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On a Special Pair of Parallel Congruences

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Abstract. The subject of this paper is the study of a special pair of parallel line congruences, i.e., line congruences with a common spherical image, in a three-dimensional Euclidean space.

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1. Introduction

In a three-dimensional Euclidean space E^3, we consider an oriented line congruence S whose middle surface and middle envelope are the surfaces $\overline{OP} = P(u,v)$ and $\overline{OM} = M(u,v)$, respectively. Then we construct a line congruence S' with the same spherical image and with the surface of reference $\overline{OQ} = Q(u,v)$, defined by the relation $\overline{OQ} = \overline{MP}$. Some results concerning S', firstly, as a special case of an image congruence and secondly, in the case S is isotropic, were proved by L. Vermeire [8] and G. Stamou [3], respectively. In the present paper we find the middle surface, the middle envelope and elements of S' in terms of elements of S and study extensively their properties (Section 2). Moreover, we search for relations among certain ruled surfaces in S, S' and special nets on $M(u,v)$ (Section 3). Finally, we make some remarks on S', when its middle envelope is a part of a sphere (Section 4).

Let S be an oriented line congruence in E^3, represented by the vector equation

$$\overline{x}\,(u,v,t) = \overline{OP} + t\,\overline{e}_3, \quad -\infty < t < +\infty, \tag{1.1}$$

where $\overline{OP} = P(u,v)$ is the position vector for the surface of reference and $\overline{e}_3(u,v)$ is the unit vector in the direction of the straight lines of S. Suppose $D = \{\overline{e}_i(u,v)\,|\,i = 1,2,3\}$ is an orthonormal, positively oriented moving frame of S and $\overline{OM} = M(u,v)$ is the middle envelope of S.

We assume that S satisfies the following conditions:

(a) The vector functions $P(u,v)$, $M(u,v)$ and $\overline{e}_i(u,v)$, $i = 1,2,3$, are defined on a simply connected domain G in the (u,v)-plane and are of class C^4.

(b) The spherical representation of S is one-to-one.

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(c) The middle envelope $M(u,v)$ is a regular surface having no parabolic or umbilical points.

(d) There is a one-to-one mapping between the points of the middle surface and the points of the middle envelope.

Referring to the moving frame D, we may write

$$dP = \sum_{i=1}^{3} \sigma_i \,\overline{e}_i, \tag{1.2}$$

$$d\,\overline{e}_j = \sum_{i=1}^{3} \omega_{ji}\,\overline{e}_i, \quad \omega_{ij} + \omega_{ji} = 0, \quad i,j = 1,2,3, \tag{1.3}$$

where σ_i, ω_{ij} are linear differential forms for $i,j = 1,2,3$. We denote by $d\omega$ the exterior derivative of a linear differential form ω and by "$\wedge$" the wedge product of two differential forms. According to condition (b) the differential forms ω_{31}, ω_{32} are linearly independent, i.e.,

$$\omega_{31} \wedge \omega_{32} \neq 0. \tag{1.4}$$

Thus, we may put

$$d\omega_{31} = q\,\omega_{31} \wedge \omega_{32}, \quad d\omega_{32} = \widetilde{q}\,\omega_{32} \wedge \omega_{31}, \tag{1.5}$$

where q, $\widetilde{q}$ are functions of u and v defined on G. Then it is known [6, p. 268] that

$$\omega_{12} = q\,\omega_{31} - \widetilde{q}\,\omega_{32}\,. \tag{1.6}$$

Moreover, the surface of reference $\overline{OP} = P(u,v)$ is the middle surface of S if and only if [6, p. 268]

$$\omega_{31} \wedge \sigma_2 + \sigma_1 \wedge \omega_{32} = 0. \tag{1.7}$$

From now on, we assume that $\overline{OP} = P(u,v)$ is the middle surface of S. Then there exist functions l, m, n of u and v defined on G such that

$$\sigma_1 = -m\omega_{31} - n\omega_{32}, \quad \sigma_2 = l\omega_{31} + m\omega_{32}. \tag{1.8}$$

The curvature k, the mean curvature h and the limit distance $2z$ of S are given by the formulae

$$k = l\,n - m^2, \quad 2h = l + n, \tag{1.9}$$

$$2z = \sqrt{(l-n)^2 + 4m^2} = 2\sqrt{h^2 - k}. \tag{1.10}$$

Next, since the normal lines to the middle envelope $M(u,v)$ are parallel to the corresponding lines of the congruence, we may consider $\overline{e}_3(u,v)$ as the unit normal vector of $M(u,v)$ and D as the moving frame on $M(u,v)$. Therefore, there exist linear differential forms ρ, σ and functions α, β, δ of u and v defined on G, such that

$$dM = \rho\,\overline{e}_1 + \sigma\,\overline{e}_2\,, \tag{1.11}$$

$$\rho = \alpha\omega_{31} + \beta\omega_{32}, \qquad \sigma = \beta\omega_{31} + \delta\omega_{32}\,. \tag{1.12}$$

Again, for the Gaussian curvature K and the mean curvature H of $M(u,v)$ we have

$$K = \frac{1}{r_1 r_2} = \frac{1}{\alpha\delta - \beta^2}, \quad 2H = \frac{1}{r_1} + \frac{1}{r_2} = -\frac{\alpha + \delta}{\alpha\delta - \beta^2} \tag{1.13}$$

and

$$\frac{2H}{K} = r_1 + r_2 = -(\alpha + \delta), \tag{1.14}$$

where r_1, r_2 are the principal radii of curvature of $M(u, v)$.

A tangent plane of the middle envelope is a middle plane of the congruence. Hence there are functions $a(u, v)$, $b(u, v)$ for $(u, v) \in G$ such that

$$\overline{OP} - \overline{OM} = a\,\overline{e}_1 + b\,\overline{e}_2\,. \tag{1.15}$$

Let us now denote the Pfaffian derivatives with respect to the forms ω_{31}, ω_{32} by ∇_i, $i = 1, 2$. The functions a, b satisfy the condition [4, p. 14]

$$\nabla_1\, a + \nabla_2\, b - \widetilde{q}a - qb = r_1 + r_2\,, \tag{1.16}$$

and the relations [5, p. 159]

$$\sigma_3 = -a\,\omega_{31} - b\,\omega_{32}, \tag{1.17}$$

$$l = \nabla_1 b + qa + \beta, \tag{1.18}$$

$$m = -(\nabla_1 a - qb + \alpha) = \nabla_2 b - \widetilde{q}a + \delta, \tag{1.19}$$

$$n = -(\nabla_2 a + \widetilde{q}b + \beta) \tag{1.20}$$

are valid.

It is well-known that two fundamental invariant quadratic forms

$$I := (d\,\overline{e}_3)^2 = \omega_{31}^2 + \omega_{32}^2 \tag{1.21}$$

and

$$II := (\overline{e}_3, d\,\overline{e}_3, dP) = l\omega_{31}^2 + 2m\omega_{31}\omega_{32} + n\omega_{32}^2 \tag{1.22}$$

are assigned to the line congruence S. Furthermore, N.K. Stephanidis [6, p. 271] introduced the third quadratic form of S

$$III := (dP, dM, \overline{e}_3) = A\omega_{31}^2 + 2B\omega_{31}\omega_{32} + \Gamma\omega_{32}^2, \tag{1.23}$$

where

$$A = -(\alpha l + \beta m)\,, \quad 2B = -2\beta h + m\,(r_1 + r_2)\,,\; \Gamma = -(\beta m + \delta n)\,, \tag{1.24}$$

as well as the mixed curvature K^*

$$K^* = A\Gamma - B^2 \tag{1.25}$$

and the mixed mean curvature H^* of S

$$2H^* = A + \Gamma. \tag{1.26}$$

Note that *whenever we use K^*, H^*, we consider that there are no isotropic lines in S, i.e.,*

$$(l - n)^2 + 4m^2 \neq 0 \;\; \forall (u, v) \in G. \tag{1.27}$$

2. Middle surface and middle envelope

Suppose S is a line congruence defined on G by (1.1) and $\overline{OP} = P(u,v)$ is its middle surface. We assume that *the middle envelope* $\overline{OM} = M(u,v)$ *of* S *is different from its middle surface,* that is, *we exclude the case that* S *is the normal congruence of a minimal middle envelope.*

Consider the oriented segment $\overline{MP}$ joining the corresponding points of the surfaces $M(u,v)$ and $P(u,v)$. Via the relation

$$\overline{OQ} = \overline{MP} \tag{2.1}$$

a point Q is assigned to every pair of points M, P. Thus, in general, we get a surface $\overline{OQ} = Q(u,v)$ with position vector (2.1).

Let us now introduce the line congruence S', which is represented by the vector equation

$$\overline{y}(u,v,t') = \overline{OQ} + t'\overline{e}_3, \qquad -\infty < t' < +\infty, \tag{2.2}$$

where $\overline{OQ} = Q(u,v)$ stands for a surface of reference.

Obviously S, S' have the same spherical representation, i.e., they are *parallel congruences.* We denote by $\overline{OP'} = P'(u,v)$ the middle surface of S' and consider D as the moving frame of S'. Then, there exist a function $w(u,v)$ and linear differential forms σ_i', $i = 1,2,3$, such that

$$\overline{OP'} = \overline{OQ} + w\,\overline{e}_3\,, \tag{2.3}$$

$$dP' = \sum_{i=1}^{3} \sigma_i' \overline{e}_i\,. \tag{2.4}$$

After differentiating both sides of (2.3) and using (2.1), (1.3), (1.15), we get

$$dP' = (da - b\omega_{12} + w\omega_{31})\,\overline{e}_1 + (db + a\omega_{12} + w\omega_{32})\,\overline{e}_2 + (dw - a\omega_{31} - b\omega_{32})\,\overline{e}_3. \tag{2.5}$$

Then, since

$$d\varphi = \nabla_1\varphi\,\omega_{31} + \nabla_2\varphi\,\omega_{32} \tag{2.6}$$

holds for every differentiable function $\varphi(u,v)$, combining (2.4) with (2.5) we find

$$\sigma_1' = (\nabla_1 a - qb + w)\,\omega_{31} + (\nabla_2 a + \widetilde{q}\,b)\,\omega_{32}\,, \tag{2.7}$$

$$\sigma_2' = (\nabla_1 b + qa)\,\omega_{31} + (\nabla_2 b - \widetilde{q}\,a + w)\,\omega_{32}\,, \tag{2.8}$$

$$\sigma_3' = (\nabla_1 w - a)\,\omega_{31} + (\nabla_2 w - b)\,\omega_{32}\,. \tag{2.9}$$

Moreover, in view of (1.7), $P'(u,v)$ is the middle surface of S' iff

$$\omega_{31} \wedge \sigma_2' + \sigma_1' \wedge \omega_{32} = 0 \quad \forall (u,v) \in G, \tag{2.10}$$

or equivalently, by virtue of (2.7) and (2.8), iff

$$\nabla_1 a + \nabla_2 b - \widetilde{q}a - qb + 2w = 0. \tag{2.11}$$

Taking into account both relations (1.14) and (1.16), we derive from (2.11)

$$w = -\frac{H}{K}\,. \tag{2.12}$$

Then, we apply (2.12) to (2.3) and, by means of (2.1) and (1.15), we obtain the following results:

Proposition 2.1 *The middle surface of the congruence S' is defined by the vector equation*

$$\overline{OP'} = a\,\overline{e}_1 + b\,\overline{e}_2 - \frac{H}{K}\,\overline{e}_3\,. \tag{2.13}$$

Corollary 2.1 *The surface $\overline{OQ} = Q(u,v)$ is the middle surface of the congruence S' if and only if the middle envelope of S is a minimal surface.*

The last result has been found in [3, p. 10] too.

Suppose now, S is a hyperbolic congruence ($k < 0$). It is well-known [1, p. 16] that its focal surfaces $\overline{OF_i} = F_i(u,v)$, $i = 1,2$, are represented by

$$\overline{OF_i} = \overline{OP} \pm \rho\,\overline{e}_3, \quad i = 1,2, \tag{2.14}$$

where 2ρ is the focal distance of S and

$$\rho = \sqrt{-k}. \tag{2.15}$$

Consider as reference surface of S' one of the surfaces $\overline{OF_i^*} = F_i^*(u,v)$, $i = 1,2$, where

$$\overline{OF_i^*} = \overline{MF_i}, \qquad i = 1,2. \tag{2.16}$$

Evidently

$$\overline{MF_i} = \overline{MP} + \overline{PF_i} = a\,\overline{e}_1 + b\,\overline{e}_2 \pm \rho\,\overline{e}_3, \qquad i = 1,2, \tag{2.17}$$

hold true. Then, by virtue of (2.15)–(2.17), the relation (2.13) leads to the ensuing

Proposition 2.2 *The middle surface $P'(u,v)$ of the congruence S' coincides with one of the surfaces $F_i^*(u,v)$, $i = 1,2$, iff*

$$k = -\frac{H^2}{K^2}\,. \tag{2.18}$$

Similarly, we may consider the surfaces $\overline{OG_i^*} = G_i^*(u,v)$, $i = 1,2$, defined by

$$\overline{OG_i^*} = \overline{MG_i}, \quad i = 1,2,$$

where $\overline{OG_i} = G_i(u,v)$, stand for the limit surfaces of S [1, p. 10], i.e.,

$$\overline{OG_i} = \overline{OP} \pm z\,\overline{e}_3\,.$$

From this, we can get further the following:

Proposition 2.3 *The middle surface $P'(u,v)$ of the congruence S' coincides with one of the surfaces $G_i^*(u,v)$, $i = 1,2$, if and only if*

$$z = \pm\frac{H}{K}\,. \tag{2.19}$$

Now we denote the elements of S' with accentuated symbols of the corresponding elements of S. Thus, there exist functions l', m', n', a', b' of u and v, such that

$$\sigma_1' = -m'\omega_{31} - n'\omega_{32}, \quad \sigma_2' = l'\omega_{31} + m'\omega_{32}, \quad \sigma_3' = -a'\omega_{31} - b'\omega_{32} \tag{2.20}$$

and, according to (2.7)–(2.9) and (2.12), it follows that

$$\begin{aligned} l' &= \nabla_1 b + qa, && (2.21)\\ m' &= -\nabla_1 a + qb + \frac{H}{K} = \nabla_2 b - \widetilde{q}a - \frac{H}{K}\,, && (2.22)\\ n' &= -\nabla_2 a - \widetilde{q}b, && (2.23)\\ a' &= a + \nabla_1\left(\frac{H}{K}\right), && (2.24)\\ b' &= b + \nabla_2\left(\frac{H}{K}\right). && (2.25) \end{aligned}$$

Moreover, on account of (1.18)–(1.20), the relations (2.21)–(2.23) become

$$\begin{aligned} l' &= l - \beta, && (2.26)\\ m' &= m + \alpha + \frac{H}{K} = m + \frac{\alpha-\delta}{2}\,, && (2.27)\\ n' &= n + \beta. && (2.28) \end{aligned}$$

Hence, in view of (1.9), for the curvature k' and the mean curvature h' of S' we have

$$\begin{aligned} k' &= l'n' - m'^2 = k + \beta(l-n) - \beta^2 - m(\alpha-\delta) - \frac{(\alpha-\delta)^2}{4}\,, && (2.29)\\ h' &= \frac{l'+n'}{2} = h. && (2.30) \end{aligned}$$

The relation (2.30) verifies the proposition:

The mean curvatures h, h' of the congruences S, S' respectively are equal [8, p. 18].

Furthermore, let A (resp. A') be the pitch of an arbitrary closed ruled surface of S (resp. S') defined on the boundary ∂G^* of a simply connected domain $G^* \subset G$. We know [4, p. 18] that

$$A = -2\iint\limits_{G^*} h\,\omega_{31}\wedge\omega_{32} \quad (\text{resp.} \quad A' = -2\iint\limits_{G^*} h'\omega_{31}\wedge\omega_{32}\,.$$

Therefore, we get immediately:

The pitches A of a closed ruled surface of S and A' of the corresponding surface of S' are equal ($A = A'$).

Assume now that the vectors $\overline{e}_1(u,v)$, $\overline{e}_2(u,v)$ are tangent to the lines of curvature on $M(u,v)$, i.e., the lines of curvature are the parameter curves $\omega_{31} = 0$, $\omega_{32} = 0$. Then

$$\beta = 0, \quad \alpha = -r_1, \quad \delta = -r_2 \tag{2.31}$$

are valid, and (2.29) takes the form

$$k' = k - m\,(r_2 - r_1) - \frac{(r_2-r_1)^2}{4}\,. \tag{2.32}$$

In addition, the S-principal ruled surfaces of the congruence S' are the parameter surfaces $\omega_{31} = 0$, $\omega_{32} = 0$ iff

$$m = 0 \quad \forall (u,v) \in G. \tag{2.33}$$

Also, by virtue of (1.13),

$$\frac{(r_2 - r_1)^2}{4} = \frac{H^2 - K}{K^2} \tag{2.34}$$

holds true. Then, (2.32)–(2.34) lead to

Proposition 2.4 *Suppose the S-principal ruled surfaces of S correspond to the lines of curvature of $M(u, v)$. Then*

$$k' = k - \frac{H^2 - K}{K^2}\,. \tag{2.35}$$

An immediate consequence of (2.35) is

$$k' < k. \tag{2.36}$$

So *if S is hyperbolic, then S' is also a hyperbolic congruence.*

Besides, it is known [6, p. 277] that the S-principal ruled surfaces of S correspond to the lines of curvature on $M(u, v)$ iff

$$K^* = \frac{k}{K}\,. \tag{2.37}$$

Thus, (2.35) may be rewritten as

$$k' = KK^* - \frac{H^2 - K}{K^2}\,. \tag{2.38}$$

Similarly, using (2.30) and (2.35), we obtain from (1.10):

Proposition 2.5 *If the S-principal ruled surfaces of S correspond to the lines of curvature on $M(u, v)$, then for the limit distance $2z'$ of S'*

$$z'^2 = z^2 + \frac{H^2 - K}{K^2} \tag{2.39}$$

holds true.

Note that, according to (2.39) and (2.35), we deduce: The formulae

$$z'^2 > z^2, \tag{2.40}$$

$$z'^2 - z^2 = k - k' = \frac{H^2 - K}{K^2} \tag{2.41}$$

are valid.

Furthermore, in the case that $K < 0$, we may express (2.35) in terms of the angle φ between the asymptotic curves on $M(u, v)$. Since

$$\cos\varphi = \frac{H}{\sqrt{H^2 - K}}, \quad 0 < \varphi < \pi, \tag{2.42}$$

we easily find

$$k' = k + \frac{1}{K\sin^2\varphi}\,. \tag{2.43}$$

We focus now on the middle envelope $M'(u, v)$ of the congruence S'. The position vector $\overline{OM'}$ of $M'(u, v)$ satisfies the equation

$$\overline{OM'} = \overline{OP'} - a'(u, v)\overline{e}_1 - b'(u, v)\overline{e}_2\,, \quad (u, v) \in G. \tag{2.44}$$

Therefore, taking equations (2.13), (2.24) and (2.25) into account, we find

Proposition 2.6 *The middle envelope $M'(u,v)$ of the congruence S' is given by*

$$\overline{OM'} = -\nabla_1\left(\frac{H}{K}\right)\overline{e}_1 - \nabla_2\left(\frac{H}{K}\right)\overline{e}_2 - \frac{H}{K}\,\overline{e}_3\,. \tag{2.45}$$

From (2.45) the already known results [8, p. 19] follow at once:

Corollary 2.2 *The congruence S is an M-congruence, i.e., the middle envelope $M(u,v)$ of S is minimal, iff the middle envelope $M'(u,v)$ of S' degenerates into the point O.*

Corollary 2.3 *If $\frac{H}{K} = c$ with $c \in \mathbb{R}\backslash\{0\}$, then the middle envelope of S' is part of a sphere centered at O and with radius c.*

Again, there exist linear differential forms ρ', σ' and functions α', β', δ', such that

$$dM' = \rho'\overline{e}_1 + \sigma'\overline{e}_2, \tag{2.46}$$

$$\rho' = \alpha'\omega_{31} + \beta'\omega_{32}\,, \quad \sigma' = \beta'\omega_{31} + \delta'\omega_{32}\,. \tag{2.47}$$

Therefore, differentiating both sides of (2.45) and using (1.3), (1.6), (2.6), (2.46), (2.47), we obtain

$$\alpha' = -\nabla_1\nabla_1\left(\frac{H}{K}\right) + q\,\nabla_2\left(\frac{H}{K}\right) - \frac{H}{K}\,, \tag{2.48}$$

$$\beta' = -\nabla_2\nabla_1\left(\frac{H}{K}\right) - \widetilde{q}\,\nabla_2\left(\frac{H}{K}\right) = -\nabla_1\nabla_2\left(\frac{H}{K}\right) - q\,\nabla_1\left(\frac{H}{K}\right), \tag{2.49}$$

$$\delta' = -\nabla_2\nabla_2\left(\frac{H}{K}\right) + \widetilde{q}\,\nabla_1\left(\frac{H}{K}\right) - \frac{H}{K}\,. \tag{2.50}$$

Let us now denote by K' and H' the curvature and the mean curvature of $M'(u,v)$ respectively. A relation similar to (1.14) holds true:

$$\frac{2H'}{K'} = -\left(\alpha' + \delta'\right). \tag{2.51}$$

We replace α', δ' by the right-hand side of (2.48), (2.50) and we infer

Proposition 2.7 *The formula*

$$\frac{2H'}{K'} = \Delta\left(\frac{H}{K}\right) + \frac{2H}{K} \tag{2.52}$$

is valid, where Δ stands for the second differential operator of Beltrami with respect to the first fundamental form of S.

Direct consequences of Proposition 2.7 are

Corollary 2.4 *The following properties are equivalent:*

(a) $\frac{H}{K}$ *is a harmonic function,*

(b) $\frac{H}{K} = \frac{H'}{K'}$.

Corollary 2.5 *The middle envelope of S' is minimal or degenerates into a point if and only if*

$$\Delta\left(\frac{H}{K}\right) + \frac{2H}{K} = 0. \tag{2.53}$$

When $M(u,v)$ is a non-minimal surface of constant curvature K, then S' is an M-congruence iff

$$\Delta H + 2H = 0.$$

Moreover, taking into account Propositions 2.2 and 2.3:

Corollary 2.6 *In the case one of the surfaces $F_i^*(u,v)$, $i=1,2$, (resp., $G_i^*(u,v)$, $i=1,2$) is the middle surface of S', a necessary and sufficient condition for the middle envelope $M'(u,v)$ of S' to be minimal or to degenerate into a point is*

$$\Delta\rho + 2\rho = 0 \quad (\textit{resp. } \Delta z + 2z = 0),$$

where 2ρ and $2z$ are the focal and the limit distance of S, respectively.

3. Ruled surfaces

Some relations among certain ruled surfaces in S, S' and special nets on $M(u,v)$ were established by L. Vermeire [8] and G. Stamou [3] (in the case S is isotropic). Our aim is to extend their investigation.

We recall the differential equations of special ruled surfaces in S [4, p. 10–12]:

$$l\omega_{31}^2 + 2m\omega_{31}\omega_{32} + n\omega_{32}^2 = 0 \qquad \text{(developable surfaces)}, \tag{3.1}$$

$$m\left(\omega_{31}^2 - \omega_{32}^2\right) - (l-n)\omega_{31}\omega_{32} = 0 \qquad \text{(S-principal ruled surfaces)}, \tag{3.2}$$

$$(l-n)\left(\omega_{31}^2 - \omega_{32}^2\right) + 4m\omega_{31}\omega_{32} = 0 \qquad \text{(K-principal ruled surfaces)}. \tag{3.3}$$

The corresponding ones in S' are similar but l, m, n are replaced by l', m', n'. The latter, by virtue of (2.26)–(2.28), become respectively

$$(l-\beta)\omega_{31}^2 + 2\left(m + \frac{\alpha-\delta}{2}\right)\omega_{31}\omega_{32} + (n+\beta)\omega_{32}^2 = 0, \tag{3.4}$$

$$\left(m + \frac{\alpha-\delta}{2}\right)\left(\omega_{31}^2 - \omega_{32}^2\right) - (l-n-2\beta)\omega_{31}\omega_{32} = 0, \tag{3.5}$$

$$(l-n-2\beta)\left(\omega_{31}^2 - \omega_{32}^2\right) + 4\left(m + \frac{\alpha-\delta}{2}\right)\omega_{31}\omega_{32} = 0. \tag{3.6}$$

In this section we assume that S *is not isotropic,* $\frac{H}{K} \neq c$, $c = $ *const., and the moving frame* D, *is such that* $\bar{e}_1(u,v)$, $\bar{e}_2(u,v)$ *are the principal directions on* $M(u,v)$. Thus, the relations (2.31) are valid.

Proposition 3.1 *Suppose S satisfies the conditions:*

(i) $2z = |r_1 - r_2|$,

(ii) *The K-principal ruled surfaces of S correspond to the lines of curvature on $M(u,v)$.*

Then either S' is isotropic or $z' = 2z$.

Proof: According to (2.31),

$$2z = |\alpha - \delta| \tag{3.7}$$

and (2.26), (2.28) take the form

$$l' = l, \quad n' = n, \tag{3.8}$$

respectively. Moreover, since the lines of curvature on $M(u,v)$ are defined by $\omega_{31} = 0$, $\omega_{32} = 0$, due to (3.3), condition (ii) yields $l = n$. This means, by virtue of (3.8), that

$$l' = n' \tag{3.9}$$

and, by (3.7), (1.10), we have

$$2m = \pm(\alpha - \delta)\,. \tag{3.10}$$

Applying (3.10) to (2.27), it follows that

$$m' = 0 \quad \text{or} \quad m' = \alpha - \delta. \tag{3.11}$$

Evidently, (3.9) and (3.11) lead to the required assertion. □

Note that *the K-principal ruled surfaces of S correspond to the lines of curvature on $M(u,v)$ iff* $H^* = \frac{H}{K}h$ [2, p. 381]. Thus, condition (ii) in Proposition 3.1 can be replaced by $H^* = \frac{H}{K}h$.

Conversely:

Proposition 3.2 *If S' is isotropic, then the conditions* (i) *and* (ii) *of Proposition 3.1 hold true.*

Proof: For S' isotropic

$$l' = n' \quad \text{and} \quad m' = 0.$$

Hence, using (2.26)–(2.28) and (2.31), we find

$$l = n, \quad 2m = \delta - \alpha. \tag{3.12}$$

Inserting (3.12) into (1.10), we get (i). Since $\beta = 0$ and $l = n$, the K-principal surfaces of S correspond to the lines of curvature on $M(u,v)$, and consequently the equivalent condition (ii) is valid. □

From now on, S' is considered to be *nonisotropic.*

Proposition 3.3 *Consider the congruences S, S'. The following properties are equivalent:*

(a) *The K-principal ruled surfaces of S correspond to the lines of curvature on $M(u,v)$.*

(b) *The K-principal ruled surfaces of S correspond with those of S'.*

(c) $H^* = \frac{H}{K}h$.

Proof: Assume property (a). Since $\beta = 0$, the relation $l = n$ holds true. Thus, from (2.26) and (2.28), it follows $l' = n'$, i.e., the property (b). Conversely, if (b) is satisfied, then on account of (3.3) and (3.6)

$$(l - n)(\alpha - \delta) = 0.$$

However, under condition (c) in Section 1, the surface $M(u,v)$ has no umbilical points. So

$$\alpha - \delta \neq 0,$$

hence (a) is valid. Again, as we know, (c) is a necessary and sufficient condition for (a) to be valid [2, p. 381]. Therefore (c) is equivalent to (b) too. □

Note that further immediate consequences of (a) and (b) are the equivalent properties:

(d) *The S-principal ruled surfaces of S correspond to the lines bisecting the lines of curvature on $M(u,v)$.*

(e) *The S-principal ruled surfaces of S correspond to those of S'.*

Next, consider the differential equation

$$\beta' \left(\omega_{31}^2 - \omega_{32}^2\right) + (\alpha' - \delta')\, \omega_{31}\omega_{32} = 0 \tag{3.13}$$

of the spherical image of the lines of curvature on $M'(u,v)$. The relation (3.13) reduces to $\omega_{31} = 0$, $\omega_{32} = 0$ iff $\beta' = 0$, which, by means of (2.49), leads to

Proposition 3.4 *A necessary and sufficient condition for the spherical image of the lines of curvature on $M(u,v)$ and $M'(u,v)$, respectively, to be the same is*

$$\nabla_1 \nabla_2 \left(\frac{H}{K}\right) + q\, \nabla_1 \left(\frac{H}{K}\right) = 0. \tag{3.14}$$

Moreover the *III*-principal ruled surfaces of S' are defined by the differential equation [7, p. 54]

$$B' \left(\omega_{31}^2 - \omega_{32}^2\right) - (A' - \Gamma')\, \omega_{31}\omega_{32} = 0, \tag{3.15}$$

where

$$A' = -\left(\alpha'\, l' + \beta'\, m'\right), \quad 2B' = -2\beta'\, h' - m'\left(\alpha' + \delta'\right), \quad \Gamma' = -\left(\beta'\, m' + \delta'\, n'\right). \tag{3.16}$$

We shall prove the ensuing

Proposition 3.5 *Let S, S' have the following properties:*

(a) *The K-principal ruled surfaces of S correspond to the lines of curvature on $M(u,v)$,*

(b) *the S-principal ruled surfaces of S and the III-principal ruled surfaces of S' have the same spherical image,*

(c) $\nabla_1 \nabla_2 \left(\frac{H}{K}\right) + q\, \nabla_1 \left(\frac{H}{K}\right) \neq 0.$

Then, either the S-principal ruled surfaces of S correspond to the lines of curvature on $M'(u,v)$ or S is normal.

Proof: Since (a) holds true, we have $\beta = 0$, $l = n$ and, by virtue of (2.26) and (2.28), $l' = n'$. Thus, the S-principal ruled surfaces of S are defined by the differential equation $\omega_{31}^2 - \omega_{32}^2 = 0$ and because of (b), (3.15), (3.16), it follows that

$$A' - \Gamma' = l'\delta' - \alpha' l' = 0$$

or equivalently

$$l'\left(\delta' - \alpha'\right) = 0, \tag{3.17}$$

i.e., $l' = 0$ or $\delta' - \alpha' = 0$. On one hand, this means $h' = l' = n' = 0$, hence S', as well as S, is normal. On the other hand, since (c) is valid, i.e., $\beta' \neq 0$, (3.13) becomes $\omega_{31}^2 - \omega_{32}^2 = 0$. □

Proposition 3.6 *If the K-principal ruled surfaces of a congruence S correspond to the lines of curvature on $M(u,v)$, then the following properties are equivalent:*

(a) *The K-principal ruled surfaces of S' correspond to the lines of curvature on $M'(u,v)$.*

(b) *The corresponding lines of curvature on $M(u,v)$ and $M'(u,v)$ have the same spherical image.*

(c) $2H^{*\prime} = h\left[\Delta\left(\frac{H}{K}\right) + \frac{2H}{K}\right].$

Proof: By assumption, $\beta = 0$ and $l = n$. Consequently, on account of (2.26) and (2.28), we have $l' = n'$. Thus, (a) leads to $\beta' = 0$. However, $\beta = \beta' = 0$ means (b) and vice versa. In addition, (a) is equivalent to the relation [2, p. 381]

$$H^{*\prime} = \frac{H'h'}{K'}\,. \tag{3.18}$$

Using now (2.30) and (2.52), we find (c). □

Furthermore, we shall show that

Proposition 3.7 *If three of the following properties are valid, then all four are satisfied.*

(a) *The K-principal ruled surfaces of S correspond to the lines of curvature on $M(u,v)$.*

(b) *The K-principal ruled surfaces of S' correspond to the lines of curvature on $M'(u,v)$.*

(c) $\dfrac{H}{K}$ *is a harmonic function.*

(d) $H^{*\prime} = H^*$.

Proof: As mentioned above, (a) is equivalent to

$$H^* = \frac{H}{K}\,h\,, \tag{3.19}$$

(b) is equivalent to (3.18), and (c), in view of Corollary 2.4, is equivalent to

$$\frac{H}{K} = \frac{H'}{K'}\,. \tag{3.20}$$

Thus, when (a), (b), (c) are valid, by virtue of (2.30), we obtain (d). Similarly, in case (a), (b), (d) hold true, we get (3.20) and on account of Corollary 2.4, property (c). Suppose now the properties (a), (c), (d) are valid. Then from (3.19), (3.20) and (2.30), we get (3.18) and consequently (b). Finally, let (b), (c), (d) hold true simultaneously. By means of (3.20) and (3.18), there follows (3.19) or, equivalently, (a). □

In view of Proposition 3.6 and Corollary 2.5 we derive:

Corollary 3.1 *Suppose that the properties (a) and (b) of Proposition 3.7 hold true. Then $H^{*\prime} = 0$ if and only if one of the following conditions is valid:*

(a) *S is normal,*

(b) *$M'(u,v)$ is a minimal surface.*

4. A special case

In this section we deal with the special case

$$\frac{H}{K} = c, \qquad c = \text{const.} \neq 0. \tag{4.1}$$

That is, according to Corollary 2.3, the case $M'(u,v)$ is a part of a sphere. Using (2.18) and (2.19), we obtain easily:

(A) *Let S be hyperbolic. If the middle surface of S' is one of the surfaces $F_i^*(u,v)$ (resp., $G_i^*(u,v)$), $i = 1,2$, then the curvature k (resp., the limit distance $2z$) of S is constant.*

Similarly, from (2.52) we arrive at

(B) *The relation* $\frac{H}{K} = \frac{H'}{K'}$ *is valid.*

Besides, if we insert (2.26)–(2.28) and (2.48)–(2.50) into (3.16), we derive

$$A' = \frac{H}{K}\,(l-\beta)\,, \quad B' = \frac{H}{K}\left(m + \frac{\alpha-\delta}{2}\right), \quad \Gamma' = \frac{H}{K}\,(n+\beta)\,. \tag{4.2}$$

Thus, the third quadratic form of S' can be written

$$III' = \frac{H}{K}\left[(l-\beta)\,\omega_{31}^2 + 2\left(m + \frac{\alpha-\delta}{2}\right)\omega_{31}\omega_{32} + (n+\beta)\,\omega_{32}^2\right] \tag{4.3}$$

and, by virtue of (3.15), the differential equation of the III-principal ruled surfaces of S' takes the form

$$\left(m + \frac{\alpha-\delta}{2}\right)(\omega_{31}^2 - \omega_{32}^2) - (l - n - 2\beta)\omega_{31}\omega_{32} = 0. \tag{4.4}$$

Immediate consequences of relations (4.1)–(4.4) are:

(C) *Let II', III' be the second and third quadratic form of S', respectively. The following formulae*

$$(i)\quad III' = cII', \qquad (ii)\quad III' = c\,(II - IV_M)$$

hold true, where IV_M stands for the fourth quadratic form of $M(u,v)$.

(D) *The III-principal ruled surfaces and the S-principal ruled surfaces of S' coincide.*

Additionally, for the mixed mean curvature $H^{*\prime}$ and the mixed curvature $K^{*\prime}$ of S', we have

$$2H^{*\prime} = A' + \Gamma', \quad K^{*\prime} = A'\Gamma' - B'^2.$$

Thus, making use of (4.1), (4.2), (2.29), (2.30), and (2.48)–(2.50), we conclude that

(E) *The formulae*

$$H^{*\prime} = c\,h', \qquad K^{*\prime} = c^2 k' \tag{4.5}$$

are valid.

Therefore

(F) $H^{*\prime} = 0$ *iff S (as well as S') is normal.*

(G) $K^{*\prime} = 0$ *iff S' is parabolic.*

Finally, we note that

(H) *In case S' is a hyperbolic congruence ($k' < 0$), we have*

- (i) $K^{*\prime} < 0$,
- (ii) *the surfaces $III' = 0$ coincide with the developable surfaces of S' and*
- (iii) *the spherical image of the surfaces $III' = 0$ is an orthogonal net iff S is a normal congruence.*

Indeed (i) is easily proved by using the second equation of (4.5). Similarly (ii) is a direct consequence of (3.4) and (4.3). Finally, (iii) follows from (ii), since the angle ω of the spherical image of the developable surfaces is $\pi/2$ iff S' (as well as S) is a normal congruence.

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