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Kinematic Analysis of a Pentapod Robot

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Abstract. The investigated milling robot with an axial spindle as platform is a parallel manipulator. Five legs carry and control the spindle. An algebraic solution of the direct kinematic problem is given by the help of vector calculus. The solutions are determined by the roots of 5 polynomials of degree 4. Therefore, together with a quadratic normalizing condition the number of solutions is not greater than 2048. Compared to our first result this number is strongly reduced but still large. However, numerical solutions of the polynomial equation system show a stable and fast convergence using Newton methods.
Then, the inverse kinematic problem is solved. Four of the five leg lengths are determined by solutions of two quadratic equations. Some geometrical considerations and additionally technical restrictions allow to prove that a unique solution exists.
Furthermore, the velocity and shakyness is studied. Using Ball's screw we show how for a given rate of change of the leg lengths the velocities are determined. The special design of the spindle causes that the Pentapod robot is architecturally shaky with respect to a revolution about the spindle axis. This fact is no technological lack because the spindle axis is identical with the actuated milling axis. Finally, all singular positions are characterized.

Key Words: Robotics, spatial mechanisms, constraint parallel manipulators

MSC: 53A17

1. Introduction

The investigated parallel manipulator (milling robot) [1] is depicted in Fig. 1. The frame is shaped like an icosahedron. There, five active joints are adjusted which drive five legs. The legs carry and control the cutter spindle which is the platform or end effector of this constraint robot. A work can be adjusted at the round table below the cutter spindle and processed by a cutter which is rotating about the spindle axis. In [9] we solved the direct kinematic problem by the help of the Study parameterisation of a displacement. Here, a numerically advanced solution is shown by the help of vector calculus. Furthermore, the inverse kinematic problem is solved.

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Figure 1: The milling robot P800

2. Direct and inverse kinematics

2.1. Kinematic design of the milling robot

The mechanical design of the cutter spindle is displayed in Fig. 2. The first leg and the spindle are directly connected by a revolute joint with centre Q_1. The other legs end also with a revolute joint with centre Q_i , $i = 2, \ldots, 5$. These joints are mounted on rings that can rotate about the spindle axis s. By this design, all joint axes at Q_i , $i = 1, \ldots, 5$, are perpendicular to s. When the robot moves by altering the lengths of its legs then each joint centre Q_i describes a circle about the spindle axis with fixed radius ρ. The anchor point M_i of a leg is the centre of a universal pair (see Fig. 3). Consequently, the centre lines M_iQ_i of the legs meet the axis s at all positions of the robot. This very important geometric property is called the *spindle condition*.

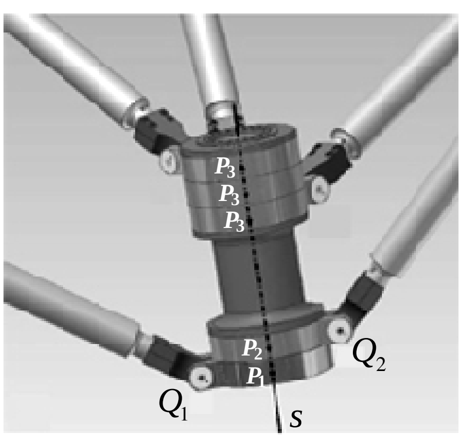

Figure 2: Cutter spindle

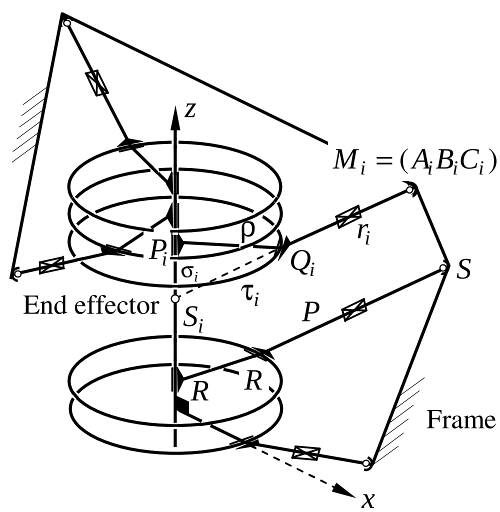

Figure 3: Kinematic structure

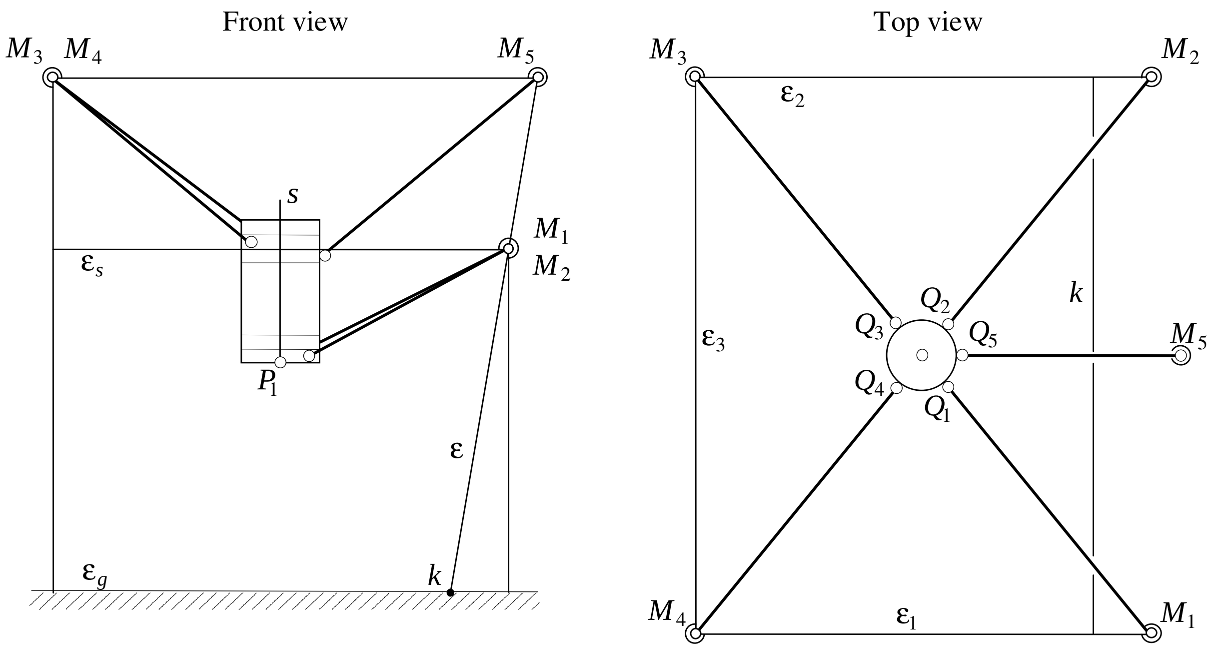

Figure 4: Dimensions of the milling robot

With reference to the introduced coordinate systems as shown in Fig. 4, the coordinates of the points $M_i = (A_i, B_i, C_i)$ and $P_i = (0, 0, c_i)$ are given in Table 1.

Table 1: Coordinates of joint points

i	A	B	C	c
1	k_1	k_2	k_3	0
2	0	k_2	k_3	k_7
3	0	0	0	$k_6 + 3 * k_7$
4	k_1	0	0	$k_6 + 2 * k_7$
5	k_4	k_5	0	$k_6 + 1 * k_7$

$k_1 = 1569,62 \quad k_2 = 1263,07 \quad k_3 = -482,41 \quad k_4 = 784,81$
$k_5 = 1359,33 \quad k_6 = 220 \quad k_7 = 42,5$

2.2. Algebraic solution by vector calculus

We chose the moving spindle frame (end effector frame) in such a way that P_1 is the origin and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ denote the base vectors with

$$\|\mathbf{w}_1\| = 1 \tag{1}$$

$$\|\mathbf{w}_3\| = 1 \tag{2}$$

$$\mathbf{w}_1 \cdot \mathbf{w}_3 = 0 \tag{3}$$

$$\mathbf{w}_2 = \mathbf{w}_3 \times \mathbf{w}_1.$$

Furthermore, with respect to an arbitrary coordinate system we designate by

$\mathbf{m}_i$	the position vector of the anchor point M_i
$\mathbf{p}_1$	the position vector of the origin P_1
$\mathbf{v}_i$	the unit direction vector of the straight line $M_iQ_i = \ell_i$ which is called *leg axis*
$\mathbf{s}_i$	the position vector of the intersection point S_i between the leg axis ℓ_i and the spindle axis
r_i	the leg length, i.e., the distance $\overline{M_iQ_i}$, $i = 1, \ldots, 5$.

For convenience we use $\mathbf{t}_i := r_i\mathbf{v}_i = \overrightarrow{M_iQ_i}$ and $\mathbf{u}_i := \overrightarrow{P_iQ_i}$. Then we have

$$\overrightarrow{S_iQ_i} = \lambda_i \mathbf{t}_i \text{ with } \lambda_i < 0, \tag{4}$$

because point Q_i lies between S_i and M_i on the leg axis ℓ_i.

In the following an algebraic solution for the direct kinematic problem is shown. For given design parameters of the robot and for adjusted leg lenghts the position and orientation of the spindle frame are to determine. So, let us determine $\mathbf{p}_1$, $\mathbf{w}_1$ and $\mathbf{w}_3$. By the design of the Pentapod, each leg allows to establish two vector loop conditions

$$\mathbf{m}_i + \mathbf{t}_i = \mathbf{p}_1 + c_i\mathbf{w}_3 + \mathbf{u}_i \tag{5}$$

$$\mathbf{m}_i + \mathbf{t}_i = \mathbf{p}_1 + \mathbf{s}_i\mathbf{w}_3 + \lambda_i r_i \mathbf{v}_i. \tag{6}$$

The first loop runs over P_i with $c_i := \overline{P_1P_i}$. The second loop runs over S_i with $s_i := \overline{P_1S_i}$. The following design conditions hold:

$$\mathbf{t}_i \cdot \mathbf{t}_i = r_i^2 \tag{7}$$

$$\mathbf{u}_i \cdot \mathbf{u}_i = \rho^2 \tag{8}$$

$$\mathbf{u}_i \cdot \mathbf{w}_3 = 0. \tag{9}$$

The special design of the first leg yields

$$\mathbf{v}_1 \cdot \mathbf{w}_2 = \mathbf{v}_1 \cdot (\mathbf{w}_3 \times \mathbf{w}_1) = 0 \tag{10}$$

and by (5) we get

$$(\mathbf{p}_1 - \mathbf{m}_1 + \rho\mathbf{w}_1) \cdot (\mathbf{w}_3 \times \mathbf{w}_1) = 0 \tag{11}$$

We rearrange eqs. (5) and (6) into

$$\mathbf{t}_i = \mathbf{p}_1 + c_i\mathbf{w}_3 + \mathbf{u}_i - \mathbf{m}_i \tag{12}$$

$$(1 - \lambda_i)\mathbf{t}_i = \mathbf{p}_1 + s_i\mathbf{w}_3 - \mathbf{m}_i. \tag{13}$$

By elimination of vector $\mathbf{t}_i$ it follows

$$(1 - \lambda_i)\mathbf{u}_i = \lambda_i(\mathbf{p}_1 - \mathbf{m}_i) + (s_i - c_i(1 - \lambda_i))\,\mathbf{w}_3. \tag{14}$$

Considering (9), we multiply eq. (14) with $\mathbf{w}_3$ and get

$$s_i = c_i(1 - \lambda_i) - \lambda_i(\mathbf{p}_1 - \mathbf{m}_i) \cdot \mathbf{w}_3. \tag{15}$$

Considering (8), we multiply eq. (14) with $(1-\lambda_i)\mathbf{u}_i$ and get

$$(\gamma_i^2 - \beta_i^2 - \rho^2)\lambda_i^2 + 2\lambda_i\rho^2 - \rho^2 = 0 \tag{16}$$

where

$$\beta_i := (\mathbf{p}_1 - \mathbf{m}_i)\cdot \mathbf{w}_3$$
$$\gamma_i^2 := (\mathbf{p}_1 - \mathbf{m}_i)\cdot(\mathbf{p}_1 - \mathbf{m}_i).$$

Inserting the solution (15) into the squared vector equation (13) and then using design condition (9) we obtain

$$\left((\beta_i + c_i)^2 - r_i^2\right)\lambda_i^2 - 2\lambda_i\left((\beta_i + c_i)^2 - r_i^2\right) + \gamma_i^2 + 2\beta_i c_i + c_i^2 - r_i^2 = 0. \tag{17}$$

We choose the coordinates of joint points according to Table 1 and eliminate the unknown λ_i from the two quadratic polynomials (16) and (17) and find the resultant

$$S_i(\mathbf{p}_1, \mathbf{w}_3)R_i(\mathbf{p}_1, \mathbf{w}_3) = 0 \tag{18}$$

where

$$\begin{gathered} S_i(\mathbf{p}_1, \mathbf{w}_3) = (\beta_i^2 - \gamma_i^2)^2 \\ R_i(\mathbf{p}_1, \mathbf{w}_3) := (2\beta_i c_i + c_i^2 + \gamma_i^2 - r_i^2)^2 + 2\left(2\beta_i(\beta_i + c_i) + c_i^2 - \gamma_i^2 - r_i^2\right)\rho^2 + \rho^4, \\ i = 1, \ldots, 5\,. \end{gathered}$$

Amazingly, the resultant is factorized. We note that

$$S_i(\mathbf{p}_1, \mathbf{w}_3) = 0 \tag{19}$$

holds iff $((\mathbf{p}_1 - \mathbf{m}_i)\cdot\mathbf{w}_3)^2 = (\mathbf{p}_1 - \mathbf{m}_i)^2$, i.e., the vectors $\mathbf{p}_1 - \mathbf{m}_i$ and $\mathbf{w}_3$ are linear dependent. Geometrically spoken, it holds $S_i(\mathbf{p}_1, \mathbf{w}_3) = 0$ iff the spindle axis is parallel to the straight line P_1M_i. In order to find a solution $(\mathbf{p}_1, \mathbf{w}_3)$ of the direct kinematics of the Pentapod $S_i(\mathbf{p}_1, \mathbf{w}_3) = 0$ must hold for $i = 1, \ldots, 5$, but that is impossible by the Pentapod design.

Therefore, the problem reduces to find all solutions of the polynomials

$$R_i(\mathbf{p}_1, \mathbf{w}_3) = 0, \quad i = 1, \ldots, 5. \tag{20}$$

The 6 unknowns are designated by $\mathbf{p}_1 = (x_1,\ x_2,\ x_3)^{\mathrm{T}}$ and $\mathbf{w}_3 = (x_4, x_5, x_6)^{\mathrm{T}}$. The computation of R_1 yields

$$\begin{aligned} R_1(x_1, \ldots, x_6) \;=\; & (-r_1^2 + (-k_1 + x_1)^2 + (-k_2 + x_2)^2 + (-k_3 + x_3)^2)^2 \\ & +2\,(-r_1^2 - (-k_1 + x_1)^2 - (-k_2 + x_2)^2 - (-k_3 + x_3)^2 \\ & +2\,((-k_1 + x_1)x_4 + (-k_2 + x_2)x_5 + (-k_3 + x_3)^2\, x_6)\,^2)\,\rho^2 + \rho^4. \end{aligned} \tag{21}$$

Analogously we obtain $R_i(x_1, \ldots, x_6)$ for $i = 2, \ldots, 5$, which are polynomials of degree 4 in the unknowns $x_1, \ldots, x_6$. Therefore, the total degree of the system (20) and (2) and the Bezout's count of the number of solutions is $4^5 \cdot 2^1$. For each solution we have to find vector $\mathbf{w}_1$ in order to solve the task completely. By a solution $(x_1, \ldots, x_6)$ the coefficients of the quadratic eq. (16) are determined. Generally, (16) has two solutions λ_{i1} and λ_{i2}. If λ_{i1} is a negative (real) solution then λ_{i1} and λ_{i2} is a positive solution and is dropped due to (4). This assertion can be proved by Vieta's Theorem stating that in our case

$$\lambda_{i1}\lambda_{i2} = \frac{-\rho^2}{\gamma_i^2 - \beta_i^2 - \rho^2}\,.$$

The right hand side of this equation is always negative because the robot design fulfills $\gamma_i^2 > \beta_i^2 + \rho^2$. Thus, by (13) $\mathbf{t}_i$ is uniquely determined. Inserting $\mathbf{t}_i$ into (12) we find $\mathbf{u}_i$.

Because of $\mathbf{w}_1 = \frac{1}{\rho}\mathbf{u}_1$ and $\mathbf{w}_2 = \mathbf{w}_3 \times \mathbf{w}_1$, finally all base vectors and the origin P_1 of the spindle frame are determined. Thus we have shown

Proposition 1 *For given leg lengths $r_1, \ldots, r_5$ of the Pentapod the number of solutions of the direct kinematic problem is not greater than* $4^5 \cdot 2^1 = 2048$.

Remark: Compared to our first result in [9] this number is strongly reduced but still large. Therefore, we performed numerical solutions of the polynomial equation system (20) using the FindRoot function of Mathematica® and the dimensions of the Pentapod given in Table 1. We specified only one starting value for each variable $x_1, \ldots, x_6$. In this case FindRoot searches for a solution using Newton methods.
This numerical simulation of the direct kinematic problem showed a stable and fast convergence for a wide range of starting values.

2.3. Inverse kinematics

The following computations are a consequence of a planar figure in quadrangle $P_1P_iQ_iM_i$. Therefore, the computation of the leg lengths is straightforward.The inverse kinematics deals with the problem to determine the leg length r_i if the end effector position is given by $\mathbf{p}_1, \mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$. For the first leg the problem is easily solved by

$$r_1 = \|\mathbf{q}_1 - \mathbf{m}_1\| = \|\mathbf{p}_1 - \mathbf{m}_1 + \rho\mathbf{w}_1\| \,. \tag{22}$$

For the other legs we have to consider that each joint centre Q_i can move on the circle about the spindle axis with centre P_i and radius ρ. This consideration means

$$\mathbf{u}_i = \mathbf{q}_i - \mathbf{p}_i = \mu_i\mathbf{w}_1 + \nu_i\mathbf{w}_2, \quad i = 2, \ldots, 5, \tag{23}$$

where

$$\mu_i = \rho\cos\varphi_i \quad \nu_i = \rho\sin\varphi_i\,, \quad \mu_i^2 + \nu_i^2 = \rho^2.$$

The spindle design causes that each two of the vectors $\mathbf{w}_3$, $\mathbf{q}_i - \mathbf{p}_i$, and $\mathbf{q}_i - \mathbf{m}_i$ span the plane $M_i s$ which includes the point M_i and the spindle axis s. Therefore,

$$\det(\mathbf{w}_3,\ \mathbf{q}_i - \mathbf{p}_i,\ \mathbf{q}_i - \mathbf{m}_i) = 0.$$

With (5) and (23) we calculate this determinant and obtain

$$d_{i1}\mu_i + d_{i2}\nu_i + 2\mu_i\nu_i = 0 \tag{24}$$

where

$$d_{ij} = \det(\mathbf{w}_3, \mathbf{w}_j,\ \mathbf{p}_1 - \mathbf{m}_i), \quad j = 1, 2.$$

The unknowns μ_i and ν_i are determined by the solution of two quadratic equations (23) and (24). So we algebraically expect 4 solutions (μ_i, ν_i). From the geometric point of view only two solutions are real because the plane $M_i s$ generally meets the above mentioned circle in two points Q_{ik}. With the two solutions (μ_{ik}, ν_{ik}), $k = 1, 2$, we obtain two leg vectors

$$\mathbf{q}_{ik} - \mathbf{m}_{ik} = \mathbf{p}_1 - \mathbf{m}_i + c_i\mathbf{w}_3 + \mu_{ik}\mathbf{w}_1 + \nu_{ik}\mathbf{w}_2,$$

and therefore two solutions for each leg length:

$$r_{ik} = \|\mathbf{q}_{ik} - \mathbf{m}_{ik}\|, \quad k = 1, 2. \tag{25}$$

The spindle axis is intended to go approximately through the centre of gravity of the anchor points $M_1, \ldots, M_5$ whereas all joint centres Q_i lie below$M_1, \ldots, M_5$. As a result of this geometric consideration we choose the numerically smallest r_{ik} to be the solution:

$$r_i = \min\{r_{i1}, r_{i2}\}, \quad i = 2, \ldots, 5. \tag{26}$$

So we have proved:

Theorem 1 *The inverse kinematic problem of the Pentapod has a unique solution according to eqs. (22) – (26).*

3. Velocity and shakyness

3.1. Velocity and Ball's screw

Let us assume that the lengths $r_1, \ldots, r_5$ of the five legs are given by functions of a time parameter t. Then an instantaneous change of these lengths is described by a joint velocity vector

$$\dot{\mathbf{r}} = (\dot{r}_1, \ldots, \dot{r}_5)^{\mathrm{T}}$$

and causes Ball's screw $(\boldsymbol{\omega}, \widehat{\boldsymbol{\omega}})$ of the end effector. This instantaneous screw determines the velocity

$$\mathbf{v}(X) = \boldsymbol{\omega} \times \mathbf{x} + \widehat{\boldsymbol{\omega}} \tag{27}$$

of each point X of the end effector described by coordinates $\mathbf{x}$ with respect to the basic frame. This screw is given by the rotation matrix $\mathbf{w} = (\mathbf{w}_1\ \mathbf{w}_2 \mathbf{w}_3)$ and the velocity $\dot{\mathbf{p}}_1$ of the origin in the following way. The velocity matrix

$$\Omega = \dot{\mathbf{W}}\,\mathbf{W}^{\mathrm{T}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{28}$$

determines the Darboux-vector $\boldsymbol{\omega}$ of the screw. The second part is

$$\widehat{\boldsymbol{\omega}} = \dot{\mathbf{p}}_1 + \boldsymbol{\omega} \times \mathbf{p}_1. \tag{29}$$

These formulae practically do not allow to compute a screw because the joint velocities $\dot{r}_i$ are indistinctly involved.

Looking for a different way to determine $(\boldsymbol{\omega}, \widehat{\boldsymbol{\omega}})$ we differentiate eq. (7) and obtain

$$2\dot{\mathbf{t}}_i \cdot \mathbf{t}_i = 2r_i\dot{r}_i. \tag{30}$$

The position vector of a joint point Q_i is

$$\mathbf{q}_i = \mathbf{m}_i + \mathbf{t}_i. \tag{31}$$

Hence, we have $\mathbf{t}_i = \mathbf{q}_i - \mathbf{m}_i$ and

$$\dot{\mathbf{t}}_i = \dot{\mathbf{q}}_i - \dot{\mathbf{m}}_i = \dot{\mathbf{q}}_i.$$

Using the screw this velocity is now expressed by

$$\dot{\mathbf{t}}_i = \dot{\mathbf{q}}_i = \boldsymbol{\omega} \times \mathbf{q}_i + \widehat{\boldsymbol{\omega}}. \tag{32}$$

Due to eqs. (30) – (32) we get

$$(\boldsymbol{\omega} \times \dot{\mathbf{q}}_i + \widehat{\boldsymbol{\omega}}) \cdot (\mathbf{q}_i - \mathbf{m}_i) = r_i \dot{r}_i,$$

and therefore

$$(\mathbf{m}_i \times \mathbf{q}_i) \cdot \boldsymbol{\omega} + (\mathbf{q}_i - \mathbf{m}_i) \cdot \widehat{\boldsymbol{\omega}} = r_i \dot{r}_i.$$

Now, the unit direction vector $\mathbf{v}_i = \frac{1}{r_i}\mathbf{t}_i$ of the leg axis is introduced and by the help of eq. (31) we obtain a system of five linear equations for the six unknown coordinates of a screw:

$$(\mathbf{m}_i \times \mathbf{v}_i) \cdot \boldsymbol{\omega} + \mathbf{v}_i \cdot \widehat{\boldsymbol{\omega}} = \dot{r}_i \quad i = 1, \ldots, 5. \tag{33}$$

Rearranging the system we get

$$\mathbf{Q}\boldsymbol{\sigma} = \dot{\mathbf{r}} \tag{34}$$

where

$$\mathbf{Q} = \begin{pmatrix} \widehat{\mathbf{v}}_1^{\mathrm{T}} & \mathbf{v}_1^{\mathrm{T}} \\ \vdots & \vdots \\ \widehat{\mathbf{v}}_5^{\mathrm{T}} & \mathbf{v}_5^{\mathrm{T}} \end{pmatrix}, \quad \widehat{\mathbf{v}}_i = \mathbf{m}_i \times \mathbf{v}_i, \quad i = 1, \ldots, 5$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\omega} \\ \widehat{\boldsymbol{\omega}} \end{pmatrix}.$$

The corresponding homogenous system $\mathbf{Q}\,\boldsymbol{\sigma} = \mathbf{o}$ is equivalent to

$$(\mathbf{m}_i \times \mathbf{v}_i) \cdot \boldsymbol{\omega} + \mathbf{v}_i \cdot \widehat{\boldsymbol{\omega}} = 0. \tag{35}$$

Due to $1 \le \operatorname{rank} \mathbf{Q} \le 5$ it is always solved by

$$\boldsymbol{\sigma}_h = \sum_{k=1}^{n} \alpha_k \boldsymbol{\sigma}_k \tag{36}$$

where $\alpha_k \in \mathbb{R}$, $n = 6 - \operatorname{rank} \mathbf{Q}$, $1 \le n \le 5$. The fundamental solutions $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n$ are linearly independent.

Proposition 2 *The homogeneous system* $\mathbf{Q}\boldsymbol{\sigma} = \mathbf{o}$ *is solved by*

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} \boldsymbol{\omega}_1 \\ \widehat{\boldsymbol{\omega}}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{w}_3 \\ \mathbf{p}_1 \times \mathbf{w}_3 \end{pmatrix}. \tag{37}$$

Proof: Inserting eq. (37) into (35) we get $(\mathbf{m}_i \times \mathbf{v}_i) \cdot \boldsymbol{\omega}_1 + \mathbf{v}_i \cdot \widehat{\boldsymbol{\omega}} = \det(\mathbf{m}_i - \mathbf{p}_1, \mathbf{v}_i, \ \mathbf{w}_3) = 0$ for $i = 1, \ldots, 5$. The determinant vanishes because the vector arguments $\mathbf{m}_i - \mathbf{p}_1$, $\mathbf{v}_i$ and $\mathbf{w}_3$ are parallel to the plane $M_i s$ for $i = 1, \ldots, 5$, due to the spindle condition. □

Note that this solution $\boldsymbol{\sigma}_1$ satisfies the Plücker condition

$$\boldsymbol{\omega}_1 \cdot \widehat{\boldsymbol{\omega}}_1 = \mathbf{w}_3 \cdot (\mathbf{p}_1 \times \mathbf{w}_3) = 0. \tag{38}$$

Therefore, $\boldsymbol{\sigma}_1$ are normed Plücker coordinates of the spindle axis s. Eq. (38) describes the fact that all leg axes meet the spindle axis in line geometric notation. (Fundamentals of line geometry are given in [8].)

From the theory of linear equations systems it follows

Theorem 2 *For given joint velocities* $(\dot{r}_1, \ldots, \dot{r}_5) \ne (0, \ldots, 0)$ *at a position* $(r_1, \ldots, r_5)$, *there is always a solution* $(\boldsymbol{\omega}_0, \widehat{\boldsymbol{\omega}}_0)$ *of eq. (33), e.g., determined by* Gaussian *elimination.*

3.2. Shakyness and singular positions

The Pentapod is shaky for some position $(r_1, \ldots, r_5)$ iff an instantaneous displacement of the end effector exists although the leg lengths are fixed. Therefore, the Pentapod is shaky iff a non zero screw $\boldsymbol{\sigma} = (\boldsymbol{\omega}, \widehat{\boldsymbol{\omega}})$ exists, corresponding to an instantaneous displacement, although $(\dot{r}_1, \ldots, \dot{r}_5) = (0, \ldots, 0)$. A shaky position is also called a singular position, c.f. [4]. Therefore, the Pentapod is shaky for $(r_1, \ldots, r_5)$ iff the homogeneous system $\mathbf{Q}\,\boldsymbol{\sigma} = \mathbf{o}$ has a nontrivial solution. By Proposition 2 we conclude

Theorem 3 *The Pentapod is shaky in every position (i.e., architecturally shaky) with respect to a revolution about the spindle axis s.*

Remark: The architectural shakyness is no technological lack for the Pentapod because the spindle axis is identical with the actuated milling axis.

In the following the Pentapod is said to be in a *singular position* iff a non zero screw $(\boldsymbol{\omega}, \widehat{\boldsymbol{\omega}})$ exists which is different from $(\boldsymbol{\omega}_1, \widehat{\boldsymbol{\omega}}_1)$. We want to characterize all singular positions. Considering the (5,6)-coefficient matrix $\mathbf{Q}$ in eq. (34) we see that a row $(\widehat{\mathbf{v}}_i^{\mathrm{T}}, \mathbf{v}_i^{\mathrm{T}})$ of the matrix is a normalized dual Plücker vector of the leg axis ℓ_i because $\mathbf{v}_i$ is a unit direction vector and $\widehat{\mathbf{v}}_i^{\mathrm{T}} = \mathbf{m}_i \times \mathbf{v}_i$ is a moment vector of ℓ_i.

It follows: $\operatorname{rank}\mathbf{Q}$ is the number of linearly independent Plücker vectors

$$\boldsymbol{\ell}_i = \begin{pmatrix} \mathbf{v}_i \\ \widehat{\mathbf{v}}_i \end{pmatrix} \in \mathbb{R}^6, \quad i = 1, \ldots, 5\,,$$

which represent the leg axes $\ell_1, \ldots, \ell_5$.

In the case of $\operatorname{rank}\mathbf{Q} = 5$ five linearly independent leg axes belong to the singular linear complex with axis $\boldsymbol{\sigma}_1$ at every position. In the case of $\operatorname{rank}\mathbf{Q} = 4$ at the considered position the five leg axes belong to a congruence which is either hyperbolic or parabolic because the spindle axis $\boldsymbol{\sigma}_1$ is one focal line of this congruence. Vice versa: If $\ell_1, \ldots, \ell_5$ belong to a line congruence, then $\operatorname{rank}\mathbf{Q} = 4$.

By eq. (36) we obtain

Proposition 3 *If* $\operatorname{rank}\mathbf{Q} = 4$, *the Pentapod is in a singular position with respect to all screws*

$$\boldsymbol{\sigma} = \alpha_1 \boldsymbol{\sigma}_1 + \alpha_2 \boldsymbol{\sigma}_2, \quad \alpha_1, \alpha_2 \in \mathbb{R}, \quad \alpha_2 \neq 0,$$

where $\boldsymbol{\sigma}_2 = (\boldsymbol{\omega}_2, \widehat{\boldsymbol{\omega}}_2)$ is a second fundamental solution of (35).
In the special case $\boldsymbol{\omega}_2 \cdot \widehat{\boldsymbol{\omega}}_2 = 0$, except for the spindle axis s, the legs $\ell_1, \ldots, \ell_5$ meet the straight line s_2 ($s_2 \neq s$) with Plücker coordinates $\boldsymbol{\sigma}_2$. Then, s and s_2 are the focal lines of a hyperbolic congruence which includes the leg axes.

For the following we need a

Definition *The work space of the Pentapod is the set of all inner points of the polyhedron surrounded by the "ground"-plane ε_g, the "ceiling"-plane ε_s, the vertical "wall"-planes $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and the inclined "wall"-plane $\varepsilon = M_1 M_2 M_5$ as given in Fig. 4.*

Proposition 4 *The case of* $\operatorname{rank}\mathbf{Q} < 3$ *does nor occur in the work space. In the case of* $\operatorname{rank}\mathbf{Q} = 3$ *the Pentapod is in a singular position and the five legs belong to a regulus.*

Proof: It was shown in [9] that for all positions rank $\mathbf{Q} < 3$ can be excluded. From line geometry it is known that in the case of rank $\mathbf{Q} = 3$ the five legs belong to a regulus, a pair of crossed pencils, or to a line bundle. In the following we show that the last two cases do not occur because the dimensions of the Pentapod design prevent it. The bundle case was already excluded in [9]. Now we consider the case of crossed pencils. In preparation for our argumentation we consider three different anchor points M_i, M_j, and M_k. These points determine the plane δ_{ijk} in which they lie. It is easy to see that all such planes δ_{ijk} do not have inner points in common with the work space. Each anchor point M_i determines a plane $M_i s$ in which M_i and s lie. The plane $M_i s$ contains the leg axis ℓ_i. If three different leg axes ℓ_i, ℓ_j, ℓ_k lie in a plane, then this plane is δ_{ijk}, and furthermore, δ_{ijk} contains s. In the case of crossed pencils, the set of five leg axes splits up into two subsets spanning a pencil each. So we have to consider the pairings (4,1) or (3,2). The first one is not possible because four leg axes in a plane demand four M_i in a plane. The second pairing demands three leg axes in a plane. Hence, this plane is identical with some plane δ_{ijk} which contains the spindle axis s. This situation is not possible in the given work space. $\square$

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