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# One a Possible Constructive Geometrical Derivation of Mercator's Conformal Cylindrical Map Projection Based on Some Historical Facts

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**Abstract.** In modern cartography, Mercator's conformal cylindrical map projection is mathematically described by equations which involve logarithmic and trigonometric functions. It is evident that calculus and differential geometry are necessary for the derivation of these equations. The very fact that Mercator's map was published in 1569, long before logarithms and calculus were invented, confirms that Mercator created his famous map projection somehow directly — by some simple and plain geometrical constructions. This paper exposes one possible constructive graphical method by which the direct geometrical synthesis of Mercator's conformal cylindrical map projection can be accomplished. We believe that the original inventive process used by Gerhard Kremer Mercator in creation of his famous world map was at least similar or compatible to the method exposed in this paper.

*Key Words:* cartography, Mercator, conformal, loxodrome, map, stereographic

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## 1. Introduction

It is well known that Mercator's map projection belongs to the family of cylindrical, conformal and non perspective cartographic projections (Fig. 1).

In modern cartography, its mathematical description represents the formula which transforms longitude-latitude $(\lambda, \varphi)$ of the Earth's sphere to Cartesian $(x, y)$ in such a way that these Cartesian coordinates are:

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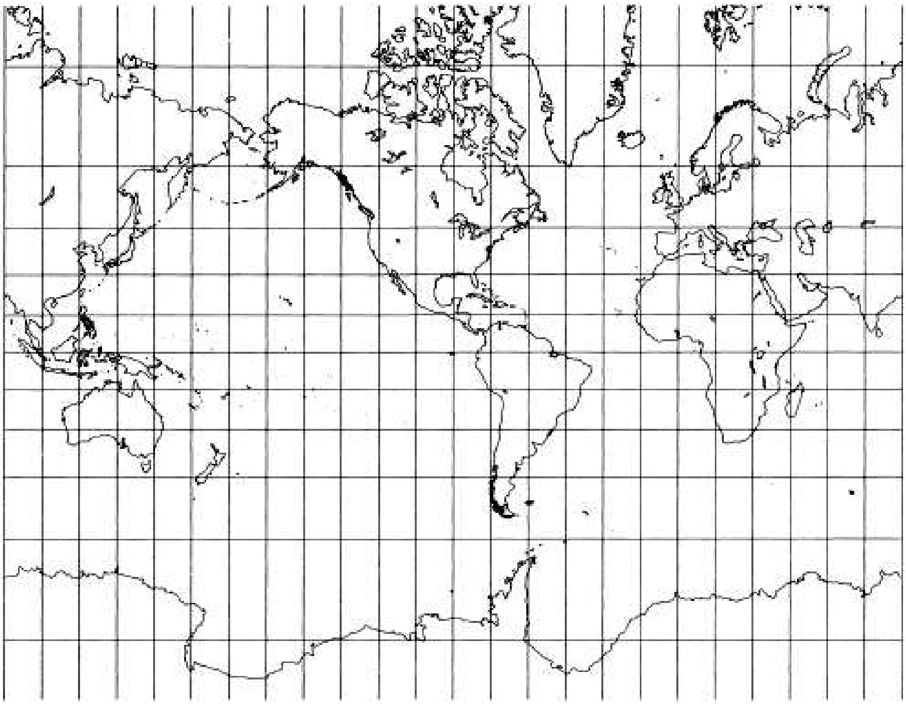

Figure 1: Mercator's conformal cylindrical map projection

- coordinates on a cylinder tangent to the Earth's equator and
- conformal.

It is evident that calculus and differential geometry are necessary for the derivation of this formula which involves logarithmic and trigonometric functions.[1] The historical fact that Gerhard Kremer Mercator created his conformal map in 1569, long before logarithms were invented by John Napier in 1614 and calculus by Isaac Newton in 1687 leads to the conclusion that Mercator's heuristic method was *not algebraic and analytical.* Indeed, it is very astonishing how exactly Mercator created and developed his famous map.

## 2. Nautical navigation in the 16th century

Gerhard Kremer Mercator was a great Flemish geographer and cartographer who lived from 1512 to 1594. It was the time of glorious expeditions and explorations of unknown regions, rapid European colonization of the "New World" and expansion of maritime trade. Under such historical circumstances, accurate maps were extremely important for the correct nautical navigating and secure sailing. Besides astrolabe, quadrant and telescope (Fig. 2), in the 16th century the compass was the most significant navigational instrument which sailors could use to determine and control ship courses in the open sea. It was convenient at that time to sail such a course that the angle measured between the ship axis and the reference direction of magnetic north is constant, i.e., to sail a course with a *constant heading.* The constant heading line is the spiral curve on the Earth's globe named *loxodrome* (also called a *rhumb line*) which intersects all meridians under a constant angle. (Despite the fact that a distance of the constant heading line is longer than the great circle distance, sailors chose the heading line course because this route was easier controlled and maintained by the compass.)

[1]The mapping equations $(\lambda,\varphi)\mapsto(x,y)$ are: $x=\lambda-\lambda_0$, $y=\ln(\tan\varphi+\sec\varphi)=\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$.

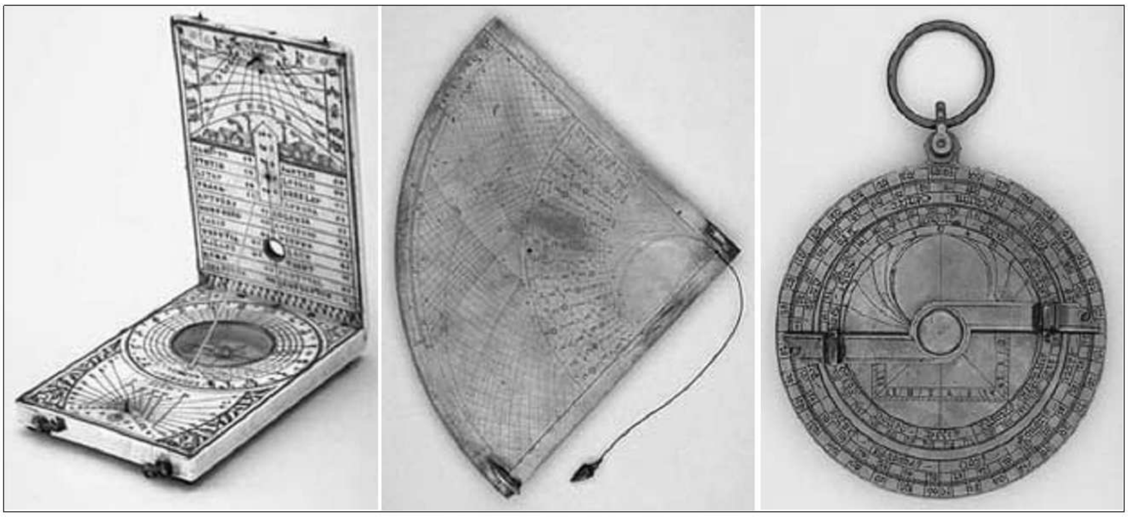

Figure 2: Compass, Quadrant, Astrolabe, 15th century

The course with a constant heading can be defined and measured on the conformal maps solely. The historical facts confirm that exclusively one type of conformal geographic map was known in the 16th century. That was *Hipparchus' stereographic map* whose circular grid was used on another extremely important navigational instrument, the *planispherical astrolabe*, sometimes called *"the precious jewel of medieval navigation"*. The projection of a loxodrome on the stereographic polar map represents the golden or logarithmic spiral (spira mirabilis). Since it is hard and impractical to construct, draw and manipulate this spiral curve on the Hipparchus' stereographic map, Mercator decided to develop a new conformal map by which the loxodromes would be represented as straight lines.

## 3. Mercator's heuristic method

It was emphasized in the previous paragraphs that Mercator did not possess the knowledge of algebra and modern algebraic notations, logarithmic and other transcendental functions, as well as the knowledge of differential and integral calculus. All mathematical structures and operations mentioned above hadn't been invented in the 16th century. Consequently to these historical facts, it is evident that Mercator did not recognize the logarithmic spiral, its algebraic notation, neither any other algebraic symbolical inscription. This also means that Mercator did not know all the important features of the logarithmic spiral which were considered, analyzed and published by René Descartes, Evangelista Torricelli and Jacob Bernoulli many decades later. Yet, he created the conformal map successfully on which each loxodrome projection would be a straight line.

The aforementioned facts may confuse and surprise us but only at the first moment. If we take into the serious consideration the certainty that, instead of algebraic and analytical approach, Mercator could use a constructive-synthetic and direct geometrical method even to produce a new conformal map, the skepticism and astonishment disappear quickly. The geometrical and constructive-synthetic method avoids algebraic inscriptions, symbolical marks, especially conferred signs and progressively synthesizes complex conclusions from basic, elementary and plain facts. It is a historical evidence that this direct heuristic approach was invented and established by geometricians and philosophers of ancient Greece and well known in West Europe in the 16th century. From the above, the presumption that MERCATOR could know and could use the constructive synthetic methods of the ancient Greek geometricians to develop his conformal map is highly reasonable and perfectly acceptable.


Figure 3: Gerhard KREMER MERCATOR (1512–1594)

In the following paragraphs it will be explained in every detail how MERCATOR could obtain the conformal, cylindrical map on which the loxodrome projections are straight lines. The process of map synthesis will be divided in two main phases. Firstly, the geometrical construction of the loxodrome projection will be accomplished on Hipparchus' stereographic polar map. And secondly, the process of cylindrical conformal map creation will be obtained by the rectification of the loxodrome projection. We believe that the original inventive process used by Gerhard KREMER MERCATOR when creating his famous world map was at least similar or compatible to the method which will be disclosed in the following paragraphs.

## 4. The geometrical synthesis of loxodrome

For the geometrical synthesis and other progressive heuristic methods it matters little whether the polar stereographic projection of a loxodrome is named logarithmic spiral, miraculous spiral or "spira mirabilis". Neither the algebraic formalism nor the symbolic inscriptions are of great significance for the synthetic approach. The geometrical synthesis of the conformal map grid on which the loxodrome projection is a straight line can be accomplished from the simple fact that the loxodrome is a curve on the Earth's globe which intersects all meridians at a constant angle.

As shown in Fig. 4, the geometrical construction of a loxodrome projection can be performed on Hipparchus' stereographic map, since the angles and the scale of $\lambda$ are preserved under this type of map projection. Historical facts confirm that MERCATOR possessed the knowledge of Hipparchus' stereographic map and that he could use it in an identical or similar way.

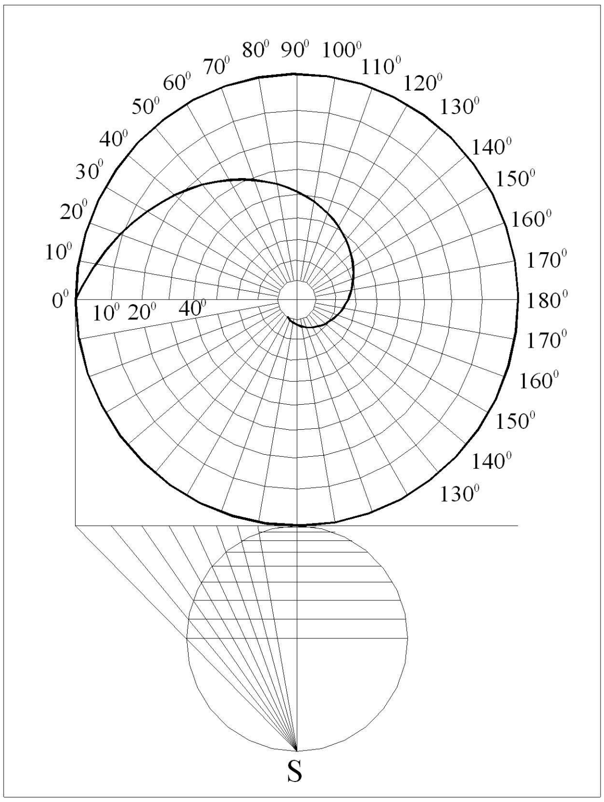


Figure 4: Loxodrome on Hipparchus' polar stereographic map

The geometric synthesis of the loxodrome cannot be absolutely accurate since its stereographic projection represents transcendental curve. Yet, it will be disclosed that its construction can be accomplished with the desired degree of accuracy which is defined beforehand, in several stages — iteratively.

**1)** The first stage of the loxodrome construction represents the approximation of this curve by its chord array in such a way that every chord intersects the corresponding meridian at a constant angle. As shown in Fig. 5, the chords $A_1A_2$, $A_2A_3$, $\ldots$ intersect the stereographic projection of meridians $OA_1$, $OA_2$, $\ldots$ at the angles $\sphericalangle\,(A_1O,\, A_1A_2)$, $\sphericalangle\,(A_2O,\, A_2A_3)$, $\ldots$, respectively. According to the presumption, these angles are equal. If the uniform meridian disposition is assumed, the angles $\sphericalangle\,(OA_1,\, OA_2)$, $\sphericalangle\,(OA_2,\, OA_3)$, $\ldots$ are also equal. From the above, one can draw the conclusion that the triangles $OA_1A_2$, $OA_2A_3$, $OA_2A_3$, $OA_3A_4$, $\ldots$ are similar to each other, and that this array of similar triangles is infinite.

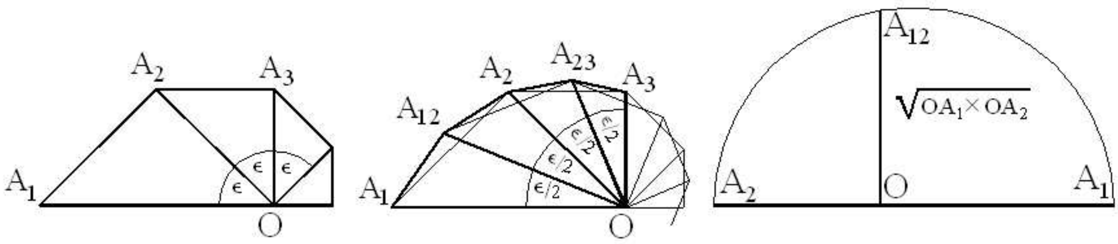


Figure 5: Iterative method for loxodrome construction

The triangles similarity implies:

$$OA_1 : OA_2 = OA_2 : OA_3 = \ \ldots \ = q_1 \,,$$

and the fact that the measure of the line segment $OA_2$ is the geometric mean of the line segments measures $OA_1$ and $OA_3$. Moreover, it can be concluded that the segments measures $OA_1$, $OA_2$, $OA_3$, ... represent a geometrical progression whose geometric ratio is $q_1$ .

**2)** The second stage of the loxodrome construction represents the next step of iteration by which the higher accuracy of the loxodrome approximation can be obtained. This can be accomplished if the first stage of the loxodrome construction is reiterated on the stereographic polar map on which the doubled number of meridians is drawn. Measures of the line segments $OA_{12}$, $OA_{23}$ ... are the geometric means of the measures of line segments $OA_1$ and $OA_2$, $OA_2$ and $OA_3$, ..., respectively. Triangles $OA_1A_{12}$, $OA_{12}A_2$, $OA_2A_{23}$, $OA_{23}A_3$ ... represent the infinite array of similar triangles from which the following proportion results:

$$OA_1 : OA_{12} = OA_{12} : OA_2 = OA_2 : OA_{23} = OA_{23} : OA_3 = \ \ldots \ = q_2 \,,$$

and the fact that the measures of segments $OA_1$, $OA_{12}$, $OA_2$, $OA_{23}$, $OA_3$, ... represent a geometrical progression whose geometric ratio is $q_2$ .

It can be easily noticed that all new points $A_{12}$, $A_{23}$ ... belong to the same loxodrome which includes already the old points $A_1$, $A_2$, $A_3$, ..., and that the geometric ratios obey $q_2 = \sqrt{q_1}$ . It is evident that all loxodrome points are obtained by the construction which accuracy is absolute. In the contrary to the above, the angles between the meridian projections and chords obtained in the first and second iteration, respectively, are not equal. Since new chords are closer to the corresponding loxodrome tangents than the old ones, the angles between meridian projections and the newly constructed chords are closer to the actual angle between meridians and the loxodrome.

**3)** The third stage of the loxodrome construction can be described as the phase of curve approximation in which the iteration error is measured and controlled.

The error of one iteration step represents the angle between two corresponding chords constructed in that step and the previous one. If this angle is greater than a specific angle defined beforehand, the second stage of loxodrome construction must be performed again. Otherwise, a desired degree of accuracy is already achieved and therefore the iteration process could be terminated immediately. The maximal acceptable error of the loxodrome approximation can be defined in advance as the maximal accuracy of the navigational instruments by which a constant heading line is controlled.

## 5. Geometrical synthesis of Mercator's conformal map

Let us presume that an unspecified loxodrome is constructed in Hipparchus' stereographic polar map by the iteration method exposed in the previous chapter in such a way that this curve intersects the Earth's equator in the point $K$ ($\varphi = 0°,\ \lambda = 0°$). Also, let us presume that the constant angle between the loxodrome and all meridians is determined in the point $K$ with the desired degree of accuracy. Under all these presumptions, which are shown in Fig. 6, the direct geometrical synthesis of Mercator's conformal cylindrical map projection is exposed and explained in this chapter.

The synthesis process of Mercator's conformal map on which the loxodrome projection is represented by a straight line will be accomplished in several stages. Since Mercator's map

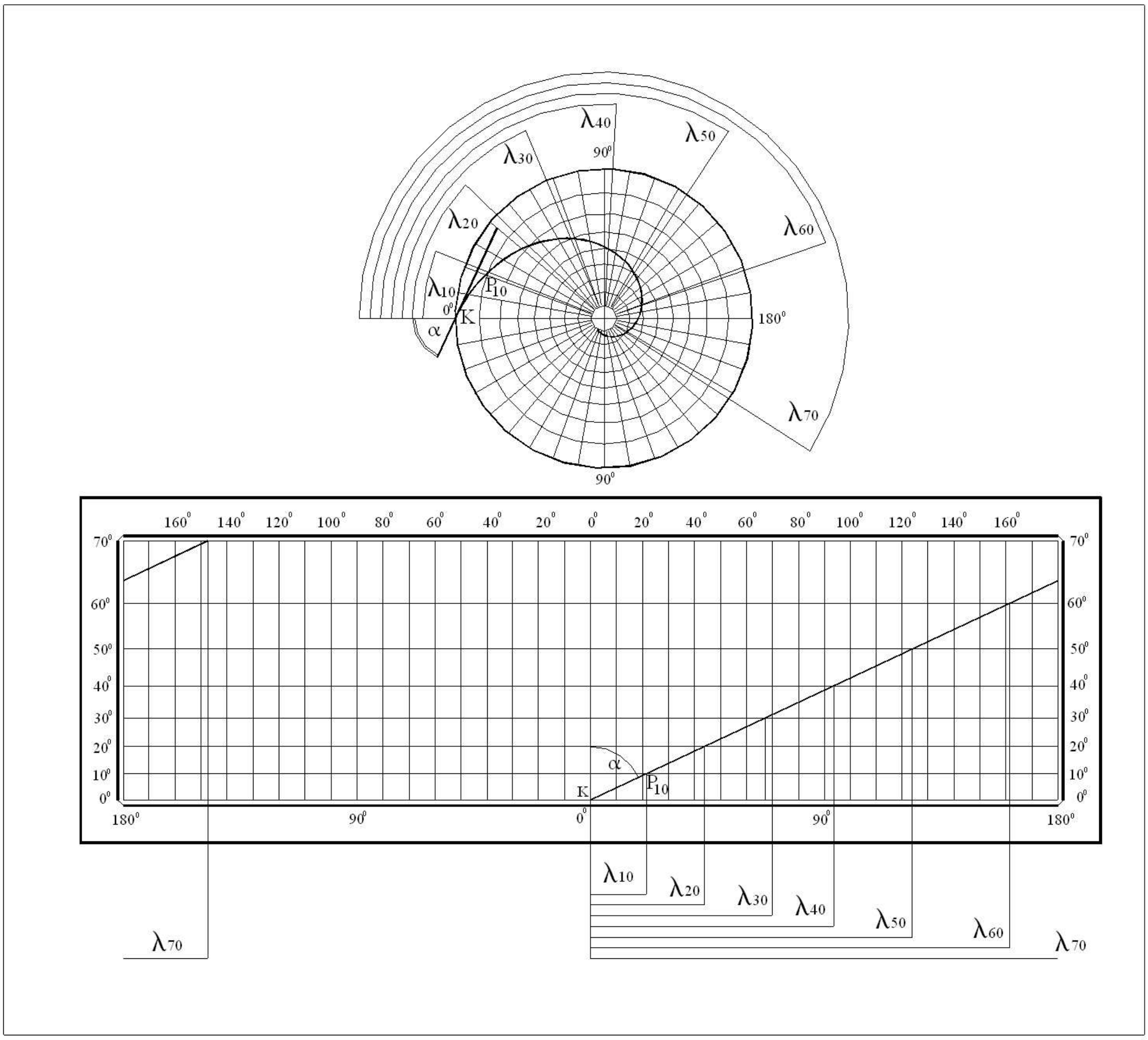


Figure 6: Geometrical synthesis of Mercators' conformal cylindrical map projection

belongs to the family of cylindrical cartographic projections, its creation can be described as the transformation of the Earth's spherical surface to the surface of a cylinder tangent to the Earth's equator and flattened onto a plane. Therefore, at the first step, it is necessary to determine the measure of the Earth's equator length in an appropriate scale. Undoubtedly, Mercator was capable of performing successfully the equator rectification whether by some geometrical and numerical methods or by some navigational and mechanical instruments. Let us presume that the angular scale is graduated uniformly on the rectified equator and that the meridians projections are drawn perpendicularly to the equator. Since under each cylindrical map the meridians are transformed onto a pencil of parallel straight lines, the north pole and the south pole are projected onto a point at infinity. The most important question in the process of this map synthesis is how to determine the positions of the parallels' projections.

It was already emphasized that Mercator's main intention was to produce a cylindrical conformal map on which any loxodrome is projected to a straight line which can be drawn by ruler. In accordance to this fact, the loxodrome is drawn as a straight line through the point $K$ which forms the constant heading angle with the corresponding meridian. This angle is determined and taken from the Hipparchus' polar stereographic map in which the loxodrome projection is constructed as the spiral curve by the method described already. After this preparation, the positions of parallels' projection can be determined easily. The distribution of these parallels will not be uniform, but they will be placed and arranged in such a way that the projection of the loxodrome is a straight line.

At the final stage of the map synthesis, the system of parallels is constructed by the longitudes of the loxodrome and each parallel intersection point which is taken from the Hipparchus' polar stereographic map and transferred to the Mercator's cylindrical map. For an example, let us presume that the intersection point $P_{10}$ between the loxodrome and the parallel $\varphi = 10°$ is given on Hipparchus' map:

At the first step, the longitude $\lambda_{10}$ of point $P_{10}$ is measured and transferred to the angular scale of Mercator's map equator. The position $P_{M10}$ of the same point is obtained on Mercator's map by the intersection between the loxodrome and the meridian defined by the longitude $\lambda_{10}$. Finally, this parallel which is orthogonal to all meridians is drawn through the point $P_{M10}$.

The intersection point of the loxodrome and any particular parallel cannot be constructed absolutely accurate on the Hipparchus' conformal map due to the fact that the loxodrome projection represents the transcendental curve on that map. Despite this verity, intersection points of loxodrome and parallels can be obtained with the desired degree of accuracy, since the loxodrome points, which are close to the intersection point, can also be constructed with the required degree of accuracy.

It is important to emphasize that constructive geometrical methods exposed in the preceding paragraphs can be interpreted numerically. Since all of these geometrical constructions can be accomplished by ruler and compasses, their numerical interpretations belong to the set of iterative arithmetic methods which involve just four basic arithmetical operations. Therefore, it is highly credible that MERCATOR, besides geometrical methods, used also the iterative numerical procedures to develop his famous conformal cylindrical map.

## 6. Conclusion

This paper exposes one possible constructive graphical method by which the direct geometrical synthesis of Mercator's conformal cylindrical map projection can be accomplished. We believe that the original inventive process used by Gerhard KREMER MERCATOR in creation of his famous world map was at least similar or compatible to the synthetic method exposed in this paper.

Besides this main intention, it is not worthless to recall and emphasize the importance of constructive geometrical, visual, graphical and synthetic methods which are neglected typically and habitually in modern branches of geometry and education. The direct perception and comprehension of the plain verities about geometrical objects and their spatial relations as well as the synthesis of complex conclusions by the progressive deduction represent the most important characteristics of these methods. Geometrical, visual and synthetic methods are still extremely useful for engineers, architects, geodesist, navigators, etc., and, despite the fact that these methods had been invented by the ancient Greeks more than 2500 years ago, they should not be disregarded or ever forgotten.

## References

[1] V. Nice: *Uvod u sintetiéku geometriju.* Školska knjiga, Zagreb 1957.

[2] *Vojna enciklopedija — IV: Kartografske projekcije.* Vojno izdavaéki zavod, Beograd 1974, pp. 265–272.

[3] S. Živanović, V. Sbutega, M. Ivanović: *Perspektiva i perspektivne kartografske projekcije.* Naucna knjiga, Beograd 1988.

[4] http://en.wikipedia.org/wiki/Gerardus_Mercator

[5] http://en.wikipedia.org/wiki/Mercator_projection

[6] http://wilhelmkruecken.de/: Ad maiorem Gerardi Mercatoris gloriam.

[7] http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Mercator_Gerardus.html

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