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The Golden Section and the Origin of this Name

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Abstract. The Golden Ratio, sometimes known as the Golden Section, is a geometrical concept, which contains a sense of mystery. Although, mathematically, it is not possible to express it as a rational number, it occurs frequently in the natural environment, as well as in many fields of human creativity. The similarity of the expression "Golden Section" in different European languages might be due to a common origin. The paper treats the possibility that the expression "Golden Section" originates in the field of carpentry, as it was practised in Antiquity and later times.

Key Words: golden ratio, golden section, geometry, timber beam, structural design

MSC: 51N05, 51-03, 01A20

1. Introduction

The concept of the Golden Section has kept scientists from different fields busy for thousand of years. According to Johannes Kepler [12], geometry has two pearls: Pythagoras' theorem, and the Golden Section. Even though the ratio corresponding to the Golden Section, i.e., the Golden Ratio, can only be expressed by means of approximations (as a decimal number), the Golden Section is present in all fields of human creativity. In art it is a symbol of good order, and expresses the conformity of the proportionality of a work of art. The Golden Section thus represents the key for composition in architecture in the past. Mathematicians have been concerned with it for thousands of years, and it is the main problem in geometrical creation.

The expression itself, i.e., Golden Section (or sectio aurea / Goldene Schnitt) contains, in various languages, the concept of "cutting" and "cross-section", which indicates that the phrase may have originated in the field of carpentry. The object of this investigation has been to try to determine the significance of the Golden Section in human life, and also to find out whether this expression could indeed have originated in this field.

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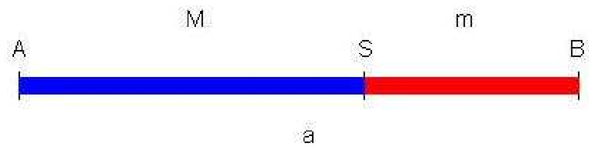
2. The Golden Section in mathematics

Viewed strictly mathematically, the Golden Section is the ratio between a part and the whole. It is designated, in honour of the Greek mathematician PHIDIAS by the Greek letter (Φ), although some mathematicians still use the letter tau (τ). The ratio of the Golden Section (Φ) can most easily be explained by means of the section $a = AB$, which contains the two sub-sections $M = AS$ and $m = SB$. If it is assumed that the ratio $AB : AS$ is equal to the ratio $AS : SB$, then we obtain the ratio corresponding to the Golden Section, whose value is obtained by solving the corresponding quadratic equation (Fig. 1).

The quantity Φ is an irrational number, and even if a large number of places beyond the decimal point are taken into account it is not possible to express it exactly, although we can reach an approximate value. As an illustration, the value of Φ to twenty decimal places is: 1.61803 39887 49894 84820 ..) [16].



$$AB/AS = AS/SB = \Phi$$

$$a/M = m/M = \Phi \qquad (a = M + m)$$

$$(m + M)m = M^2/: m^2$$

$$M/m + 1 = (M/m)^2$$

$$(M/m)^2 - M/m - 1 = 0$$

$$(M/m) = (1 \pm \sqrt{5})/2$$

$$M/m = (1 + \sqrt{5})/2$$

$$M/m = \Phi$$

$$a/M = \Phi$$

$$a/m = \Phi + 1$$

Figure 1: Mathematical determination of the equation corresponding to the Golden Section

A rational approximation to the exact value of the ratio corresponding to the Golden Section (Φ) can be obtained by means of the additive FIBONACCI's sequences. *Additive sequences* have the property that the sum of two successive neighbouring members is equal to the next member. If we compare the ratio between the successive neighbouring numbers of the first FIBONACCI sequence, then it can be seen that these values approximate the value Φ of to the Golden Section. This is also true for the second and third FIBONACCI sequences (Fig. 2).

3. The Golden Section in geometry

In geometry, the Golden Section is even more highly valued than in numerical mathematics (algebra). Its main role is the division of a certain line segment into two separate parts in the ratio corresponding to the Golden Section. The diagram given below shows the application of the Golden Section to the subdivision of the section AB into the two sub-sections AS and SB, in the ratio corresponding to the Golden Section. The rectangle which has its length of the sides in the ratio of the Golden Section is known as the Golden Rectangle. The ratio corresponding to the Golden Section also occurs in a Regular Pentagon, which provides much insight into the Golden Section, and is the most important geometrical figure connected to the Golden Section. The examples given below in Fig. 3 show some structures containing the Golden Section.

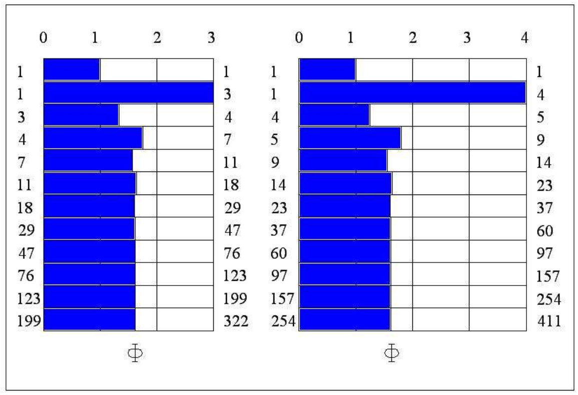


Figure 2: The ratios between the successive numbers of the second and third Fibonacci sequences

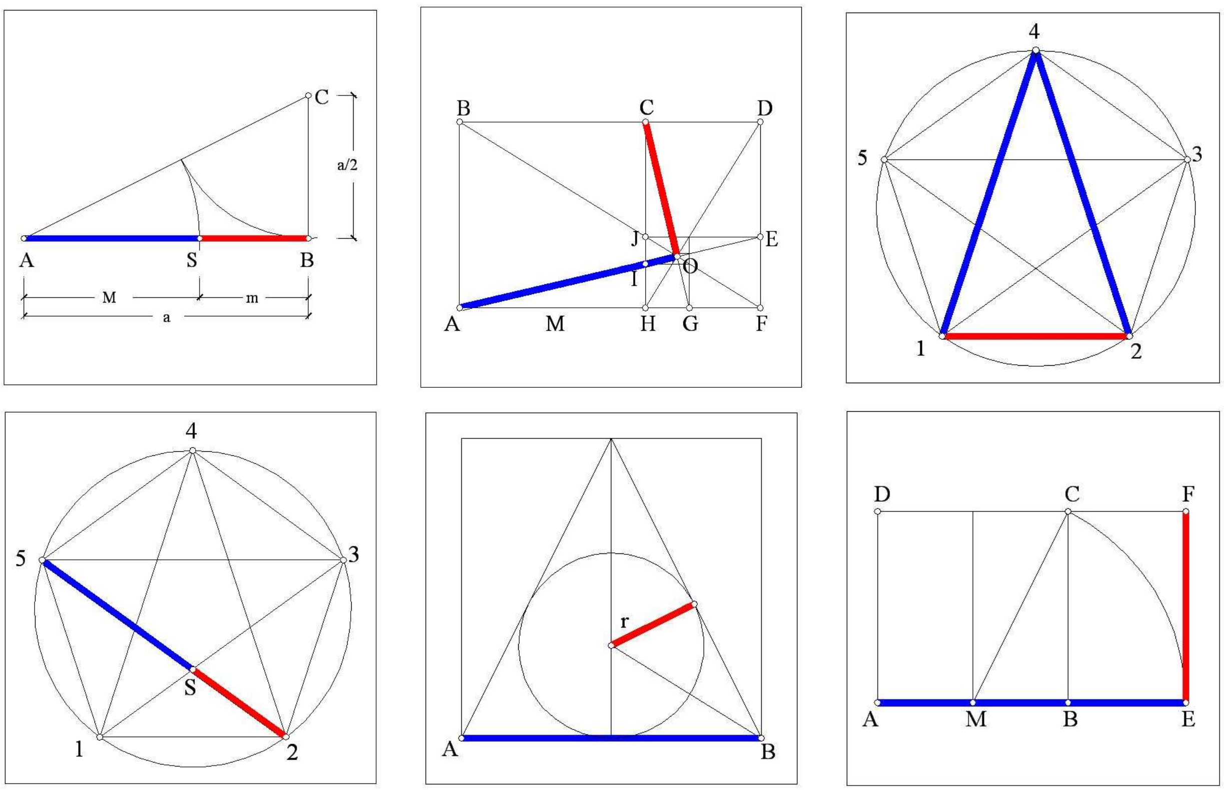


Figure 3: The Golden Section in geometrical figures

4. The Golden Section in the natural environment and in fine arts

Neufert [11] based his assertion that the Golden Section is the most attractive proportional ratio on the following items of proof. A proportional analysis of the human body shows that individual parts of the body occur in ratios defined by the Golden Section. The results of proportional analysis of flora and fauna also show that the structures of their individual parts occur in the same ratio. Fechner's psychological investigations, which were aimed at determining the "most attractive rectangle", showed that more than one third of those interviewed considered that rectangles defined by the Golden Section are the most attractive

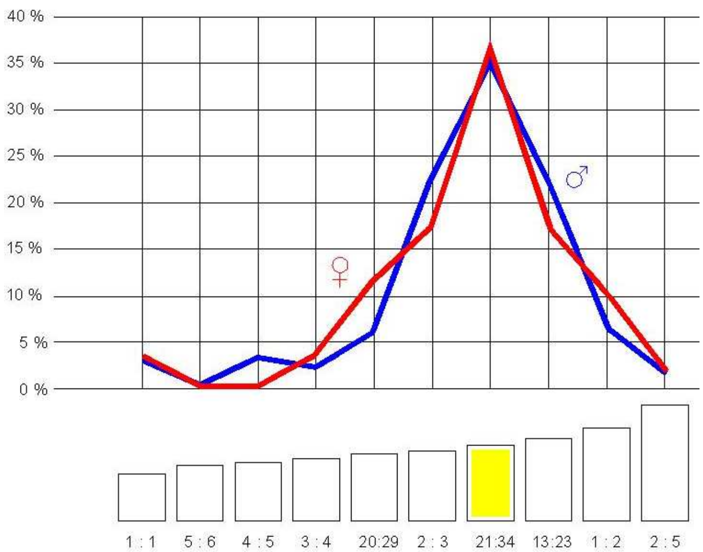


Figure 4: The results of Fechner's psychological study of the "most attractive rectangle"

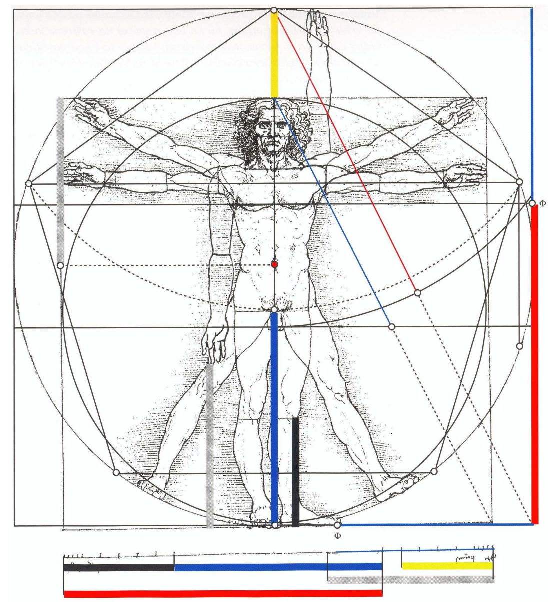


Figure 5: Proportional analysis, using the Golden Section of the study of a human body by Leonardo da Vinci [13]

(Fig. 4). It is considered that this may also be the result of the fact that the field of vision of the human eye is in the proportions of the Golden Section [15].

The results of analysis of the proportions of works of art indicate that the arrangement of different elements in a picture or in relief is not coincidental. The position of any element depends on its role, i.e., its value with respect to the work as a whole. By means of geometrical analysis it is possible to find a proportional key, which indicates how the elements are arranged in the work of art, and what roles they have. The proportional key can also show us that the important figures or details are frequently painted in proportions corresponding to the Golden Section. Compositions using the Golden Section are observed very early in works created in relief, from the Ancient East. During the Middle Ages the most well-known proportional study of the human body in the Golden Section was performed by the leading intellect of the Renaissance, Leonardo da Vinci (Fig. 5).

In music, the Golden Section has two roles. Its first role defines the difference between tones and their frequencies, since the ratios between tones can be expressed by ratios between small integers. The ratios are as follows: a first 1:1, a second 8:9, a third 4:5 (a minor third 5:6), a fourth 3:4, a fifth 2:3, a sixth 3:5 (a minor sixth 5:8), a seventh 4:7 and an octave 1:2. The minor sixth most closely approaches the ratio corresponding to the Golden Section (5:8 = 0.625) [4]. The arrangement of tones in a musical composition takes place according to the rules of musical composition, but can also be expressed mathematically [9].

5. The Golden Section in architecture

The ratios between the length, width and height of an architectural structure, as well as the ratios between individual parts of the structure, represent one of the essential questions of architecture, and this question appeared very early on in historical times. It was, of course, of primary importance that buildings satisfied the requirements of strength and safety. The

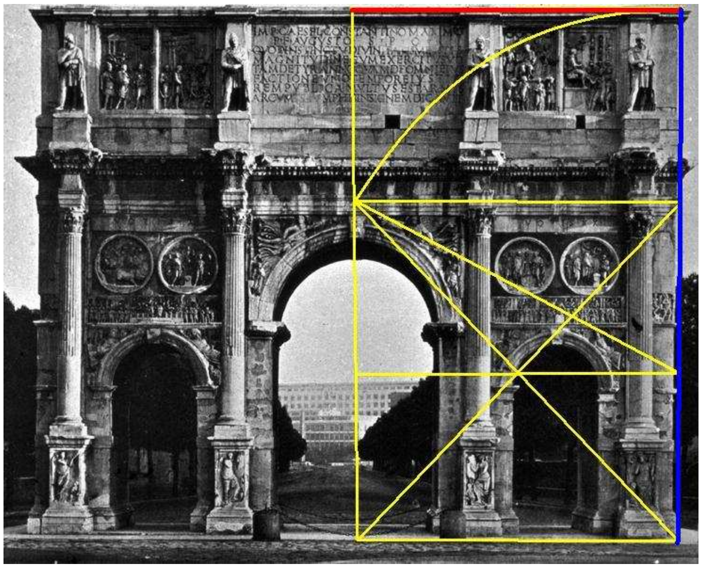


Figure 6: Proportional analysis of *Constantine's Arch* in Rome

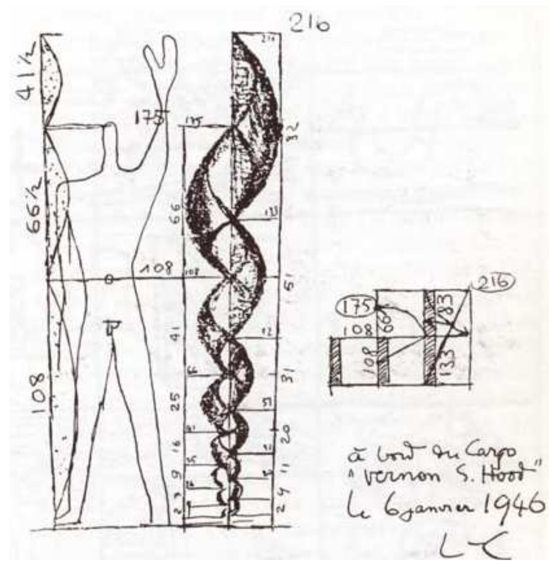


Figure 7: Le Corbusier's study of the proportions of the human body *(Modulor)* [2]

size of buildings depended on the funds available for their construction, and even to a greater extent on the expertise of builders, the state of development of building technology, and the availability of building materials. What were the ratios between the length, breadth and height of a building? The results of geometrical analysis have indicated that builders always kept to certain patterns of building. The basis for this was provided by small elements and the human body. The latter was important as a measure for dimensions (the size of the palm of the hand, the size of a foot, the length of a step, etc.), either as building "machines" (the reach of an arm, the strength needed to carry burdens, etc.) or as a user (the size of rooms, the height of ceilings, seats, etc.). Building elements were constructed taking into account the sizes of different parts of the human body. During construction works these elements repeated themselves, and were used until the required dimensions were obtained. In this way the concept of the ratio between the dimensions of a building and its parts, i.e., proportionality was introduced.

In the case of architecture, it is best to make a mathematical and geometrical study of proportions for the period when mathematics and geometry were developed sufficiently to permit people to make conscious decisions about certain proportional keys. The Ancient Greeks, for instance, were convinced that beauty is based on mathematical relationships, both rational and irrational, and that symmetry and rhythm are the basic criteria for beauty and ideal proportions [14]. So-called proportional compasses were used for design on the basis of the Golden Section (Fig. 6).

In more recent years Neufert [11] and Le Corbusier [2] have become well-known names in the field of human measures in relation to architecture. Neufert is best-known for his studies of the relationship between human measures and internal furnishing. The so-called *Modulor* system is the result of Le Corbusier's studies. This is a measuring system consisting of two curves, which is based on the proportions defined by the Golden Section. One curve leads from the apex of a man standing upright (183 cm), whereas the other is based on the figure of a man with outstretched arms (226 cm), which is crossed over by the halved

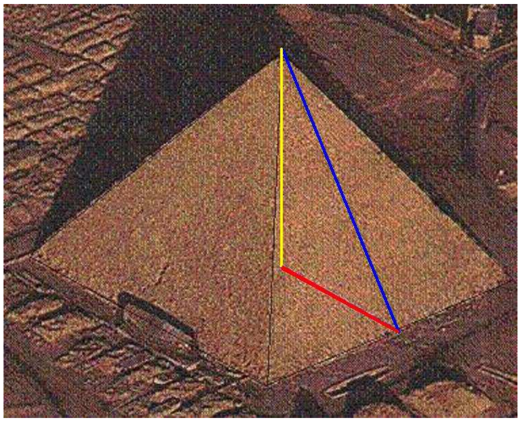


Figure 8: Proportional analysis of *Cheops' pyramid*

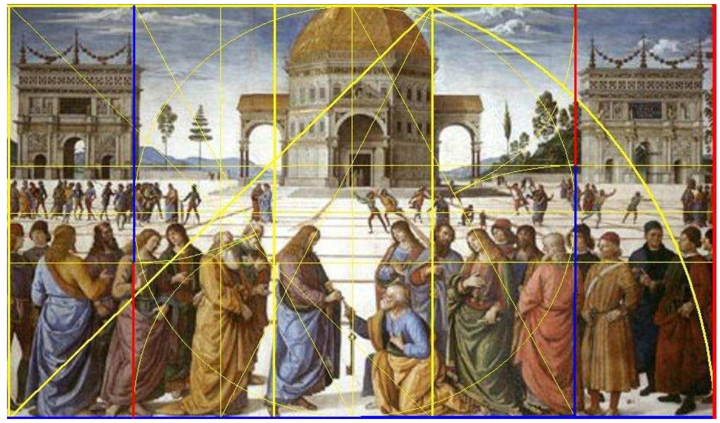


Figure 9: Proportional analysis of a picture by the artist Vanucci

value of the larger unit (113 cm). According to Le Corbusier, the human body is the basic unit for everything that is built. The size of parts of the human body, and its movements, are used as measures [10] (Fig. 7).

6. Historical overview

The Golden Section was known to the Ancient Greeks, the pentagram being the recognized sign of the School of Pythagoras. Taking into account the results of research, it can be concluded that the Ancient Egyptians, too, knew of the existence of the Golden Section, since they used it in the construction of their pyramids (Fig. 8). Proportioning on the basis of the Golden Section was also known to the Romans. For instance, at Pompei a so-called proportional compass was found, which could be used to make designs based on the ratio of the Golden Section. Although Vitruvius, in his book, mentions this ratio, he uses no name for it. Later knowledge about proportions disappeared into the secret world of the guilds, but on the basis of an analysis of the works (including works of art) that they created (Fig. 9) it can concluded that the ratio corresponding to the Golden Section was frequently used. The first written record of the ratio known today as the Golden Section goes back to the beginning of the 16$^{\text{th}}$ century, when the Venetian Luca Pacioli [13] called this ratio the *"Divina proportione"*, as well as describing knowledge about it as *"secretissima scientia"*, proving its secrecy. The name *"section aurea"* occurs very late, only in the 19$^{\text{th}}$ century, but then succeeds existing expressions very quickly, in the majority of European languages.

7. The Golden Section in the cross-sections of timber beams

The idea that there might be an original structurally-based reason for the proportions of the Golden Section sounds rather unusual, since in connection with art, which is dominated by the concept of the Golden Section, we are not used to using vocabularies from the fields of logic, physics, and mathematics. However, this idea becomes much more easily understandable and acceptable since its validity has been confirmed by engineering-based views, derived from basic technological and static quantities. At the 21st Meeting of the Structural Engineers of Slovenia, which was held at Bled in October 1999, the hypothesis [8] was put forward that builders, in Antiquity, translated their experimentally obtained findings about the most

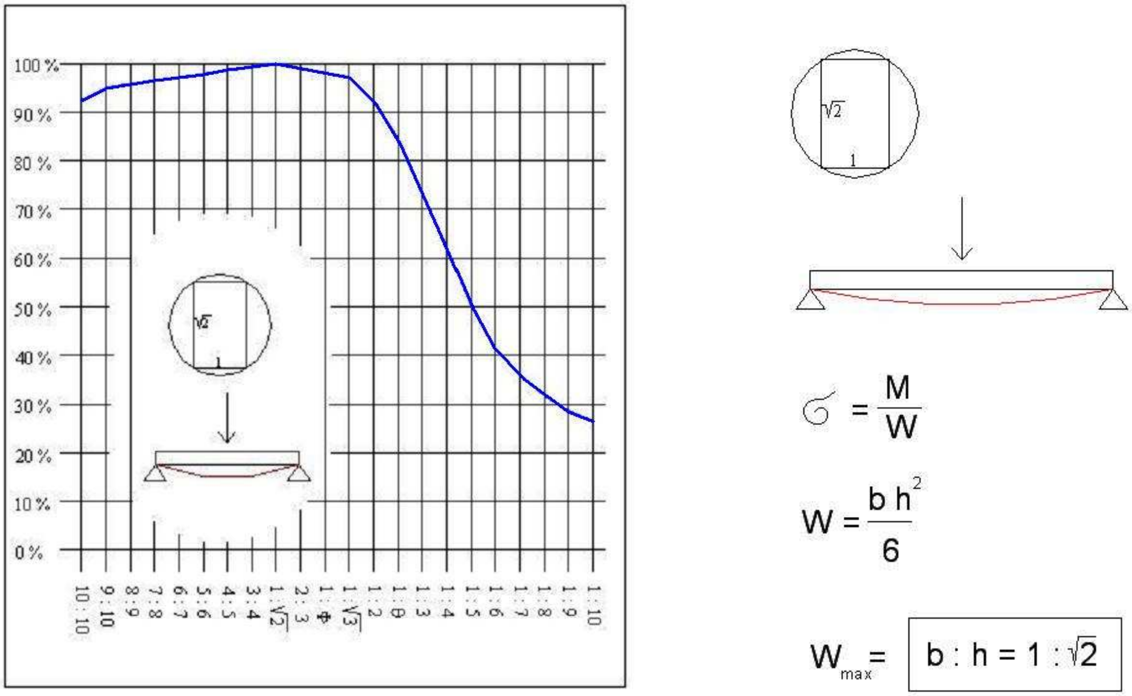


Figure 10: Diagram showing the load-carrying capacity of a beam, cut from a log, with differently shaped rectangular cross-sections, expressed as a percentage of the load-carrying capacity of the beam with the largest section modulus W_{max}

favourable cross-sections of timber beams into simple geometrical models having proportional ratios. Based on the historical and technological findings about the preparation, design, transport and installation of timber beams, and the use of up-to-date static formulae and mathematical procedures, a new, so far unknown statically-structural aspect of their design has become apparent, indicating the original reason for the use of the ratio corresponding to the Golden Section. The importance of these conclusions is particularly in the possibility of their proof by means of statics and mathematics. In the case of these assertions, one starts from the generally acknowledged requirement that beams cut out of tree-trunks must be cut up in such a way that they can optimally satisfy the foreseen loading conditions. The greatest load-carrying capacity and the least deflection of a beam with a rectangular cross-section can be expressed as follows:

- the greatest load-carrying capacity of a beam with a rectangular cross-section, cut from a log, is obtained when the beam is cut in such a way that its cross-section has the greatest section modulus W_{max} (Fig. 10),
- the smallest deflection of a beam with a rectangular cross-section, cut from a similar log, is obtained when the beam is cut in such a way that its cross-section has the greatest moment of inertia I_{max} (Fig. 11).

By implementing a program for the evaluation of the individual specific static quantities of the cross-sections of beams, it is possible to confirm the validity of the assumption, and at the same time it has been shown, by means of a diagram showing the product of the degree of exploitation of the maximum load-carrying capacity in bending, and the minimum deflection, that the shaping of beams with proportions corresponding to the Golden Section suits both criteria best. (Fig. 12).

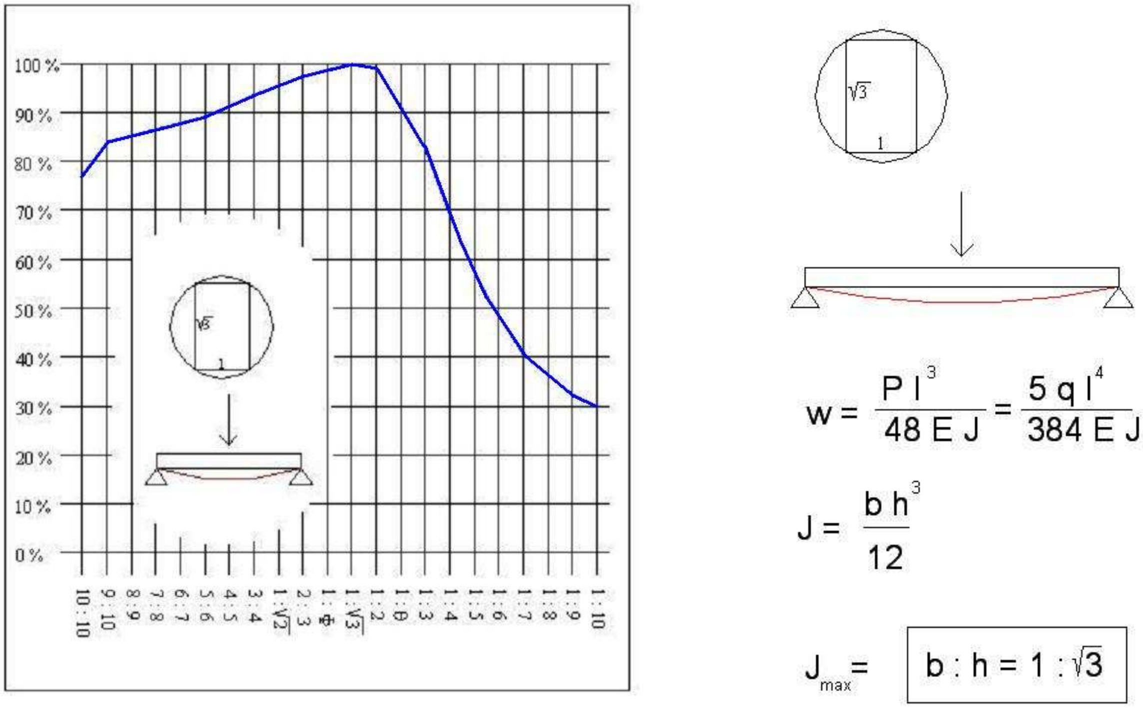


Figure 11: Diagram showing the deflection of a beam, cut from a log, with differently shaped rectangular cross-sections, expressed as a percentage of the deflection of the beam with the largest moment of inertia I_{max}

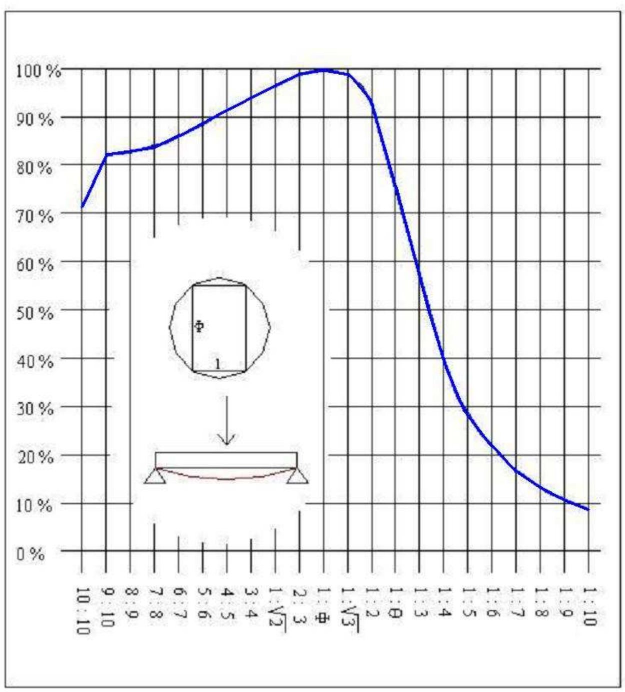


Figure 12: Diagram showing the product of the percentage of the maximum section modulus W and the percentage of the maximum moment of inertia

The result obtained was verified by means of an extreme function, which deviates from the correct value of the "Golden Section" by 2.33%. The difference between the value of the extreme function and the product of the percentages defining the maximum load-carrying capacity and deflection can be attributed to the differences between the actual stress-strain diagrams in the case of permissible bending stresses and the bending stresses at failure, where this difference is hidden due to the rheological properties of wood [7]. This is the additional minimum reserve which former builders and researchers in early times found and took into account when determining proportions on the basis of the Golden Section.

8. Conclusions

On the basis of the results of archaeological excavations and historical research it is well known that, in Antiquity, the knowledge of mathematics and geometry, as well as experimentation, was so well developed that certain loading tests and measurements could be easily performed, so that comparisons between results could be made, and conclusions drawn. In the same way we are able to conclude that the Ancients knew how to turn technical findings, obtained by means of tests, into practical applications of the proportions of rectangular geometrical shapes. For this reason the assumption that the expression "Golden Section" originates from findings about the optimum dimensions of timber beams, obtained by carpentry, sounds quite reasonable. Explanation of the origin of these proportions on the basis of their static structural aspect is based mainly in the mathematical ease of proof of such a method of thinking. The best cut, i.e., the Golden Section was turned — as technological know-how — into simple mathematical formulae, whose technical significance has, over the centuries, have been covered over by layers of other explanations.

The late arrival of the expression "Golden Section", and the use of expressions with

Figure 13: A medieval construction site [5]

the same meaning in different European languages indicates that this expression was probably used, for many years, among the members of guilds of carpenters, and was therefore widespread throughout Europe. Members of these guilds travelled widely, and thus were able to exchange knowledge and experience (Fig. 13). In this way it is possible to confirm the use of the same expression "golden cut" ("cutting", "cross-section") in different European languages. The appearance of the expression "Golden Section" among the general public is probably connected to the downfall of guilds, in the first half of the 19th century. In the Austro-Hungarian Empire guilds were officially abolished in 1859 [6]. The secrets of good design were very carefully guarded within the framework of a guild. The punishments for those giving away such secrets were very cruel, including the death of the guilty party and/or members of his family. However, at the time of the appearance of the expression "Golden Section" the strength and influence of guilds was insufficient to prevent the escape into the public eye of a once carefully guarded secret.

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