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Uniform Maps on the Klein Bottle

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Abstract. The purpose of this note is to describe and classify the uniform maps of type {4, 4}, {3, 6} and {6, 3} on the Klein bottle in a simple and uniform way. We determine all such maps and their symmetry properties.

Key Words: Klein bottle, map, uniform, symmetry

MSC: 51M20, 52B15, 53A05

1. Definitions

If Γ is a graph (or multigraph or pseudograph), a *topological form* of Γ is a 1-dimensional cell complex where the 0-cells correspond to the vertices and the 1-cells correspond to the edges of Γ , the correspondence preserving incidence. A *surface* is a connected, compact 2-manifold (without boundary). An *embedding* of Γ in a surface S is a topological form of Γ that is a subset of S such that each 1-cell has a neighborhood which is homeomorphic to a neighborhood of a segment or a circle in the Euclidean plane. The embedding is a *map* provided that each component of the complement of the graph is homeomorphic to an open disk and the boundary of the open disk is a union of 1-cells. Henceforth, we will consider the graph to be identical to its topological form. We will refer to the components of its complement as *faces* of the map.

A map is *uniform* (or *platonic*) of type $\{p, q\}$ provided that each face has exactly p sides and each vertex is incident to exactly q edge-ends. Also, to avoid trivial examples of embedded cycles and dipoles on the sphere and projective plane, it is required that both p and q are at least 3.

Enumeration of uniform maps on a given surface is one of the oldest problems in mathematics, going back to Pythagoreans, who already knew all uniform maps on the sphere (i.e., the Platonic solids). Uniform maps on the projective plane are easy to determine: there are four, each an antipodal projection of a Platonic solid. Uniform maps on the torus were determined in [2]. In this note, we determine all uniform maps on the only remaining surface with non-negative Euler characteristic, the Klein bottle.

If V, E, F are the numbers of vertices, edges and faces in the map, an easy count yields $pF = 2E = qV$. If $\mathcal{M}$ is a uniform map on the Klein bottle or on the torus, its Euler

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characteristic $F - E + V$ must be 0, and from that it is easy to deduce that $\{p, q\}$ must be one of $\{4, 4\}, \{3, 6\}, \{6, 3\}$.

For any map $\mathcal{M}$ on a surface, we can form its dual, $D(\mathcal{M})$ in a usual way: we introduce one new vertex in the interior of each face, and for each edge of $\mathcal{M}$, draw a new edge joining the new vertices in the two adjoining faces. If $\mathcal{M}$ is uniform of type $\{p, q\}$ then $D(\mathcal{M})$ is uniform of type $\{q, p\}$, and $D(D(\mathcal{M}))$ is isomorphic to $\mathcal{M}$.

Let $\mathcal{M}$ be a map of type $\{4, 4\}, \{3, 6\}$ or $\{6, 3\}$ on the Klein bottle. We can choose a universal covering projection of the Euclidean plane onto the surface such that the pre-image of $\mathcal{M}$ is the tessellation $\{4, 4\}, \{3, 6\}, \{6, 3\}$ of the plane into squares, equilateral triangles or regular hexagons, respectively. It follows that every symmetry of $\mathcal{M}$ lifts to an isometry of the plane which is a symmetry of the tessellation.

Then the map is formed from the tessellation by factoring out a fixed-point-free group G of isometries. Of the 17 possible crystallographic groups, only one of them, having international symbol **pg**, is both fixed-point-free and contains orientation-reversing elements. See [1], for example.

This group $G = \mathbf{pg}$ is generated by two glide reflections (we will just say *glides* henceforth), α and β, with parallel axes, each by the distance g in the same direction. Suppose that the distance between the axes is t. Then the set A of axes of glides in this group consists of a family of parallel lines, the distance between any two being a multiple of t.
Then $\alpha\beta^{-1}$ is a translation by $2t$ in the direction perpendicular to the axes, while $\alpha^2 = \beta^2$ is a translation by $2g$ parallel to the axes. Factoring out the group generated by these translations gives a map on the torus which is the 2-fold orientable cover of the map $\mathcal{M}$.

A symmetry of the tessellation projects to a symmetry of the map if and only if it normalizes G. This happens if and only if the symmetry sends the set A to itself. Thus the normalizer of G consists of:

(1) All translations whose component perpendicular to A is an integer multiple of t.

(2) All rotations of 180° about centers on or midway between axes.

(3) All reflections about axes perpendicular to, or in, A, or midway between axes in A.

(4) All glides which are products of translations from (1) and reflections from (3).

2. Maps of type {4, 4}

We first classify maps on the Klein bottle of type $\{4, 4\}$. Consider the grid $\{4, 4\}$ to be placed with its lines horizontal and vertical, and its line segments of length 1. If the axis of a glide is at an angle of θ to the horizontal, the image under that glide of a horizontal line is at an angle of 2θ to the horizontal. Thus, the axis of a glide which preserves the grid must be parallel to lines of the grid or at a 45° angle to them. If parallel, it may lie (a) along a line or (b) midway between two lines. In these cases, the distance g must be an integer.

If it is at 45° to the axes, it might (c) run through vertices and face-centers or (d) run through mid-points of edges. In case (c), g must be an even multiple of $r = \sqrt{2}/2$, while in case (d), it is an odd multiple of r. These possibilities are illustrated in Fig. 1.

We see that the axes of α and β must be both of type a, both of type b, one each of types a and b, both of type c or both of type d.

Let $\{4, 4\}_{|m,n|}$ be the map that results by using two glides of length m on axes that are $n/2$ apart, parallel to the lines of the grid, with at least one of the axes along a line. If n is even, both axes are of type a, and if n is odd, there are one of each type. Fig. 2 shows $\{4, 4\}_{|3,4|}$ and $\{4, 4\}_{|3,5|}$, using letter labels on vertices to show identifications of edges.

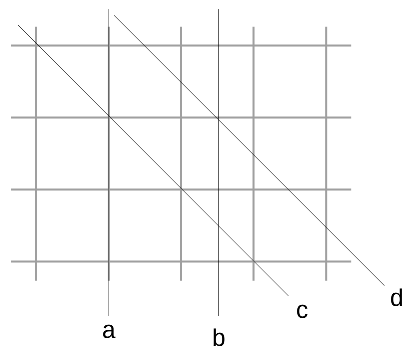

Figure 1: possible glide axes in $\{4,4\}$

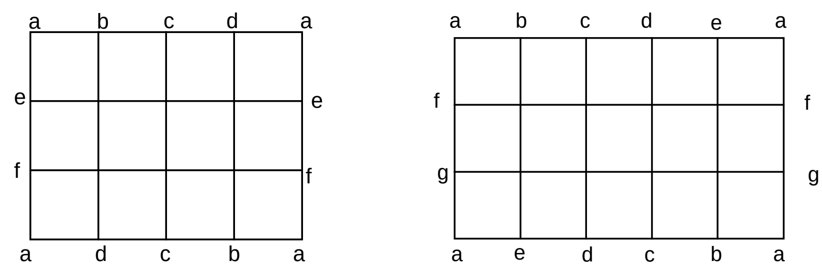

Figure 2: $\{4,4\}_{|3,4|}$ and $\{4,4\}_{|3,5|}$

In both of these drawings, one axis is along the right and left ends of the rectangle, while the other axis is vertical through the center. The third possibility in the parallel case is that the axes of are both of type b. Then n must be even. The resulting map is isomorphic to $D(\{4,4\}_{|m,n|})$. Fig. 3 shows $D(\{4,4\}_{|3,4|})$, labelled in a similar way.

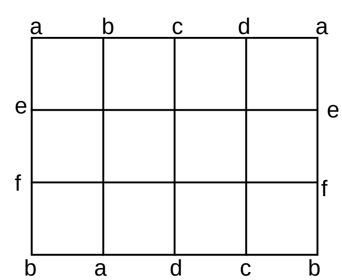

Figure 3: $D(\{4,4\}_{|3,4|})$

In Fig. 3, the axes are vertical lines through the midpoints of edges ab and cd.

If the axes are at a 45° angle from the lines of the grid, then because α and β must have the same component in the direction of their axes, both must be of type c or both of type d. If both are of type c, the distance g must be an even integer multiple of $r = \sqrt{2}/2$. If both axes are of type d, g is an odd multiple of r. In either case, t can be any integer multiple of r.

We define $\{4,4\}_{\backslash m,n\backslash}$ to be the map that results from factoring out by such a G when $g = mr$ and $t = nr$. As an example, we show $\{4,4\}_{\backslash 6,3\backslash}$ in Fig. 4, below. In this example, m is even, so the axes are of type c.

In Fig. 4, three of the axes are shown. A letter labels all faces in the tessellation which are pre-images of one face in the map. The shaded area shows one fundamental region for G. The second part of Figure 4 shows the map as a rectangle with sides identified as shown.

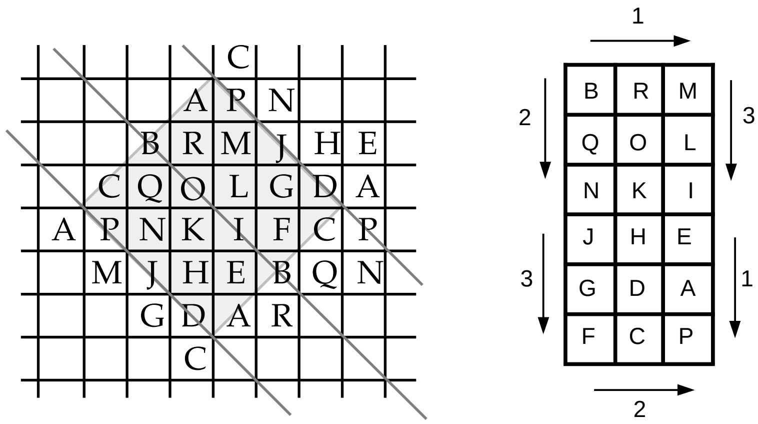

Figure 4: $\{4,4\}_{\backslash 6,3\backslash}$

Specifically, arrow 1 indicates that the edges above B, R, M are the edges to the right of E, A, P, respectively; arrow 2 indicates that the edges at the bottom of F, C, P are the edges at the left of B, Q, N, respectively, and similarly for arrow 3. The map $\{4,4\}_{\backslash m,n\backslash}$ is always self-dual. An example with m odd is $\{4,4\}_{\backslash 3,4\backslash}$, shown in Fig. 5.

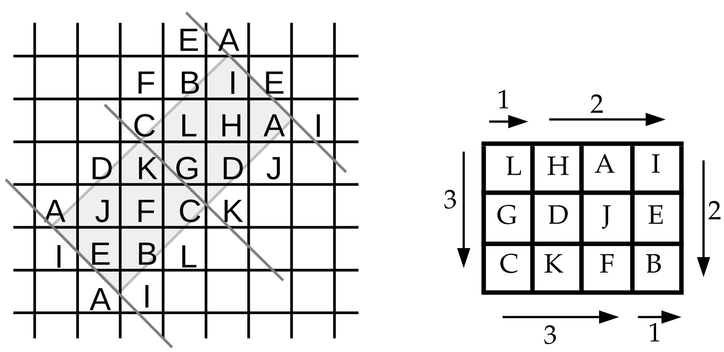

Figure 5: $\{4,4\}_{\backslash 3,4\backslash}$

As symmetries of the map come from isometries of the plane which fix A, we see that vertices in an orbit must be similarly positioned with respect to the axes and the same is true for the edges and faces. It is not difficult to work out the size of the symmetry group and the numbers of orbits of faces, vertices and edges.

For example, consider face E in the left half of Fig. 5. An axis cuts across one corner of E. The only symmetry which fixes E (other than the identity) is a reflection about a line perpendicular to that axis. So the stabilizer of E has size 2. The orbit of E consists of all faces which are cut by an axis, namely A, E, I, C, G, K. By the Orbit-Stabilizer theorem then, the order of $\mathrm{Aut}(\{4,4\}_{\backslash 3,4\backslash})$ is $2 * 6 = 12$.

Table I summarizes information about groups and orbits in these maps. From this we can see that $\{4,4\}_{|m,n|}$ is face-transitive if and only if $n = 1$, 2, 4; it is vertex-transitive if and only if $n = 1, 2$. None of the maps $\{4,4\}_{|m,n|}$ is edge-transitive.

The map $\{4,4\}_{\backslash m,n\backslash}$ is vertex transitive (and so face-transitive) if and only if $n = 1$, or m is even and $n = 2$. It is edge-transitive if and only if $n = 1$, or m is odd and $n = 2$.

Table I

$\mathcal{M}$	$\{4,4\}_{\vert m,n\vert}$		$\{4,4\}_{\backslash m,n\backslash}$	
$V=F$	mn		mn	
E	$2mn$		$2mn$	
$\vert\mathrm{Aut}(\mathcal{M})\vert$	$4t(n)m$		$4m$	
Number of face-orbits	n odd	$\frac{n+1}{2}$	$P(m,n)$	
	n even	$\frac{\frac{n}{2}+2-t(\frac{n}{2})}{2}$		
Number of vertex-orbits	n odd	$\frac{n+1}{2}$	$P(m,n)$	
	n even	$\frac{\frac{n}{2}+t(\frac{n}{2})}{2}$		
Number of edge-orbits	n odd	$n+1$	$Q(m,n)$	
	n even	$\frac{n}{2}+1$		
Medial map	$\{4,4\}_{\backslash 2m,n\backslash}$		m odd	$\{4,4\}_{\vert m,2n\vert}$
			m even	$D(\{4,4\}_{\vert m,2n\vert})$

In this table, we deal with parity by letting

$$t(x)=\begin{cases}2 & \text{if } x \text{ is even,}\\ 1 & \text{if } x \text{ is odd.}\end{cases}$$

We define $P(m,n)$ by:

	n odd	n even
m odd	$\frac{n+1}{2}$	$\frac{n+2}{2}$
m even	$\frac{n+1}{2}$	$\frac{n}{2}$

We then define Q by $Q(m,n)=n+1-P(m,n)$, which effectively interchanges the two rows.

2.1. Rectangular forms

The maps $\{4,4\}_{|m,n|}$ have a natural representation as a rectangle with sections of its perimeter identified in pairs. In Fig. 4 and 5, we see that the maps $\{4,4\}_{\backslash 6,3\backslash}$ and $\{4,4\}_{\backslash 3,4\backslash}$ also have a rectangular form.

These forms will generalize in the following way: The rectangle for $\{4,4\}_{\backslash m,n\backslash}$ is m high by n wide. If $m \leq n$, then the m rightmost segments on the top, in order left to right, are identified with the m segments on the right edge, in order from top to bottom, as arrow 2 in Fig. 5. Similarly, the m leftmost segments on the bottom, in order left to right, are identified with the m segments on the left edge (arrow 3). The remaining segments (if any) on top and bottom are identified directly (arrow 1).

If, on the other hand, $m \geq n$, then the n topmost segments on the left side are identified with the bottom n segments, as arrow 2 in Fig. 4; the bottom right with the top and the rest directly, as before.

In Figs. 6 and 7, we show how the rectangles in each case fit together as fundamental regions for the group G.

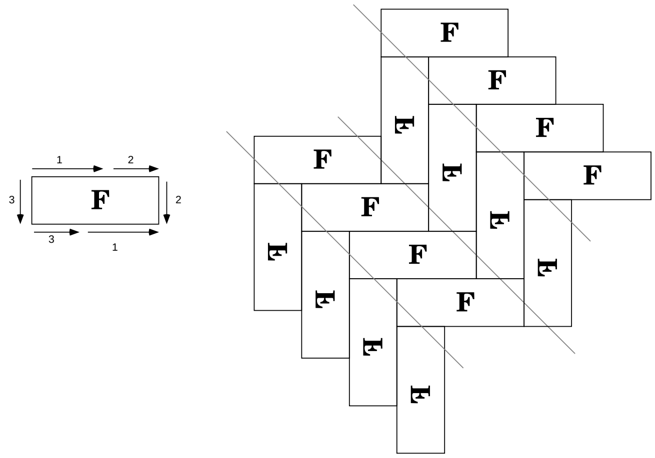

Figure 6: Fundamental regions when $m < n$

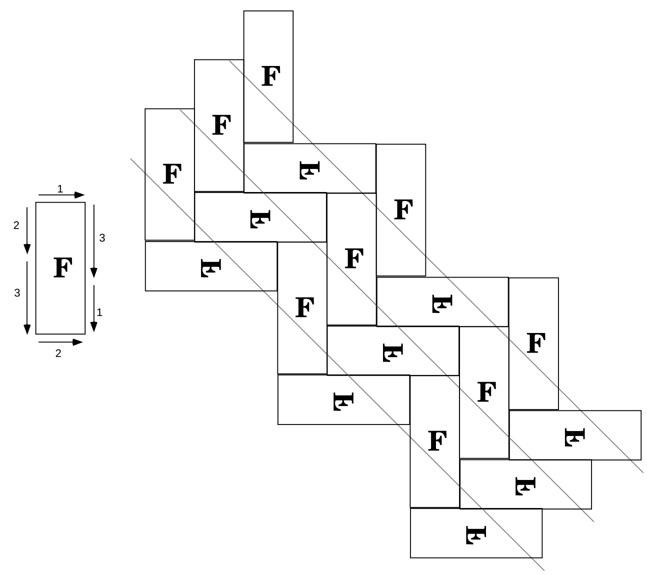

Figure 7: Fundamental regions when $m \geq n$

2.2. Previous work

The only author to previously consider maps of the Klein bottle is THOMASSEN [3]. He describes families of maps of type {4, 4}, using notation $Q_{k,m,x}$, where x is a letter denoting the family, and k and m are numbers, parameters for the family. (More precisely, he describes *graphs* embeddable on the Klein bottle and the torus, and avoids maps whose skeleton is not a graph, as well as those in which some two vertices have the same neighbors.) Table II, below, describes the correspondence between the families in [3] and the current paper.

3. Maps of type {3, 6}

The axis of a glide which preserves the triangular grid $\{3, 6\}$ must be parallel to one class of edges of the grid or perpendicular to one. If perpendicular, it may lie (a) along a line of vertices and face-centers or (b) on a line through edge-midpoints of edges in two parallel classes and quarter-points of edges in the third. In these cases, the distance g must be,

Table II

[3] notation	[3] restriction(s)	current
$Q_{k,m,a}$	k is odd	$\{4,4\}_{\|m+1,k\|}$
$Q_{k,m,a}$	k is even	$D(\{4,4\}_{\|m+1,k\|})$
$Q_{k,m,b}$	k is even	$\{4,4\}_{\|m+1,k\|}$
$Q_{k,m,c}$	k is even	$\{4,4\}_{\|\frac{k}{2},2m+2\|}$
$Q_{k,m,f}$	$k \geq 6$ is even	$\{4,4\}_{\|\frac{k}{2},2m+2\|}$
$Q_{k,m,g}$	k is even	$\{4,4\}_{\|\frac{k}{2},2m+3\|}$
$Q_{k,m,h}$	k is even	$\{4,4\}_{\backslash k+m-1,m-1\backslash}$

respectively, an even or an odd multiple of $s = \frac{\sqrt{3}}{2}$. If an axis is parallel to edges, it might (c) run through vertices or (d) run through mid-points of edges in the other two parallel classes. In case (c), g must be an even multiple of $\frac{1}{2}$, while in case (d), it is an odd multiple of $\frac{1}{2}$. These possibilities are illustrated in Fig. 8.

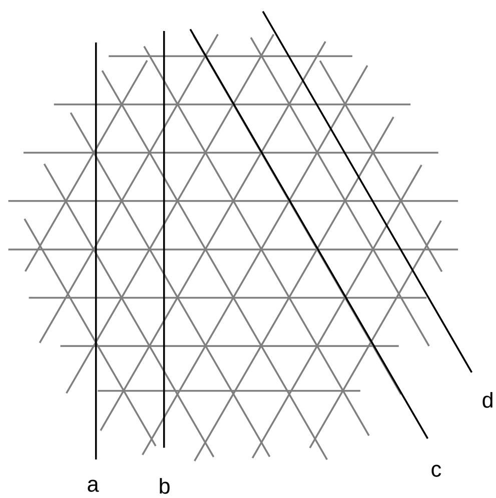

Figure 8: possible glide axes in {3, 6}

Let $\{3,6\}_{|m,n|}$ be the map that results by using two glides of length ms on axes that are $\frac{n}{2}$ apart, perpendicular to one class of parallel lines of the grid. If m is even, both axes are of type a. Fig. 9 shows $\{3,6\}_{|4,3|}$

In Fig. 9, a letter labels all faces in the tessellation which are pre-images of one face in the map. The shaded area shows one fundamental region for G. The second part of Figure 9 shows the map as a parallelogram with slant sides identified directly and horizontal sides identified as indicated by vertices.

If m is odd, both axes are of type b. For example, Fig. 10 shows $\{3,6\}_{|3,4|}$.

If the axes are parallel to lines of the grid, then because α and β must have the same component in the direction of their axes, both must be of type c or both of type d. If both are of type c, the distance g must be an even integer multiple of $\frac{1}{2}$. If both axes are of type d, g is an odd multiple of $\frac{1}{2}$. In either case, t can be any integer multiple of s.

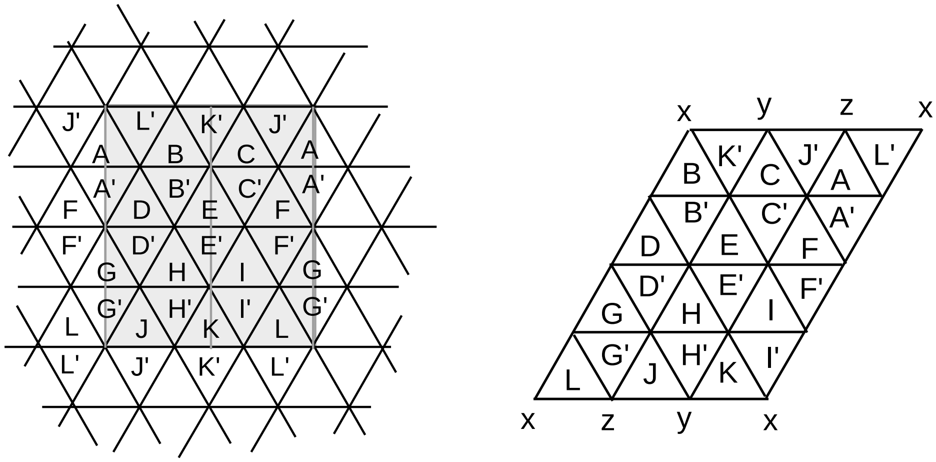

Figure 9: $\{3,6\}_{|4,3|}$

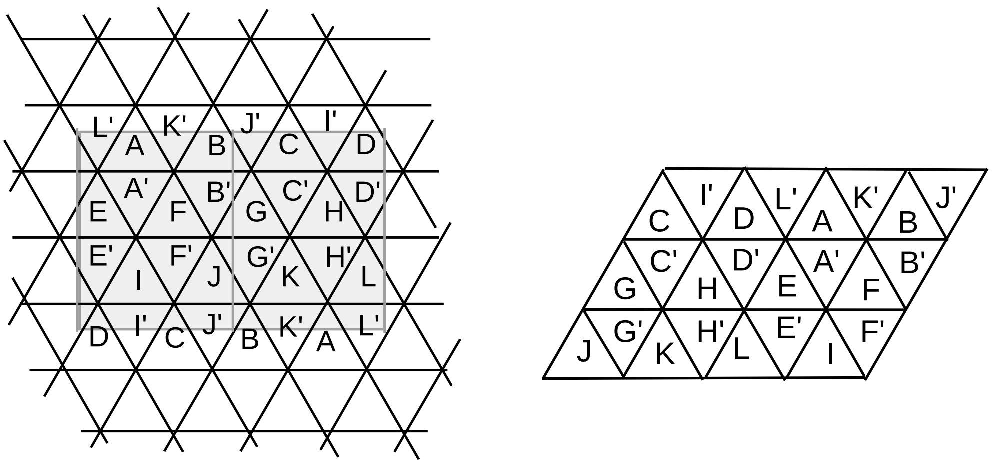

Figure 10: $\{3,6\}_{|3,4|}$

We define $\{3,6\}_{\backslash m,n\backslash}$ to be the map that results from factoring out by such a G when $g = \frac{m}{2}$ and $t = ns$. As an example, we show $\{3,6\}_{\backslash 4,3\backslash}$ in Fig. 11, below. In this example, m is even, so the axes are of type c.

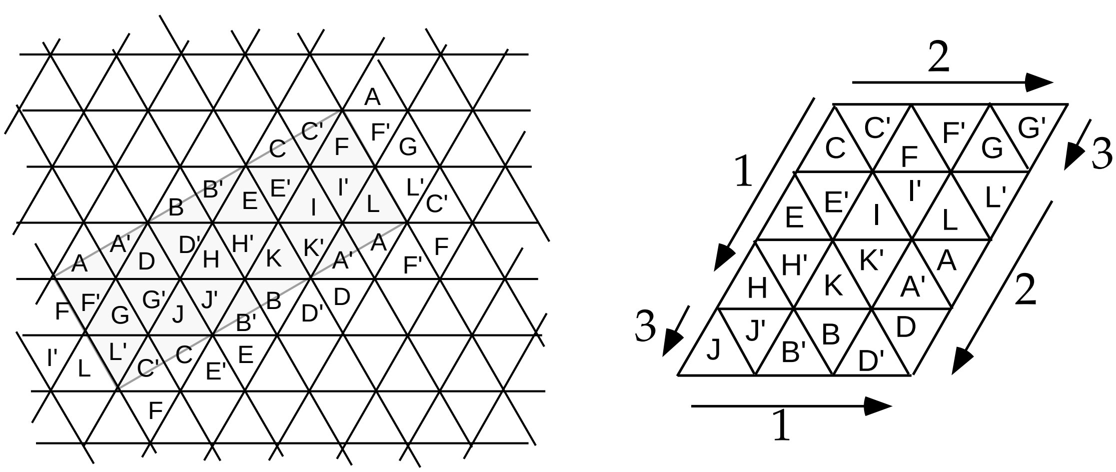

Figure 11: $\{3,6\}_{\backslash 4,3\backslash}$

An example with m odd is $\{3,6\}_{\backslash 3,4\backslash}$ shown in Fig. 12.

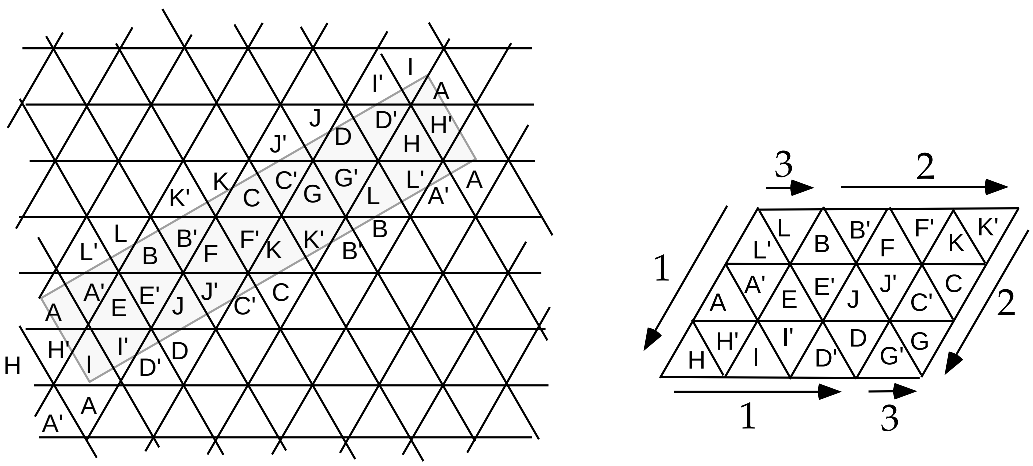

Figure 12: $\{3,6\}_{\backslash 3,4\backslash}$

As in the $\{4,4\}$ case, vertices in an orbit must be similarly positioned with respect to the axes as must edges and faces. Again, it is not difficult to work out the size of the symmetry group and the numbers of orbits of faces, vertices and edges. This information is summarized in Table III, below.

Table III

$\mathcal{M}$	$\{3,6\}_{\|m,n\|}$	$\{3,6\}_{\backslash m,n\backslash}$
F	mn	mn
V	$2mn$	$2mn$
E	$3mn$	$3mn$
$\|\mathrm{Aut}(\mathcal{M})\|$	$4m$	$4m$
Number of face-orbits	$Q(m,n)$	n
Number of vertex-orbits	$Q(m,n)$	$Q(m,n)$
Number of edge-orbits	$n+1$	$n+1$

From this Table we can see that $\{3,6\}_{|m,n|}$ is face-transitive if and only if $n=1$, or m is odd and $n=2$, and is vertex-transitive under those conditions as well. None of the maps $\{3,6\}_{|m,n|}$ is edge-transitive.

$\{3,6\}_{\backslash m,n\backslash}$ is vertex-transitive if and only if $n=1$, or m is odd and $n=2$. It is face-transitive if and only if $n=1$, and it is never edge-trasitive.

3.1. Parallelogram forms

Each of the maps $\{3,6\}_{|m,n|}$ has a natural representation as a parallelogram with sections of its perimeter identified in pairs. In Figs. 11 and 12, we see that the maps $\{3,6\}_{\backslash 4,3\backslash}$ and $\{3,6\}_{\backslash 3,4\backslash}$ also have a parallelogram form.

These forms will generalize in ways similar to those for type $\{4,4\}$. The parallelogram for $\{3,6\}_{\backslash m,n\backslash}$ is m high by n wide. If $m \leq n$, then the m rightmost segments on the top, in order left to right, are identified with the m segments on the right edge, in order from top to bottom, as arrow 2 in Fig. 12. Similarly, the m leftmost segments on the bottom, in order left to right, are identified with the m segments on the left edge (arrow 1). The remaining segments (if any) on top and bottom are identified directly (arrow 3).

If, on the other hand, $m \geq n$, then the n topmost segments on the left side are identified with the bottom n segments, as arrow 1 in Fig. 11; the bottom right with the top and the rest directly, as before.

In Figs. 13 and 14, we show how the parallelograms in each case fit together to make fundamental regions for the group G. In each figure, one glide axis is shown.

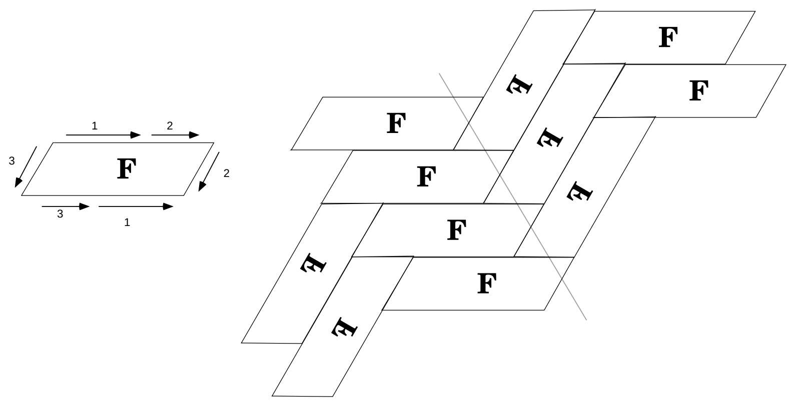

Figure 13: Fundamental regions when $m \leq n$

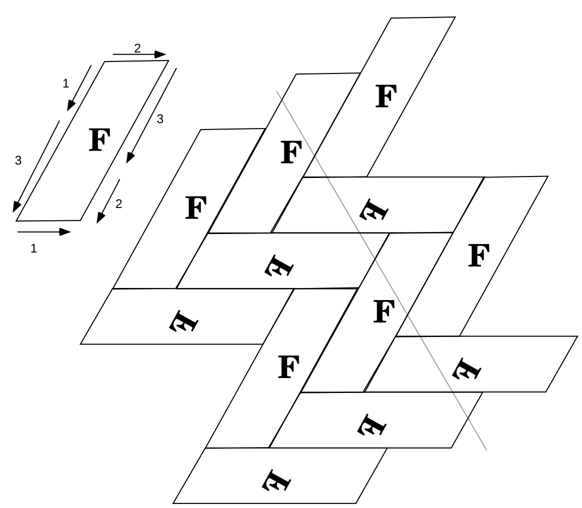

Figure 14: Fundamental regions when $m \geq n$

3.2. Previous work

Again, the only previous reference is Thomassen [3]. He describes families of maps of type $\{6,\ 3\}$, using notation $H_{k,m,x}$, where x is a letter denoting the family, and k and m are numbers, parameters for the family. Table IV, below, describes the correspondence between the families in [3] and the current paper.

Table IV

[3] notation	[3] restriction(s)	current
$DH_{k,m,a}$		$\{3,6\}_{\|m,k\|}$
$DH_{k,m,b}$	k is even, m is odd	$\{3,6\}_{\|m,k\|}$
$DH_{k,m,c}$	k is even	$\{3,6\}_{\backslash m,k\backslash}$
$DH_{k,m,f}$	k is odd	$\{3,6\}_{\backslash k,m+1\backslash}$

3.3. Finally

The reader might enjoy considering these questions to exercise his ingenuity:

(1) Consider the map $\mathcal{M} = \{4,4\}_{\backslash m,n\backslash}$; if we draw, in each square, the diagonal which is parallel to the axes, the result is a map of type $\{3,6\}$. Which map is it? We may instead draw, in each square, the diagonal which is perpendicular to the axes, and get a potentially different map of type $\{3,6\}$. Which map is this?

(2) Kurth [2] offers the best and most thorough description of uniform toroidal maps: consider a 2×2 matrix A of integers with non-zero determinant, and unit vectors $\mathbf{u}$, $\mathbf{v}$ along adjacent axes of the tessellation (at 90° or 60° from each other, respectively). Let G be the group of translations generated by $a_{11}\mathbf{u} + a_{21}\mathbf{v}$ and $a_{12}\mathbf{u} + a_{22}\mathbf{v}$. Modifying the notation of [2] slightly, let $\{4,4\}_A, \{3,6\}_A$ be the maps resulting from the grids by factoring out G. For each of the maps $\{4,4\}_{|m,n|}, \{4,4\}_{\backslash m,n\backslash}, \{3,6\}_{|m,n|}, \{3,6\}_{\backslash m,n\backslash}$, identify its orientable two-fold cover in this form.

(3) Determine which sub-rectangles of $\{4,4\}$ can serve as the "rectangular form" of $\{4,4\}_{|m,n|}$ and $\{4,4\}_{\backslash m,n\backslash}$. Similarly, determine which sub-parallelograms of $\{3,6\}$ can serve as the "parallelogram form" of $\{3,6\}_{|3,6|}$ and $\{3,6\}_{\backslash m,n\backslash}$.

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