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On Feuerbach's Theorem and a Pencil of Circles in the Isotropic Plane

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Abstract. After adapting the well-known Euler and Feuerbach theorems for the isotropic plane, the connection among the circumcircle, Euler circle, tangential circumcircle, and the polar circle of a given allowable triangle has been shown. It has been proved that all four circles belong to the same pencil of circles. There are two more interesting circles in this pencil.

Key Words: isotropic plane, triangle, Feuerbach theorem, pencil of circles

MSC: 51N25

1. Preliminaries

It has been shown in [2] that any allowable triangle ABC in the isotropic plane I_2 can be moved in the so called *standard position*, having the circumcircle equation

$$\mathcal{K}_c \ \ldots \ y = x^2, \tag{1}$$

by choosing an appropriate affine coordinate system, while its vertices are of the form

$$A = (a,\ a^2), \quad B = (b,\ b^2), \quad C = (c,\ c^2), \tag{2}$$

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where along with

$$p = abc, \quad q = bc + ca + ab \tag{3}$$

the equality

$$a + b + c = 0 \tag{4}$$

holds as well. As a consequence, other useful reletions hold too, for example,

$$a^2 = bc - q, \tag{5}$$

$$a^2 + b^2 + c^2 = -2q, \tag{6}$$

wherefrom it follows that $q < 0$. From (4) and (6) we get that the centroid G of the triangle ABC is of the form

$$G = \left(0, -\frac{2}{3}q\right). \tag{7}$$

The triangle ABC given in its standard position (the expression *standard triangle* will further on be in use) has, according to [2], *the Euler line* $\mathcal{E}$, *the inertial axis* $\mathcal{G}$, and *the orthic axis* $\mathcal{H}$ given in the equations

$$\mathcal{E} \quad \ldots \quad x = 0\,, \tag{8}$$

$$\mathcal{G} \quad \ldots \quad y = -\frac{2}{3}\,q\,, \tag{9}$$

$$\mathcal{H} \quad \ldots \quad y = -\frac{q}{3}\,. \tag{10}$$

It has been shown in [2] that the midpoints of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of the standard triangle are the points A_m, B_m, C_m respectively where, for example,

$$A_m = \left(-\frac{a}{2}, -\frac{1}{2}(q + bc)\right), \tag{11}$$

while the feet of the altitudes are the points A_h, B_h, C_h, being e.g.

$$A_h = (a, q - 2bc). \tag{12}$$

In order to prove any statement on any allowable triangle it is sufficient to prove the considered statement for the standard triangle.

2. Euler circle

We recall now the well-known theorem in Euclidean plane:

The Euler theorem or the nine-point circle theorem: *In any triangle, the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments from the orthocenter to the vertices lie on a circle.*

In the isotropic plane we have the following:

Theorem 1 *In the standard triangle the midpoints of the sides and the feet of the altitudes lie on a circle* $\mathcal{K}_e$ *having the equation*

$$\mathcal{K}_e \ \ldots \ y = -2x^2 - q. \tag{13}$$

Proof: The points A_m and A_h from (11) and (12) lie on the circle (13) since

$$-2\left(\frac{a}{2}\right)^2 - q = -\frac{1}{2}(bc-q) - q = -\frac{1}{2}(q+bc)$$

and

$$-2a^2 - q = -2(bc-q) - q = q - 2bc$$

due to (5). Analogous points B_m, C_m and B_h, C_h lie on the same circle. □

Remark: The above theorem can be called *Euler theorem* or *six-point circle theorem* in I_2. The six-point circle theorem can be understood as the nine-point circle theorem as well, taking in consideration that the three midpoints of the segments from the orthocenter (that is the absolute point F) coincide with F, and as such are incidental with $\mathcal{K}_e$. The circle $\mathcal{K}_e$ will be called *Euler circle of the triangle* ABC in I_2.

3. Inscribed circle

Theorem 2 *The inscribed circle (excircle) of the standard triangle ABC obeys the equation*

$$\mathcal{K}_i \ \ldots \ y = \frac{1}{4}x^2 - q\,, \tag{14}$$

while the points of contact with the straight lines BC, CA, AB are

$$A_i = (-2a,\ bc-2q), \quad B_i = (-2b,\ ca-2q), \quad C_i = (-2c,\ ab-2q). \tag{15}$$

Proof: The straight line BC has an equation

$$y = (b+c)x - bc \tag{16}$$

(see [2]). From (14) and (16) due to (4) and (5) we obtain an equation

$$\frac{1}{4}x^2 + ax + a^2 = 0.$$

This equation has a double solution $x = -2a$, wherefrom it follows that circle (14) touches the straight line BC at the point A_i with abscissa $-2a$ and ordinate according to (16)

$$y = 2a^2 - bc = 2(bc-q) - bc = bc - 2q\,.$$

By analogy with the above, the same properties for the straight lines CA and AB can be derived. □

4. Feuerbach theorem

Theorem 3 *Circles $\mathcal{K}_e$ and $\mathcal{K}_i$ from Theorem 1 and 2 touch each other externally in the point*

$$\Phi = (0, -q), \tag{17}$$

while the common tangent $\mathcal{F}$ has the equation

$$\mathcal{F} \ \ldots \ y = -q\,. \tag{18}$$

Proof: From equations (13) and (14), using their combination $\mathcal{K}_e + 8\mathcal{K}_i$, one gets the equation (18) of a straight line passing through all common points of the circles $\mathcal{K}_e$ and $\mathcal{K}_i$. Then, using (13) and (18) for calculating these common points, we get the equation $-q = -2x^2 - q$ with the double solution $x = 0$. □

Remark: Theorem 3 will be called *Feuerbach theorem in* I_2. Accordingly, the point Φ and the straight line $\mathcal{F}$ will be the *Feuerbach point* and the *Feuerbach line* of the considered triangle ABC. Let us point out that the statement of Theorem 3 has been proved in Yaglom [1, 121–129] in elementary way. The advantage of our method used here intending to prove the same theorem, and other theorems as well, lies in the fact that the proof is condensed, simple and short.

Corollary 1 *The Feuerbach point of an allowable triangle is parallel to its centroid; i.e., it lies on the Euler line of this triangle.*

5. Tangential circumcircle

Let's recall the Euclidean meaning of the *tangential triangle*: A tangential triangle of a given triangle ABC is a triangle determined by the three tangents to its circumcircle at the vertices A, B, and C, respectively.

In the known text book [3] some properties of the tangential triangle of an allowable triangle in I_2 have been discussed. More properties are given in:

Theorem 4 *For the tangential triangle* $A_tB_tC_t$ *of the standard allowable triangle* ABC *we have, successively, the equations of the sides given by*

$$\mathcal{T}_A \ \ldots \ y = 2ax - a^2, \quad \mathcal{T}_B \ \ldots \ y = 2bx - b^2, \quad \mathcal{T}_C \ \ldots \ y = 2cx - c^2, \tag{19}$$

the vertices

$$A_t = \left(-\frac{a}{2},\, bc\right), \quad B_t = \left(-\frac{b}{2},\, ca\right), \quad C_t = \left(-\frac{c}{2},\, ab\right), \tag{20}$$

and the equation of the circumscribed circle (tangential circumcircle)

$$\mathcal{K}_t \ \ldots \ y = 4x^2 + q. \tag{21}$$

Proof: From (1) and for example the first equation in (19) one obtains $x^2 - 2ax + a^2 = 0$ with the double solution $x = a$ which implies that $(19)_1$ represents an equation of the tangent line of the circle (1) at point A. Since for example

$$2a\left(-\frac{b}{2}\right) - a^2 = -ab - (bc - q) = ca,$$

we see that the point B_t from (20) lies on the discussed tangent. Analogously it can be shown that C_t lies on that tangent as well. Following the above procedure it can be shown that the straight lines given by $(19)_{2,3}$ have analogous properties. For the circle (21) we see that for example the point A_t lies on it, since

$$4\left(-\frac{a}{2}\right)^2 + q = a^2 + q = bc.$$ □

6. Polar circle

If the polar of a point A with respect to (*w.r.t.*) a conic passes through the point B, then the polar of B w.r.t. the same conic passes through A. Such points are called *conjugate* w.r.t. the conic. Also, if the polars of two points A and B w.r.t. a conic meet at the point C, then the line AB is the polar of C. If such a triangle ABC exists, it is called *autopolar* w.r.t. the conic, and the conic itself is called a *polar conic* of the triangle.

Theorem 5 *The polar circle of the standard triangle has the equation*

$$\mathcal{K}_p \ \ldots \ 2y = -x^2 - q. \tag{22}$$

Proof: The equation of a polar of any point (x_0, y_0) w.r.t. (22) is

$$y + y_0 = -xx_0 - q.$$

For the point $A = (a, a^2)$ we get $y = -ax - (a^2 + q)$, that is, according (5), the equation of the straight line BC. Likewise, straight lines CA and AB are polars of the points B and C w.r.t. circle (22). □

7. Pencil of circles

Any two circles with the equations given in

$$y = u_i x^2 + v_i x + w_i, \quad u_i \neq 0, \quad i = 1, 2\,, \tag{23}$$

have two common chords, one being the absolute line, the other a straight line whose equation is obtained by eliminating in (23) the terms with x^2. The latter straight line is called the *potential axis of the circles* (23). A certain family of circles is said to represent *a pencil of circles* if any two of them have the same potential axis. For that to happen, it is sufficient that one of the circles has the same potential axis with all other circles. That very potential axis is called *a potential axis of the* observed *pencil of circles*.

Theorem 6 *The circumcircle, the Euler circle, the tangential circumcircle, and the polar circle of an allowable triangle belong to the same pencil of circles.*

Proof: For the standard allowable triangle the equations of the observed circles are

$$\begin{array}{llll} \mathcal{K}_c & \ldots & y = x^2, & (1) \\ \mathcal{K}_e & \ldots & y = -2x^2 - q\,, & (13) \\ \mathcal{K}_t & \ldots & y = 4x^2 + q\,, & (21) \\ \mathcal{K}_p & \ldots & 2y = -x^2 - q\,. & (22) \end{array}$$

It's easy to check that combining $2\mathcal{K}_c + \mathcal{K}_e\,,\ 4\mathcal{K}_c - \mathcal{K}_t\,$, or $\mathcal{K}_c + \mathcal{K}_p$ always gives the same straight line with the equation

$$\mathcal{H} \ \ldots \ y = -\frac{q}{3}\,. \tag{24}$$

Therefore, the observed circles belong to the same pencil of circles with the straight line $\mathcal{H}$ as its potential axis. □

The straight line $\mathcal{H}$ from the latter proof is *the orthic axis* of the standard triangle ABC (see [2]). From equations (18) and (24) it follows straight forward

Corollary 2 *The Feuerbach line and the orthic axis of an allowable triangle are two parallel straight lines.*

The inertial axis $\mathcal{G}$ of the standard triangle ABC, as obtained in [2], has the equation

$$\mathcal{G} \ \ldots \ y = -\frac{2}{3}q. \tag{25}$$

From (18), (24) and (25) we have

Corollary 3 *The orthic axis of an allowable triangle and its Feuerbach straight line are symmetric with respect to the inertial axis of the same triangle.*

8. Orthocentroidal circle

In Euclidean geometry a theorem holds that is analogous to Theorem 6, and in the same time the considered pencil of circles contains the so-called orthocentroidal circle, having the centroid and the orthocenter of the triangle as end-points on its diameter. In our case, in the isotropic geometry, the orthocenter coincides with the point at infinity, so we'll have to modify slightly the definition of the orthocentroidal circle. For a given triangle in the isotropic plane we'll call the circle passing through its centroid and belonging to the pencil of circles listed in Theorem 6 the *orthocentroidal circle* of the triangle.

Theorem 7 *The standard triangle ABC has the orthocentroidal circle with the equation*

$$\mathcal{K}_o \ \ldots \ y = -x^2 - \frac{2}{3}q. \tag{26}$$

Proof: The circle we are looking for with the equation $y = ux^2 + vx + w$, and the circumcircle with the equation given in (1) have as their potential axis a straight line with equation $(1-u)y = vx + w$, i.e.,

$$y = \frac{v}{1-u}x + \frac{w}{1-u}.$$

This very potential axis coincides with the orthic axis whose equation is given in (24), wherefrom it follows $v = 0$ and

$$\frac{w}{1-u} = -\frac{q}{3}, \quad i.e. \quad u = 1 + \frac{3w}{q}. \tag{27}$$

Knowing that the circle is supposed to pass through the centroid $G = (0, -\frac{2}{3}q)$ it follows that $w = -\frac{2}{3}q$, and in (27) we get $u = -1$. □

The observed pencil of circles also contains a circle whose equation is of the form $x^2 + vx + w = 0$. Eliminating x^2 from the latter equation and from equation (1) we obtain their potential axis obeying $y = -vx - w$. This will be the same orthic axis (24) provided $v = 0$ and $w = \frac{q}{3}$. That's why the above mentioned circle has the equation

$$x^2 + \frac{q}{3} = 0,$$

where due to $q < 0$ it degenerates into a pair of isotropic straight lines with equations

$$x = \sqrt{-\frac{q}{3}}, \quad x = -\sqrt{-\frac{q}{3}}.$$

9. Feuerbach theorem in tangential triangle

Theorem 8 *The equation of the Euler circle of the tangential triangle $A_tB_tC_t$ of the standard triangle ABC is*

$$\mathcal{K}_{te} \ \ldots y = -8x^2, \tag{28}$$

the Feuerbach point is $\Phi_t = (0,0)$, and the Feuerbach line $\mathcal{F}_t$ is given by $y = 0$.

Proof: Points B_t and C_t given in (20) have the midpoint

$$A_{tm} = \left(\frac{a}{4}, -\frac{a^2}{2}\right)$$

for $ca + ab = -a^2$. The point A_{tm} obviously lies on the circle (28), and likewise holds for the analogous points B_{tm} and C_{tm}. The circle with the equation $y = x^2$ is the excircle of the triangle $A_tB_tC_t$. These circles meet at the point $\Phi_t = (0,0)$ having a common tangent $\mathcal{F}_t$ whose equation is $y = 0$. □

Presuming that $G = \left(0, -\frac{2}{3}q\right)$ is the centroid of a triangle ABC then the homothety $(G, -2)$ transforms any point into its *anticomplementary point*, and any straight line into its *anticomplementary straight line* with respect to the triangle ABC. $T = (x, y)$ is a point anticomplementary to the point $T' = (-2x, -2q - 2y)$ since $2y + (-2q - 2y) = 3\left(-\frac{2}{3}q\right)$, i.e., $2T + T' = 3G$. Specially, the point $\Phi_t = (0,0)$ is anticomplementary to the point $\Phi = (0, -q)$. Hence, it holds:

Theorem 9 *The Feuerbach point and the Feuerbach line of a tangential triangle of a given triangle are anticomplementary to the Feuerbach point and the Feuerbach line of that triangle.*

From Theorem 8 and Theorem 9 we read a geometrical meaning of the origin $(0,0)$ and of the x-axis of the coordinate system which has been chosen for the triangle ABC to be in the standard position.

Points A_t, B_t, C_t given in (20) have the centroid $G_t = \left(0, \frac{q}{3}\right)$ lying on the Euler line $x = 0$ of the triangle ABC. Thus, we have:

Theorem 10 *Any triangle and its tangential triangle have the same Euler line.*

The potential axis of the circles (21) and (28) is given in the equation

$$\mathcal{H}_t \ \ldots \ y = \frac{2}{3}q. \tag{29}$$

Therefore:

Theorem 11 *The equation of the orthic axis $\mathcal{H}_t$ of the tangential triangle of the standard triangle is given in (29).*

The centroid $G_t = \left(0, \frac{q}{3}\right)$ of a tangential triangle has another interesting property: It represents the fixed point for the homothety with the constant 4 and having the equations of the form

$$x' = 4x, \quad y' = 4y - q. \tag{30}$$

(30) transforms a circle having an equation $4y - q = \frac{1}{4}(x)^2 - q$, that is $y = x^2$ in a circle with the equation $y' = \frac{1}{4}(x')^2 - q$. Thus, the homothetic transformation maps the circumcircle of the standard triangle ABC to its inscribed circle. Hence, it follows:

Theorem 12 *If G_t represents the centroid of the tangential triangle of the given allowable triangle ABC, then the homothety $(G_t, 4)$ maps the circumcircle of the triangle ABC onto its incircle.*

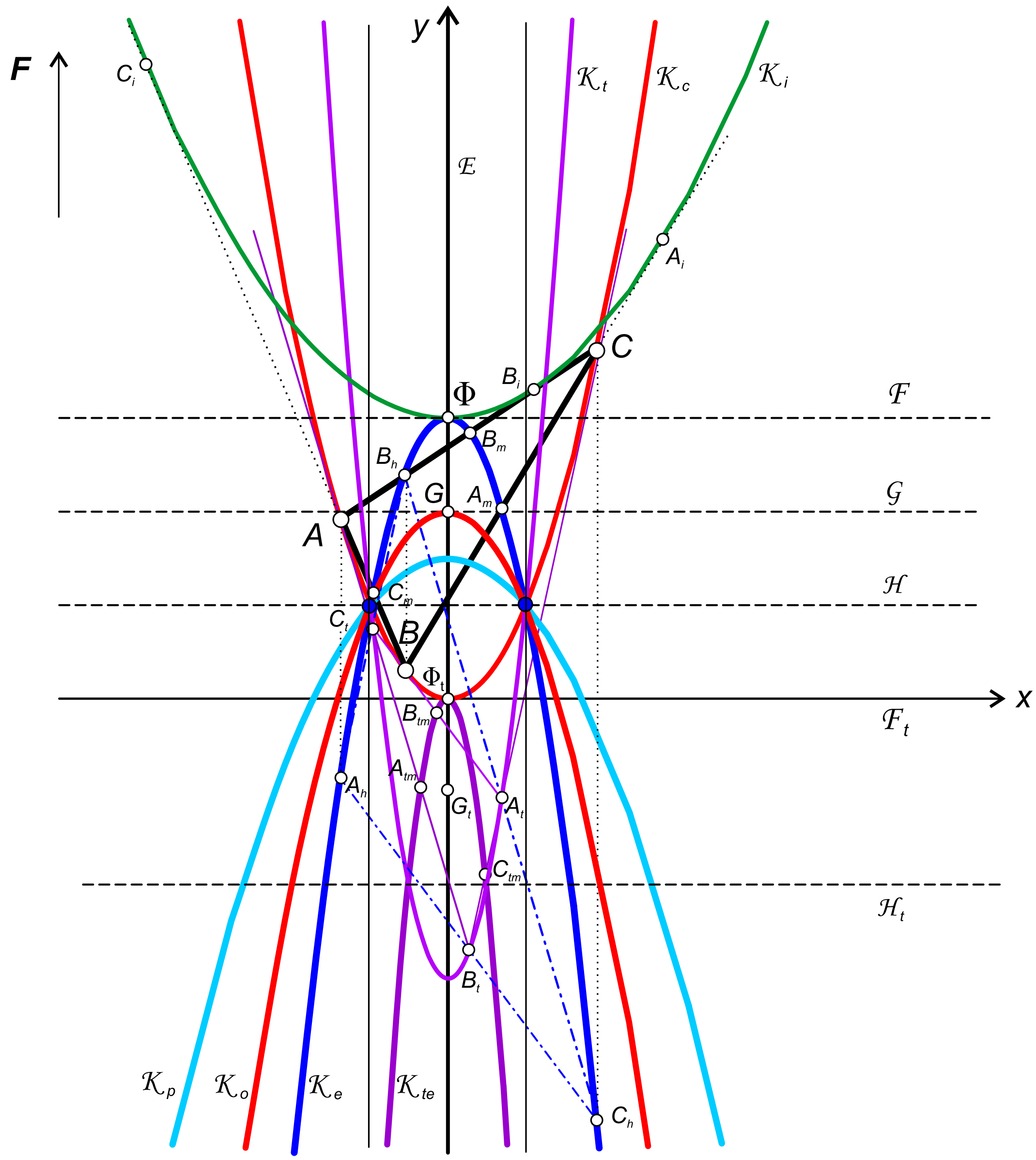

Figure 1: The circumcircle, the Euler circle, the tangential circumcircle, the polar circle, the incircle, the orthocentroidal circle, the degenerated circle, and the Euler circle of the tangential circle of the standard triangle

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