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A Note on Similar-Perspective Triangles

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Abstract. An old theorem of F. E. Wood [9] states that if two triangles in the Euclidean plane are directly similar and perspective from a point then either their sides are parallel in pairs or their circumcircles pass through the point of perspectivity. In this note, we give a simple proof using complex numbers and the notion of triangle shape.

Key Words: similar perspective triangles, Wood's theorem, triangle geometry

MSC: 51M04

1. Introduction

Throughout, we identify the Euclidean plane with the plane of complex numbers, and we define a *triangle* to be any *ordered* triple $[A, B, C]$ of complex numbers that are not all equal. We reserve the notation ABC to stand for the product of the complex numbers A, B, and C. Thus the quadrilateral having vertices A, B, C, and D will be denoted by $[A, B, C, D]$, and the line segment joining A and B by $[A, B]$. The norm of a complex number A will be denoted by $|A|$, and the zero complex number by O. The cross ratio $(A, B; C, D)$ of A, B, C, and D is defined by

$$(A, B; C, D) \quad = \quad \left(\frac{A-C}{A-D}\right)\left(\frac{B-C}{B-D}\right)$$

It is well-known that *the quadrilateral* $[A, B, C, D]$ *is cyclic if and only if the cross-ratio* $(A, B; C, D)$ *is real* (see [3, Corollary 2.2.2, page 65]).

We say that the triangles $[A, B, C]$ and $[A', B', C']$ are *directly similar* if they have the same orientation and if $|A - B| : |A' - B'| = |B - C| : |B' - C'| = |C - A| : |C' - A'|$. It is easy to see that this is equivalent to the requirement that

$$\frac{A-B}{A-C} = \frac{A'-B'}{A'-C'}$$

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as extended complex numbers, i.e., as elements in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. June A. Lester called the quantity $\frac{A-B}{A-C}$ the *shape* of the triangle (A, B, C) and she studied properties and applications of this shape function in great detail in [6], [7], and [8].

Our main theorem, Theorem 1, is an old theorem that appeared, with a purely geometrical proof, in [9]. Our simple proof makes use of the shape function and of the aforementioned characterization, given above, of cyclic quadrilaterals. Theorem 2, which follows immediately from Theorem 1, has appeared earlier; see [4] and [1, Theorem 7], where three different proofs are given. Other proofs are given in [5] and [2].

2. Wood's theorem revisited

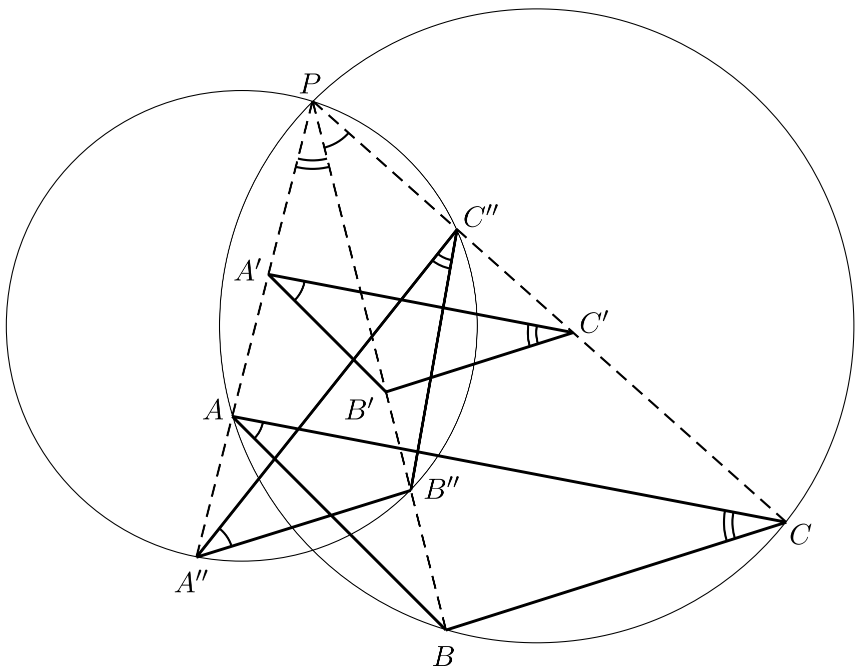

Figure 1: Theorem 1 (F.E. Wood, 1929)

Theorem 1 *Suppose that the triangles $[A, B, C]$ and $[A', B', C']$ are directly similar and perspective from a point P. Then either the sides $[A', B']$, $[B', C']$, and $[C', A']$ are parallel to the sides $[A, B]$, $[B, C]$, and $[C, A]$, respectively, or the quadrilaterals $[A, B, C, P]$ and $[A', B', C', P]$ are both cyclic (see Fig. 1).*

Proof: Without loss in generality, we may assume that $P = O$. Then $A' = xA$, $B' = yB$, and $C' = zC$ for some real numbers x, y, and z. Therefore

$$\begin{aligned}
& [A, B, C] \text{ and } [A', B', C'] \text{ are similar} \\
\Longleftrightarrow\ & \frac{A - B}{A - C} = \frac{A' - B'}{A' - C'} \\
\Longleftrightarrow\ & \frac{A - B}{A - C} = \frac{xA - yB}{xA - zC} \\
\Longleftrightarrow\ & (x - y)AB + (y - z)BC + (z - x)CA = 0 \\
\Longleftrightarrow\ & (y - z)(BC - AB) = (x - z)(CA - AB) \\
\Longleftrightarrow\ & x = y = z \text{ or } (A, B; C, O) = \frac{x - z}{y - z} \\
\Longleftrightarrow\ & x = y = z \text{ or } (A, B; C, O) \in \mathbb{R}.
\end{aligned}$$

In the first case, the sides $[A', B']$, $[B', C']$, and $[C', A']$ are parallel to the sides $[A, B]$, $[B, C]$, and $[C, A]$, respectively. In the second case, the quadrilaterals $[A, B, C, P]$ and $[A', B', C', P]$ are both cyclic, by (the case $D = O$) of Theorem 1. This completes the proof. □

Remark: The orientation preserving similarity $ABC \mapsto A''B''C''$ maps also the circumcircle of ABC onto that of $A''B''C''$. Any pair of corresponding points on these circles is aligned with P. This proves that the remaining point of intersection remains fixed under the similarity, i.e., this similarity is a stretch-rotation about this second point of intersection.

Theorem 2 *Let P be a point inside triangle $[A, B, C]$ and let the cevians through P meet the sides $[B, C]$, $[C, A]$, and $[A, B]$ at A', B', and C', respectively. If the triangles $[A', B', C']$ and $[A, B, C]$ are similar, then P is the centroid.*

Proof: Since P is inside $[A, B, C]$, it follows that $[P, A, B, C]$ cannot be cyclic. By Theorem 1, the sides $[A', B']$, $[B', C']$, and $[C', A']$ must be parallel to the sides $[A, B]$, $[B, C]$, and $[C, A]$, respectively. Therefore

$$\frac{|A - C'|}{|C' - B|} = \frac{|A - B'|}{|B' - C|}.$$

It also follows from Ceva's Theorem that

$$\frac{|A - C'|\ |B - A'|\ |C - B'|}{|C' - B|\ |A' - C|\ |B' - A|} = 1.$$

Therefore $|B - A'| = |C - A'|$, and A' is the midpoint of the line segment $[B, C]$. Similarly for B' and C', and thus P is the centroid. □

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