mathbf{a}_2, \qquad \mathbf{s}_3 = l\,\mathbf{a}_3\,, \tag{4}$$

where k and l are positive real constants and $\mathbf{a}_2$, $\mathbf{a}_3$ are unit vectors. Otherwise we set $\mathbf{a}_3 = \frac{\mathbf{s}_3}{\|\mathbf{s}_3\|}$. The choice of the constants k and l forms the shape of $\mathcal{S}$, which will be used as spine curve of Γ.

2.2. Radii of the generating circles

Let R_1 and R_2 be two functions given by

$$R_1(t) \;=\; -2(r_1 - r_0)t^3 + 3(r_1 - r_0)t^2 + r_0, \quad t \in [0, 1], \tag{5}$$

$$R_2(t) \;=\; \frac{1}{2}\left(\sqrt{(r_1 + r_0)^2 - (r_1 - r_0)^2 \sin^2 \pi t} - (r_1 - r_0)\cos \pi t\right), \quad t \in [0, 1]. \tag{6}$$

We may assume $r_0 < r_1$ because otherwise we can interchange the notation of the radii. These specifications imply:

a) $R_1(0) = R_2(0) = r_0$ and $R_1(1) = R_2(1) = r_1$.

b) Both functions are continuously differentiable on $]-\infty, +\infty[$.

c) The first derivatives of both functions at the boundary of the interval $[0, 1]$ are equal 0.

d) Both functions are monotonically increasing on the interval $[0, 1]$.

We prove item a) by substituting $t = 0$ and $t = 1$. The first derivatives are used to confirm the next properties of the functions R_1 and R_2. We obtain

$$R_1'(t) = 6(r_1 - r_0)\,t\,(1 - t),$$

and it is clear that R_1 has the properties b) and c). Since the derivative $R_1'(t)$ is positive on the interval $]0, 1[$, the function R_1 has the property d). On the other hand

$$R_2'(t) = \frac{\pi(r_1 - r_0)\sin \pi t}{2\sqrt{(r_1 + r_0)^2 - (r_1 - r_0)^2 \sin^2 \pi t}}\left(-(r_1 - r_0)\cos \pi t + \sqrt{(r_1 + r_0)^2 - (r_1 - r_0)^2 \sin^2 \pi t}\right),$$

and it is evident that the function R_2 has the properties b) and c). The function R_2 is increasing on the interval $[0, 1]$ provided

$$\sqrt{(r_1 + r_0)^2 - (r_1 - r_0)^2 \sin^2 \pi t} \;>\; (r_1 - r_0)\cos \pi t.$$

The inequality holds for $t \in [0.5, 1]$. After squaring and some simplifications for $t \in [0, 0.5[$ we have

$$(r_1 + r_0)^2 - (r_1 - r_0)^2 \sin^2 \pi t > (r_1 - r_0)^2 \sin^2 \pi t \implies (r_1 + r_0)^2 > (r_1 - r_0)^2.$$

Therefore the derivative $R_2'(t)$ is positive on the interval $]0, 1[$. Hence the function R_2 is increasing on this interval.

The functions R_1 and R_2 have all properties required in Section 1.4 and we use them as radius functions of the generating circles. Graphs of these functions R_1 and R_2 are displayed in Figs. 1 and 2. In Fig. 1 we have $r_0 = 1$ and $r_1 = 4$, in Fig. 2 $r_0 = 1$ and $r_1 = 9$.

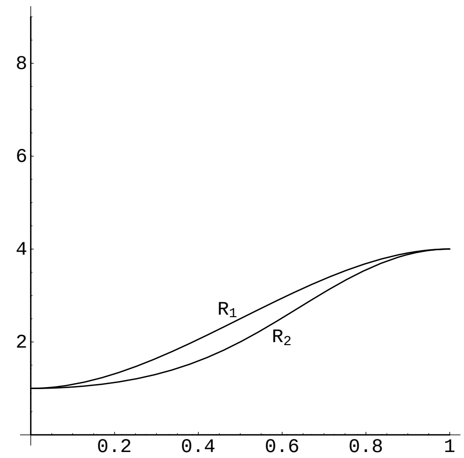

Figure 1: Graph of $R_1(t)$

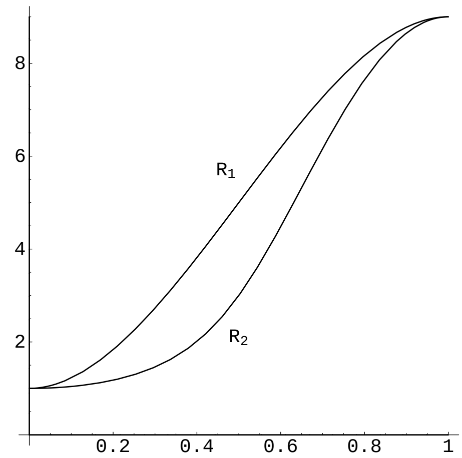

Figure 2: Graph of $R_2(t)$

The choice of a cubic function R_1 is natural as $\mathcal{S}$ is a cubic. The function R_2 is a modification of the radius function of the generating circles of a Dupin cyclides. Of course, there are many possible functions R with the required properties in general. We chose the functions R_1 and R_2 also for the reason given below in the remark.

It is clear from the figures that the functions have indeed properties a) – d) stated above. Further it is evident that in case of a big difference $r_1 - r_0$ the values of R_1 are fairly bigger than that of R_2 mainly around the point $t = \frac{1}{2}$. For the transition surface this means that if R_1 is used, the transition surface has a greater area and a greater volume of the enclosed solid than in case of using R_2. From the practical point of view it means that using R_2 saves material. It is evident that the shape of the pipe, the area of its surface and its volume have influence on liquid flow in the pipe. However, the problem of liquid flow will not be addressed here.

Remark: It is interesting that

$$R_1\left(\frac{1}{2}\right) = \frac{r_0 + r_1}{2} \quad \text{and} \quad R_2\left(\frac{1}{2}\right) = \sqrt{r_0 r_1}\,.$$

The function R_1 evaluates to the arithmetic mean of numbers r_0 and r_1 at the point $t = \frac{1}{2}$, and R_2 evaluates to the geometric mean of r_0 and r_1 at this point. Between both means the inequality

$$\frac{r_0 + r_1}{2} \geq \sqrt{r_0 r_1}$$

holds which is an equality if and only if $r_0 = r_1$. In this case the transition surface becomes a pipe surface.

2.3. Planes of the generating circles

According to Section 1.4 the generating circles lie in the normal planes of the curve $\mathcal{S}$. An orthonormal frame is needed for the parameterization of all generating circles. The first derivative

$$\frac{d\,\mathbf{y}(t)}{dt} = \mathbf{y}'(t)$$

determines the direction vectors of tangent lines of $\mathcal{S}$. Let $\mathbf{b}_1 = \mathbf{b}_1(t)$ and $\mathbf{b}_2 = \mathbf{b}_2(t)$ denote vector functions which form an orthonormal basis of the normal plane for each $t \in [0,1]$. Then the characteristic circles of the surface Γ are parameterized by

$$R(t)(\mathbf{b}_1(t)\cos u + \mathbf{b}_2(t)\sin u), \quad t \in [0,1],\ u \in [0,2\pi].$$

Let $(b_{11}(t),\, b_{12}(t),\, b_{13}(t))$ be coordinate functions of the vector function $\mathbf{y}'(t)$. The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are not uniquely determined in the normal planes. We can rotate them about the tangent line. Hence we may choose the function as follows:

$$\mathbf{b}_1(t) = \frac{1}{\sqrt{b_{11}^2(t) + b_{12}^2(t)}}\,(-b_{12}(t),\ b_{11}(t),\ 0) \tag{7}$$

and

$$\mathbf{b}_2(t) = \frac{\mathbf{y}'(t) \times \mathbf{b}_1(t)}{\|\mathbf{y}'(t) \times \mathbf{b}_1(t)\|}\,. \tag{8}$$

It corresponds to the case when both axes o_0 and o_1 lie in a plane which is then determined as the xy-plane. In this case $\mathcal{S}$ is a plane curve in the xy-plane and we have $\mathbf{b}_2(t) = (0,0,1)$ for each $t \in [0,1]$ (e.g., [8]). When the axes o_0 and o_1 are supposed to be skew, $\mathcal{S}$ is a space curve and the vector function $\mathbf{b}_2$ cannot be constant (e.g., [10]).

2.4. Parameterization of the transition surface

A transition surface is parameterized by

$$\mathbf{x}(t,u) = \mathbf{y}(t) + R(t)\,(\mathbf{b}_1(t)\cos u + \mathbf{b}_2(t)\sin u)\,, \quad t \in [0,1],\ u \in [0,2\pi], \tag{9}$$

where $\mathbf{y}$ is the vector function (3) with the notations given in (4). The real function R equals either R_1 from (5) or R_2 from (6) (or eventually another function R with the required properties). And $\mathbf{b}_1$, $\mathbf{b}_2$ are the vector functions (7) and (8).

In the first part of this paper we stated that the surface does not intersect itself if at every point of the curve $\mathcal{S}$ the radius of the osculating circle is greater than the radius of the generating circle. The radii of the osculating circles are defined by the real function

$$\varrho(t) = \frac{\|\mathbf{y}'(t)\|^3}{\|\mathbf{y}'(t) \times \mathbf{y}''(t)\|}\,, \quad t \in [0,1] \tag{10}$$

(e.g., [9, 12, 15]). In order to guarantee that the surface is free of selfintersections — at least locally — the following inequality must hold for each $t \in [0,1]$:

$$\varrho(t) \geq R_i(t), \quad i = 1,2 \quad \text{(see (5) and (6))}. \tag{11}$$

Eq. (11) does not only depend on t, it is also a condition for k and l. We must set the numbers k and l such that the condition (11) holds for each $t \in [0,1]$. After substitution from (10) and (5) or (6) in (11) we have greatly complicated the inequality. Hence a more feasible way of k and l setting is a computer-aided modeling.

3. Examples

In order to get an idea of the shape of the transition surface we model the surface on a computer. This affords us the possibility to change easily its shape by modification of the optional parameters according to our needs.

Data which define the transition surface and which cannot be influenced by the designer are

1) the radius vectors $\mathbf{s}_0$ and $\mathbf{s}_1$ of the centers of the circles k_0 and k_1 which inosculate the transition surface to the given pipes,
2) the unit direction vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ of the axes of the given pipes,
3) the radii r_0 and r_1 of the circles k_0 and k_1.

Parameters which can be set by the designer are

a) the radius function R of Γ, and
b) the positive numbers k and l which describe the lengths of the direction vectors of the pipe axes.

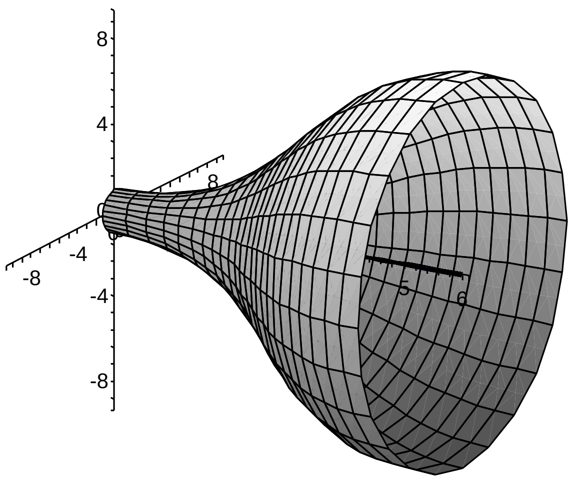

Figure 3: Surface Γ with a linear spine curve and $R = R_1$

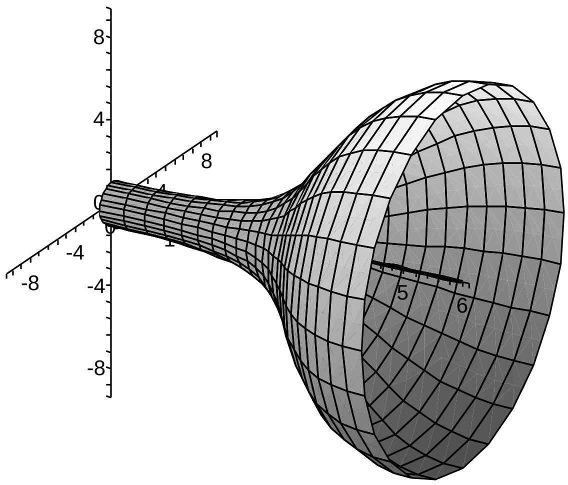

Figure 4: Surface Γ with a linear spine curve and $R = R_2$

The choice of the function R and the constants k and l depends on the designer and his practical needs. We demonstrate the influence of the specifications of R, k and l on the shape of Γ in several examples.

In Figs. 3 and 4 we display a transition surface to the following data:

$$\mathbf{s}_0 = (0,0,0), \quad \mathbf{s}_1 = (6,0,0), \quad \mathbf{a}_2 = \mathbf{a}_3 = (1,0,0), \quad r_0 = 1, \quad r_1 = 9.$$

It is evident that the curve $\mathcal{S}$ is a line segment and the choice of k and l does not influence its shape. However, we see the influence of the choice of R on the shape of the transition surface. We use the function R_1 in Fig. 3 and R_2 in Fig. 4.

Fig. 5 shows the transition surface for

$$\mathbf{s}_0 = (0,0,0), \quad \mathbf{s}_1 = (10,4,0), \quad \mathbf{a}_2 = (1,0,0), \quad \mathbf{a}_3 = \frac{(1,2,10)}{\sqrt{105}}, \quad r_0 = 1, \quad r_1 = 2.$$

We use the function R_1 to parameterize the generating circles and we set $k = 18$ and $l = 25$. However, this option of k and l is not appropriate because the transition surface intersects

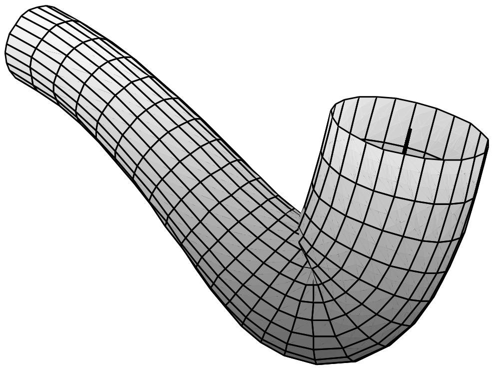

Figure 5: A transition surface with selfintersections

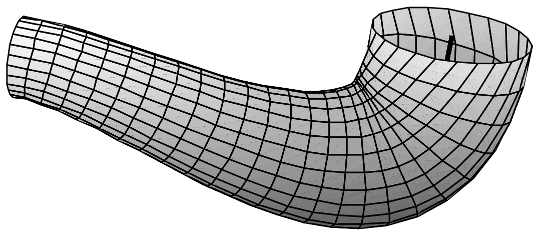

Figure 6: By modification of k and l we avoid selfintersections of Γ

itself. We eliminate this intersection by modifying k from 18 to 10 and l from 25 to 20, as we see in Fig. 6.

If planes and centers of the outer circles of a transition surface are "sufficiently close" to each other, then the transition surface can intersect itself. We eliminate this undesirable event by an appropriately selected composition of several mentioned transition surfaces, e.g., by use of a spline function.

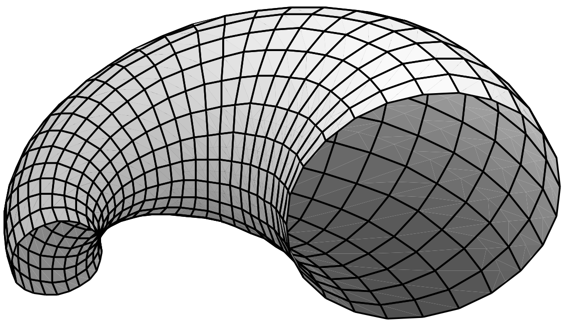

Figure 7: A Dupin cyclide is used for blending

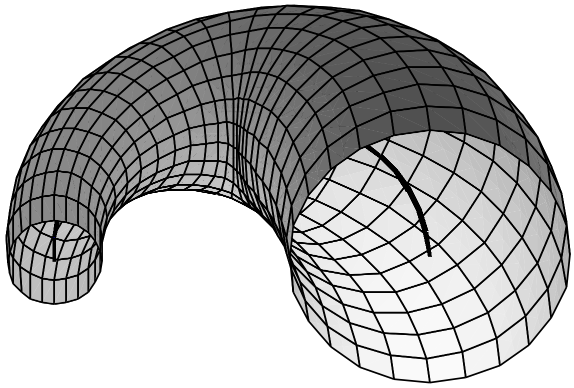

Figure 8: A cyclic blending surface Γ in comparison with Fig. 7

4. Comparison with canal surfaces

One can also use canal surfaces as blending surfaces. A *canal surface* is the envelope of one-parameter family of spheres with a varying radius (e.g., [7, 13, 14]). The generating circles — characteristic lines of the canal surface — are located in planes which in general are not normal planes of the determining curve. So the generating circles are not major circles on the spheres. This happens when the radius of the spheres is constant, i.e., for *tubular surfaces.* For a blending surface Γ this needs the condition $r_1 = r_0$.

If we use a canal surface for blending between two cylinders γ_0 and γ_1, the connections at the circles k_0 and k_1 need not be of C^1. We should choose the radii of the spheres such that the characteristics at the terminating points S_0 and S_1 are major circles of the corresponding spheres. Hence, the radius function of the sphere surface should have the same properties as the function R which have been listed above.

The derivation of a parametric representation for a general canal surface is not complicated as long as the arc length parametrization of the determining curve is given. However, most of the curves (e.g., cubics $\mathcal{S}$) do not have an elementary arc length parametrization. From this point of view the determination of a parametric representation of a blending cyclic surface Γ is simpler than that of a canal surface.

Dupin cyclides are special canal surfaces and also used as blending surfaces. We can find a number of parametric representations of Dupin cyclides, e.g., [1] – [6]. A Dupin cyclide as a blending surface is displayed in Fig. 7, where the axes of the cylinders are in a special position. For comparison, we show the blending surface Γ in Fig. 8 where we choose $k = 30$ and $l = 30$. In this case the cyclide is uniquely determined. The advantage of a blending cyclic surfaces Γ is its variability as it has been described above.

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