Journal for Geometry and Graphics
Volume 11 (2007), No. 1, 111–126.

Spire-polyhedra

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Abstract. The shapes of the spires of Western European medieval churches show almost as high variability as their interiors, but while inside — after the first Millennium — the builders began to use curved surfaces (i.e., vaults), the spires mostly kept their polyhedral (or conical) shapes. Architecture — due to its necessities and restrictions — used only a limited portion of the infinite set of potentially possible polyhedral shapes — such a small subset, that it seems conceivable to categorise them, or at least most of them. This paper suggests a method of classification of spire shapes, postulating that the more complex forms can be produced as compounds (either intersections or unions) of some basic shapes.

Key Words: spire, geometry, classification of spire shapes

MSC: 51N05

1. Basics

Strictly speaking, the subject of this paper is to describe a specific subset of polyhedra, and in this sense, its pure geometry. At the same time, the goals and means of architecture play such an important role in defining the domain and codomain of the design space, that it seemed not only reasonable, but — for the sake of brevity and clarity — almost inevitable to use some architectural terms and definitions.

1.1. Base concepts, definitions

By definition, a spire is a steeply pointed termination to a tower. In this paper I will use this term in a somewhat wider sense: not only for the most common pyramidal or conical shapes, but for any shape a roof of a tower can have. In the figures below the top parts of the towers are also depicted, but they do not belong to the spires proper: the spire ends at the horizontal plane where its bottommost sloping plane ends. In other words the incidental gables are part of the geometry of the spire (in addition to the sloping planes of the roof proper of course), but the tower walls are not, even if they are in the same vertical plane.

ISSN 1433-8157/\$ 2.50

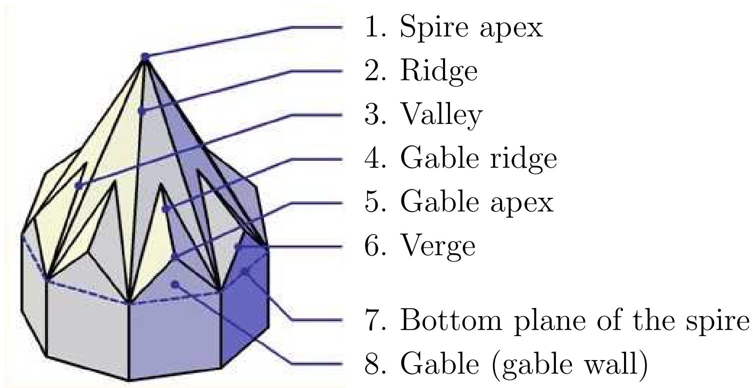


Figure 1: Parts of a spire

A *gable* is a vertical plane (a wall, actually) whose existence is inevitable whenever the bottom edges of the sloping surfaces of the roof proper are not horizontal. A *verge* is the sloping outer edge of a gable, a *gable apex* is the highest point of a verge. A *spire apex*, however, is a point located over the centre of the base, usually the highest point of the whole shape. A *valley* is a concave break between adjacent surfaces, which therefore collects the water from them; while a *ridge* is a convex break, which consequently diverts, not collects water. Finally, a *gable ridge* is a ridge starting from the gable apex, mostly (but not always) connecting it with the spire apex.

1.2. Base principles

The most obvious examples of architectural restrictions determining the applicability of shapes are the ones that are consequences of the limitations of the building materials, like the size and steepness restrictions in case of wood, stone or brick spires. Another consideration is the rationality of the plan: e.g., the claim for freer plan arrangement causes the dominancy of rectangular shapes. Finally, a third aspect is aesthetics, which — through the use of symmetry — effectively excludes every ad hoc plan and shape.

Even if we take the above restrictions into account, the set of possible shapes is still so enormous, that it takes some abstraction to produce usable categories; however, in my opinion this does not decreases the applicability of the system.

1.3. Base plans

The archetype of the medieval tower can be described as a building or part of a greater building (mostly a church or castle), whose height is considerably bigger then the dimensions of its base — which is usually a square, a polygon or a circle, or, in rare cases, a rectangle or an ellipse (arranged in order of decreasing frequency).

In this paper I focus on spires with square or regular octagonal base, partially because conclusions derived from their examinations can be easily generalized, but mainly because these are (prominently) the most frequent shapes — so much so, that sometimes we find polygonal spires even on circular towers (e.g., in Maria Laach/Germany). Furthermore, the spires of circular medieval towers are usually simple surfaces of revolution — mostly cones.

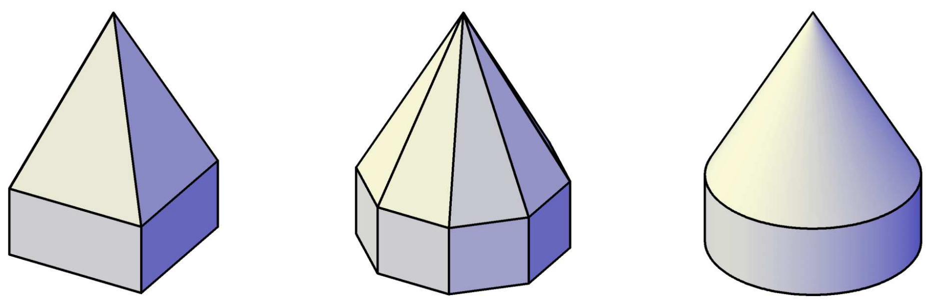
2. Basic shapes

First let us deal with the simplest shapes, as according to my hypothesis all architecturally relevant spire shapes can be generated from them.

Note here that in case of these basic shapes the change of the steepness of roof planes will not produce a topologically different spire shape, therefore a whole set of shapes can be derived from the ones being shown below using only affine transformations.

2.1. Regular n-gonal pyramid

Perhaps the simplest way to cover a regular n-gonal base is to use congruent isosceles triangles. The $\mathbf{a}_4$ and $\mathbf{a}_8$ shapes of Fig. 2 are depicting pyramids of equal height, covering regular polygonal bases which can be inscribed in circles having equal radii[1]. By duplicating the number of the faces of a pyramid we soon get a shape which — at least in architectural practice —, can be classified as a cone. Perhaps this kinship is the reason why all three shapes appear on the church of Maria Laach.



$\mathbf{a}_4 \longrightarrow \mathbf{a}_8 \longrightarrow \mathbf{a}_0$

Figure 2: Regular n-gonal pyramids

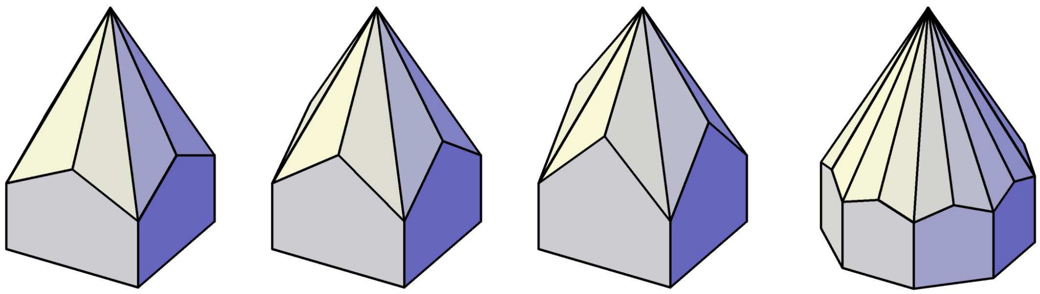
2.2. Convex $2n$-gonal base-truncated pyramid

If we grab the midpoints of the base of a regular n-gonal pyramidal spire described in 2.1, and move them slightly upward, the sloping triangular surfaces break, and because of these new ridges (connecting the spire apex with the apexes of the newly formed gables) the horizontal section of the spire becomes a $2n$-gonal polygon.

Since the spire apex is closer to the midpoints of the base then to its corners, if the gables are not too high, the new gable ridges are steeper then the ridges over the diagonals of the base ($\mathbf{b}_4^-$), but when the height of the gables exceed a certain limit (see 2.7), the diagonal ridges become the steeper of the two sets ($\mathbf{b}_4^+$).

The idiosyncrasy of shapes $\mathbf{b}_4$ and $\mathbf{b}_8$ of Fig. 3 is that all of their ridges have equal slope, and this way their horizontal sections (over the level of their gable apexes) are regular polygons. Therefore, these shapes can be described as base-truncations of regular $2n$-gonal pyramids. I call a shape "base-truncated pyramid", when a regular pyramid is truncated by a rotational symmetric set of planes which are perpendicular to the base of the pyramid. For example, if we take an $\mathbf{a}_8$ pyramid, connect every second vertex of its base, and then keep

[1]For better comparison all shapes in Section 2 have equal spire height — and therefore equal slope over the diagonals of their base polygons.



$\mathbf{b}_4^-$ $\mathbf{b}_4$ $\mathbf{b}_4^+$ $\longrightarrow$ $\mathbf{b}_8$

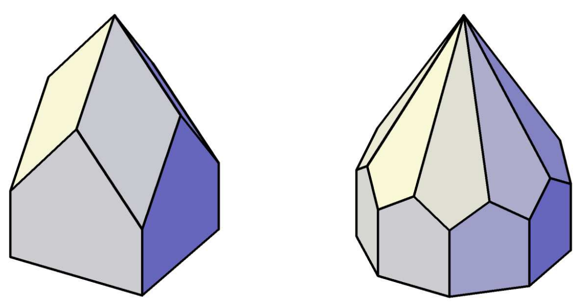
Figure 3: Convex $2n$-gonal base-truncated pyramids

only that part of the original volume which lies above this new square base, we get a shape similar (affine) to $\mathbf{b}_4$.

The most well-known examples of this shape (or more exactly its $\mathbf{b}_4^+$ subtype) are the spires of the Speyer Cathedral in Germany — but the regular $\mathbf{b}_4$ form also appears, for example on the towers of the Marienkirche in Lübeck/Germany.

2.3. Rotated n-gonal base-truncated pyramid

If we raise the gable apexes higher, until the diagonal ridges are embedded into the roof planes (see 2.7), we get a rotated base-truncated n-gonal pyramid — or in other words, the part of a pyramid which lies over the n-gonal base we get by connecting the midpoints of its original base.



$\mathbf{c}_4$ $\longrightarrow$ $\mathbf{c}_8$

Figure 4: Rotated n-gonal base-truncated pyramids

Over a square base, the $\mathbf{c}_4$ shape of Fig. 4 appears when the height of the gable apexes is exactly half of that of the spire apex. Since in this case the verges and the gable ridges are parallel, the sloping planes of the spire are rhombuses, and the shape of the roof proper can be described as a translational surface.

The $\mathbf{c}_8$ shape is obviously similar to the previous $\mathbf{c}_4$ one as it also lacks the diagonal breaks, but — just like shape $\mathbf{b}_4$ in 2.2 — it also can be described as a (different) base-truncation of a similar $\mathbf{a}_8$ pyramid. The sloping planes of this shape are deltoids.

The $\mathbf{c}_4$ form can be found for example on the western tower of the cloister church of Maria Laach, the octagonal $\mathbf{c}_8$ form on St. Martin Münster in Bonn/Germany.

2.4. Concave $2n$-gonal base-truncated pyramid

The obvious next step is to raise the gable apexes even higher, which again produces diagonal breaks in the roof surfaces — this time valleys.

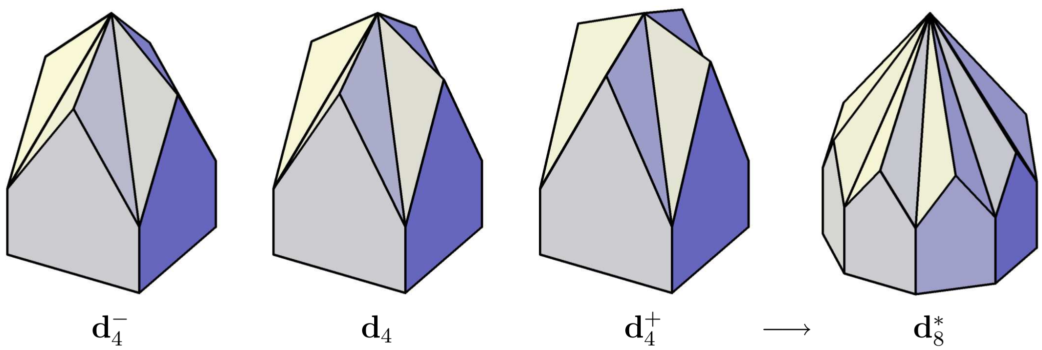


Figure 5: Concave $2n$-gonal base-truncated pyramid

It is true again that a small lowering ($\mathbf{d}_4^-$) or raising ($\mathbf{d}_4^+$) of the gable apexes does not change the basic attributes of this shape — and that we can find an equilibrium ($\mathbf{d}_4$, see 2.7), when the slopes of the verges and the diagonal valleys are equal. Furthermore, shape $\mathbf{d}_4$ of Fig. 5 is a complementary form of shape $\mathbf{b}_4$ described in 2.2, since the slopes of the verges of shape $\mathbf{b}_4$ and the slopes of the gable ridges of shape $\mathbf{d}_4$ are equal (and vice versa). This logic obviously can be applied to an octagonal base also, but — unlike its quadrilateral counterpart — this shape hasn't got any further special attributes.

However, in case of n-gons, $n > 4$, there is a much more significant shape. The appearance of this $\mathbf{d}_8^*$ shape is similar to the "normal" $\mathbf{d}_8$ form (e.g., it also has star-shaped horizontal section), but in other respects it resembles the $\mathbf{c}_4$ form described in 2.3, since its verges and gable ridges have equal slopes. It has an even more unique attribute also: the roof proper has sixteen faces, but only eight planes, as each pair of every third face is lying in a common plane. Hence, this shape can be described as the union of two isomorphic base-truncated pyramids (one obviously in rotated position) — the same way as the $\mathbf{c}_8$ shape can be seen as their intersection.

This rather attractive $\mathbf{d}_8^*$ shape can be seen for example on the eastern towers of St. Aposteln in Cologne/Germany.

2.5. Intersecting gable roofs

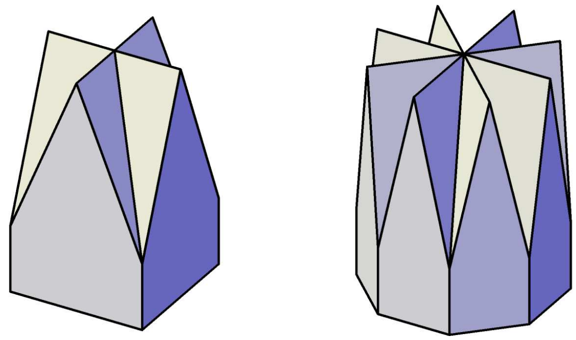
If we raise the gable apexes to the height of the spire apex, we get intersecting gable roofs.

Shape $\mathbf{e}_4$ of Fig. 6 resembles shape $\mathbf{d}_8^*$ described in 2.4, since in both cases each pair of every third verge is lying in a common plane passing through the spire apex, and the intermediate gables seem to emerge from these planes.

In strict architectural sense these shapes are probably not spires — but they are used as components of several compound spire shapes. Besides, these forms can indeed be seen on towers: after World War II (before its restoration), the $\mathbf{e}_4$ shape appeared on the Marienkirche in Lübeck.

2.6. Additional basic shapes

There are shapes not described above that can be generated by similar logic.

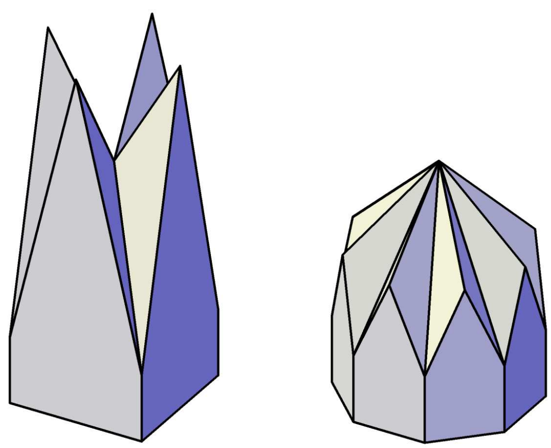


$\mathbf{e}_4$ $\longrightarrow$ $\mathbf{e}_8$

Figure 6: Intersecting gable roofs

Shape $\mathbf{f}_4$ of Fig. 7 illustrates that the gable apexes (theoretically at least) can be even higher then the spire apex: in this figure the slopes of the diagonal valleys and the gable ridges are equal (resembling $\mathbf{b}_4$).

Shape $\mathbf{d}_8^{**}$ of Fig. 7 is another special case. On shape $\mathbf{c}_8$ each pair of every second verge, on $\mathbf{d}_8^{*}$ each pair of every third verge, and on $\mathbf{e}_8$ each pair of every seventh is lying in a common plane passing through the spire apex. Therefore it is quite obvious, that there must be a shape, where each pair of every fifth verge is lying in a common plane, with two intermediate gables emerging from these planes.



$\mathbf{f}_4$ $\mathbf{d}_8^{**}$

Figure 7: Additional basic forms

2.7. Comparison of basic shapes

Before moving on to the more complicated forms, let us have a look at the more significant shapes summarized in Fig. 8, and clarify algebraically what the above geometrical solutions mean.

Let us designate the height of the spire as H, the radius of the circumcircle of its base n-gon as R, and let h_v be the height of the gable apex ($v := a, b, c, d, e$) (see Fig. 9).

As we have seen in 2.2, a $\mathbf{b}_n$ spire is basically a base-truncated $2n$-gonal pyramid whose diagonal ridges (starting from the spire apex) reach the base plane, while its gable ridges — due to the base-truncation — do not. Since both sets of ridges have equal slopes, their

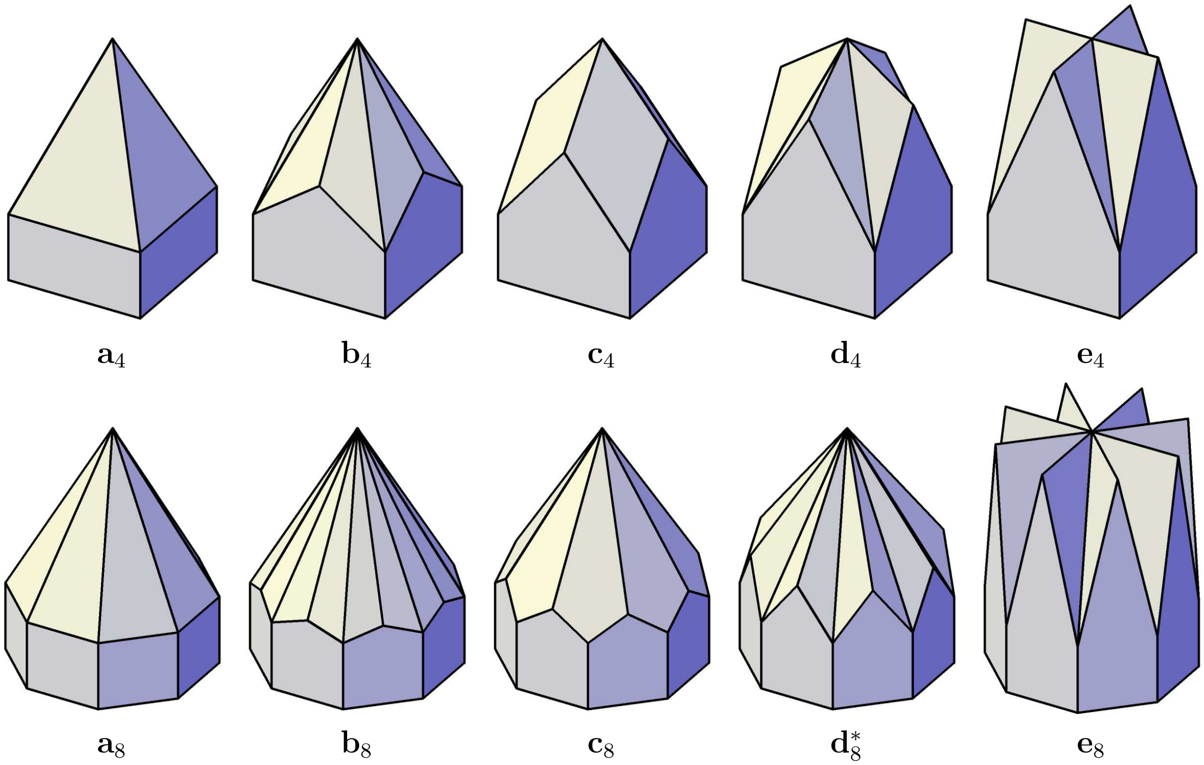


Figure 8: Basic square and octagonal spire shapes of equal spire apex height, arranged in order of ascending gable apex height: $\mathbf{a}_n$ regular n-gonal pyramid, $\mathbf{b}_n$ convex $2n$-gonal base-truncated pyramid, $\mathbf{c}_n$ rotated n-gonal base-truncated pyramid, $\mathbf{d}_n$ concave $2n$-gonal base-truncated pyramid, $\mathbf{e}_n$ intersecting gable roofs

height-difference is proportional with the length-difference of their horizontal projections — and hence the radii of the circumcircle and incircle of the base. If we designate the radius of the incircle of the base n-gon as r, we can write:

$$r = R \cdot \cos\tfrac{\pi}{n}\,, \quad \frac{H}{R} = \frac{h_b}{R-r} \implies \frac{H}{R} = \frac{h_b}{R - R\cdot\cos\frac{\pi}{n}} \implies h_b = H\cdot\left(1-\cos\tfrac{\pi}{n}\right).$$

The $\mathbf{c}_n$ spire is also a base-truncated pyramid — this time an n-gonal one —, hence R can be seen as the radius of the incircle of the un-truncated pyramid. As this way the gable ridges are simply the remains of the original pyramid's sloping edges (the length of their horizontal projections is again r), if we designate the radius of the circumcircle of the un-truncated pyramid (hence the length of the horizontal projection of the pyramid's sloping edges) as S, we can write:

$$R = S\cdot\cos\tfrac{\pi}{n}\,, \quad \frac{H}{S} = \frac{h_c}{S-r} \implies \frac{H}{S} = \frac{h_c}{S - S\cdot\cos^2\frac{\pi}{n}} \implies h_c = H\cdot(1-\cos^2\frac{\pi}{n}).$$

The idiosyncrasy of a $\mathbf{d}_n^*$ spire ($n > 4$) is that the horizontal projections of the sloping triangular faces of the un-truncated pyramids have $4\pi/n$ central angles (not $2\pi/n$, as in the $\mathbf{b}_n$ and $\mathbf{c}_n$ cases) — hence, shape $\mathbf{d}_{2n}^*$ and shape $\mathbf{b}_n$ have equal gable apex height. If we designate the radius of the circumcircle of of the un-truncated pyramid as T, we can write:

$$r = T\cdot\cos\tfrac{2\pi}{n}\,, \quad \frac{H}{T} = \frac{h_d^*}{T-r} \implies \frac{H}{T} = \frac{h_d^*}{T - T\cdot\cos\frac{2\pi}{n}} \implies h_d^* = H\cdot(1-\cos\tfrac{2\pi}{n}).$$

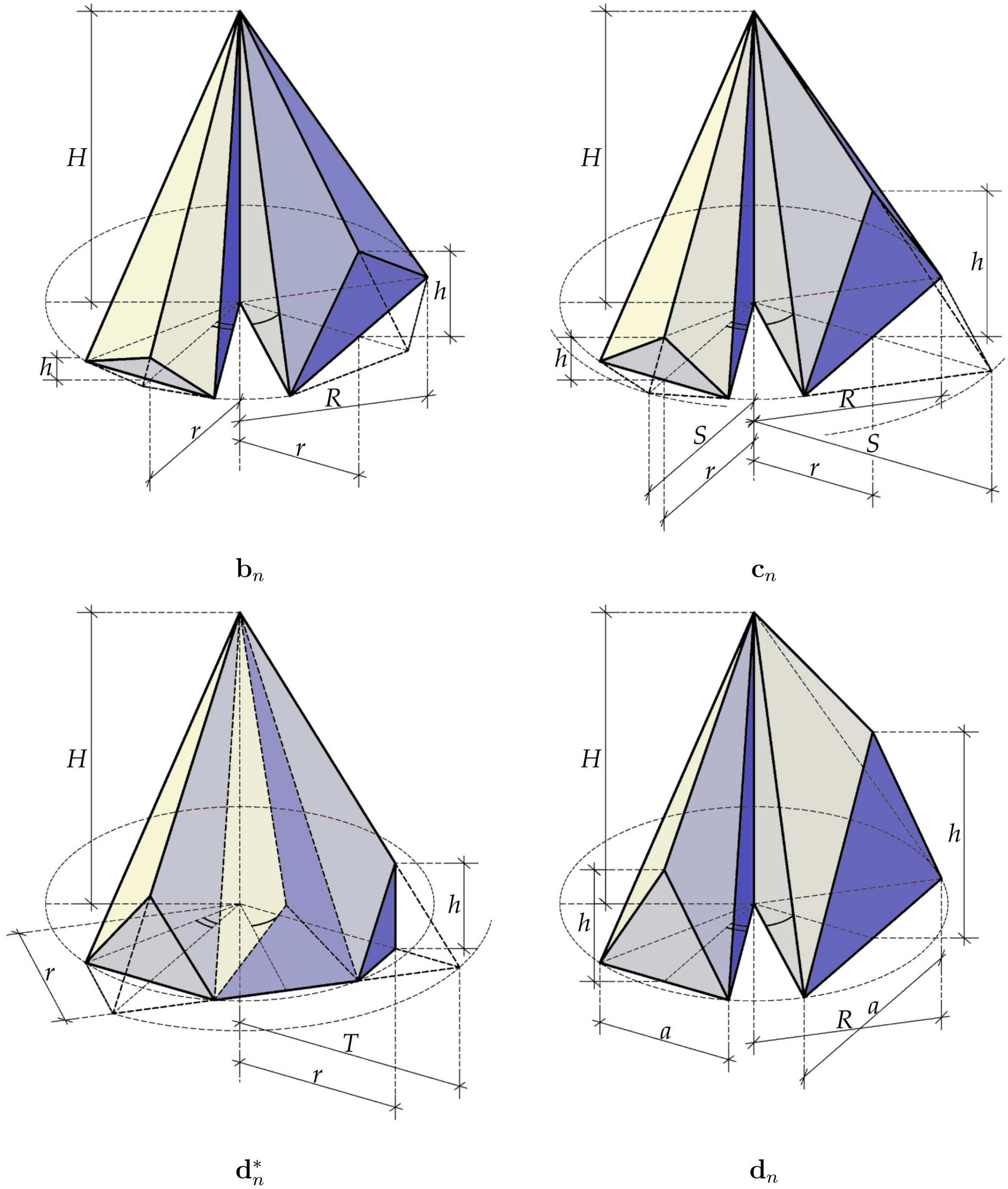


Figure 9: Finding the height of the gable apex

Since in $n = 4$ case the above equation would produce an unsatisfactory result (shape $\mathbf{e}_4$), in 2.4 I suggested another specification for the $\mathbf{d}_n$ designation: a shape whose verges and diagonal valleys have equal slopes. The advantage is that this type can be interpreted in $n \leq 4$ cases also — the disadvantage is its irrelevancy in $n > 4$ cases. If we designate the length of the side of the base n-gon as a, we can write:

$$\tfrac{a}{2} = R \cdot \sin \tfrac{\pi}{n}\,, \quad \frac{H}{R} = \frac{h_d}{a/2} \implies \frac{H}{R} = \frac{h_d}{R \cdot \sin \frac{\pi}{n}} \implies h_d = H \cdot \sin \tfrac{\pi}{n}\,.$$

As (obviously) $h_a = 0$, and $h_e = H$, we now have all necessary limits set:[2]

$$\begin{array}{ll} \text{for } h = h_v & \text{we have } v_n,\ (v := a, b, c, d, e) \\ \text{for } h_a < h < h_b & \text{we have } b_n^-, \\ \text{for } h_b < h < h_c & \text{we have } b_n^+, \\ \text{for } h_c < h < h_d & \text{we have } d_n^-, \\ \text{for } h_d < h < h_e & \text{we have } d_n^+, \\ \text{and for } h_e < h & \text{we have } f_n. \end{array}$$

3. Compound shapes

I use the term "compound" for the shapes described in this section because they can be produced as combinations of the basic elements described in Section 2.

Note that this process adds a new level of variability: if we combine the same types of elements, but choose different relative heights for them, we get shapes that are — although still topologically similar — not affine transformations of each other, unlike the basic shapes described above.

3.1. Combinations of basic shapes having square base

All shapes of Fig. 10 can be produced as either a union ($\cup$) or an intersection ($\cap$) of two of the $\mathbf{a}_4$, $\mathbf{c}_4$, or $\mathbf{e}_4$ shapes. Each cell of the figure contains a combination of the elements that can be found in the first column of its row, and in the first row of its column. In the top left, middle, and bottom right cells we can see the basic elements themselves (as either unions or intersections of two congruent shapes). Above this diagonal we can see the results of the unions, below it the intersections of the basic shapes.

Obviously, the shapes of Fig. 10 are not equally frequently used in the architectural practice — probably we cannot even find examples for all of them. (However, the rarely used forms in the second row of the table get a greater role if we add another $\mathbf{a}_8$ pyramid to the compound, as we will see it in 3.3).

It seemed pointless to repeat the same table of combinations for octagonal shapes, as it would show even fewer relevant forms. One noticeable exception is the octagonal variation of the top right shape ($\mathbf{e}_8 \cup \mathbf{a}_8$), which can be seen for example on the cathedral of Limburg an der Lahn in Germany.

3.2. Combinations of pyramids

Not only different types of shapes can be combined: a considerable number of spires can be described as unions of pyramids.

Shape $\mathbf{a}_4 \cup \mathbf{a}_8$ of Fig. 11 illustrates the union of two pyramids (a square and an octagonal one) whose bases are circumscribed about the same circle (e.g., St. Vigor, Cerisy-la-Forêt/France). But it's the exception, not the rule: the typical solutions are unions of a steeper pyramid having smaller base dimensions and a less steep one having a bigger base.

The number of the faces of the two pyramids can be equal, as in the $\mathbf{a}_4 \cup \mathbf{a}_4'$ and $\mathbf{a}_8 \cup \mathbf{a}_8'$ cases (e.g., in Marmoutier/France), but it can also be different: in this case usually the higher

[2] Two examples: $h_{c4} = h_{d6}^* = 0.5H$, and $h_{b4} = h_{d8}^* = 2 \cdot h_{c8} = H - h_{d4} = H \cdot (1 - \frac{\sqrt{2}}{2}) \approx 0.2929H$.

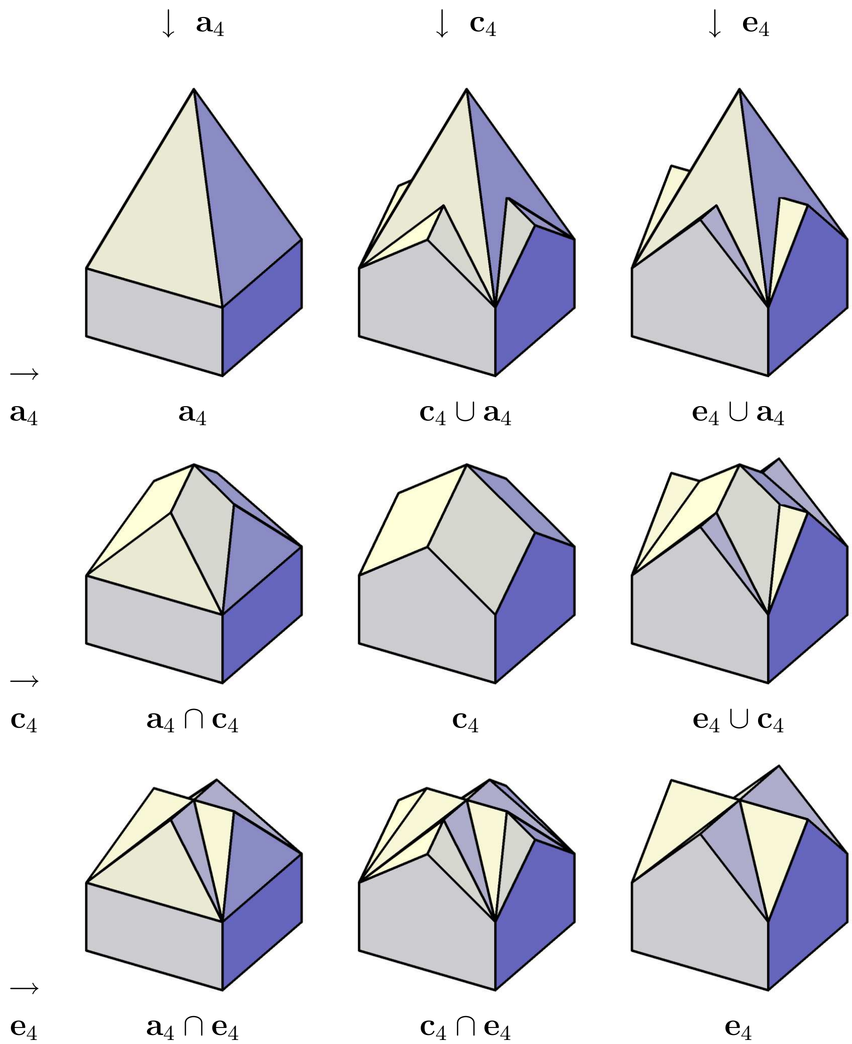


Figure 10: Combinations of basic shapes having square base

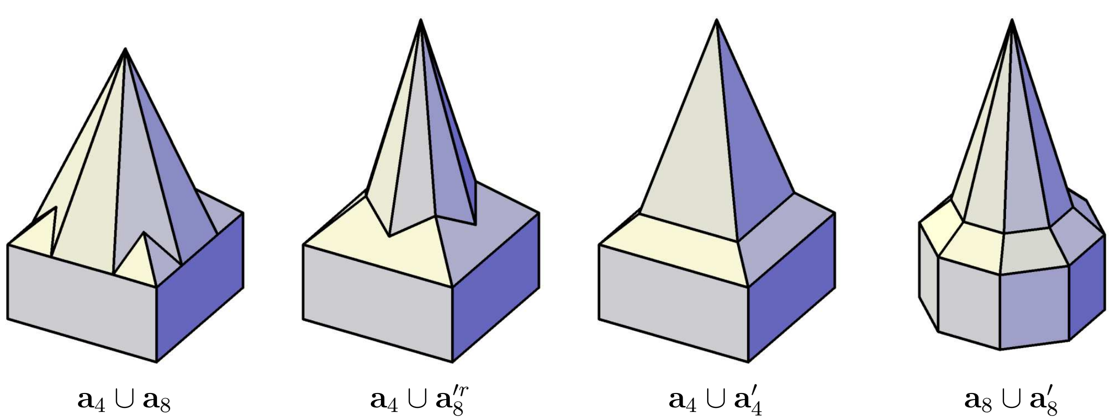


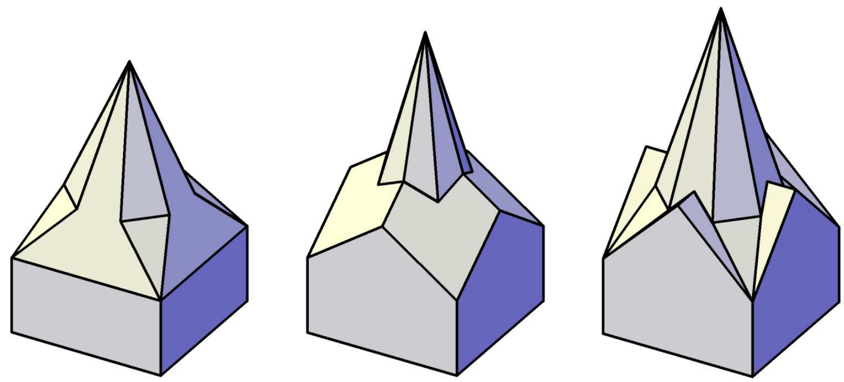
Figure 11: Unions of square and octagonal pyramids

pyramid has more faces, probably because — due to its more acute angles at its top — it seems to be even higher this way.

Shape $\mathbf{a}_4 \cup \mathbf{a}_8'^r$ demonstrates another possibility also: the upper pyramid can be in rotated position (e.g., Nyírbátor/Hungary, bell tower). This can be done in case of square pyramids also, but it's more likely in case of octagonal ones, since this way — as it can be seen in the figure — the ridges can run directly from the spire apex to the corners of the base.

3.3. Combinations of pyramids and other shapes

As I mentioned in 3.1, the shapes in the second row of Fig. 10 are more frequently used in union width an additional $\mathbf{a}_8$ pyramid[3].



$\mathbf{a}_4 \cap \mathbf{c}_4 \cup \mathbf{a}_8$ $\mathbf{c}_4 \cup \mathbf{a}_8'^r$ $\mathbf{e}_4 \cup \mathbf{c}_4 \cup \mathbf{a}_8$

Figure 12: The shapes of the second row of Fig. 10 with an added $\mathbf{a}_8$ pyramid

Shape $\mathbf{c}_4 \cup \mathbf{a}_8'^r$ of Fig. 12 is related to shape $\mathbf{a}_4 \cup \mathbf{a}_8'^r$ of Fig. 11: both compounds are unions of a basic ($\mathbf{c}_4$ or $\mathbf{a}_4$) shape and a rotated $\mathbf{a}_8$ pyramid (e.g., Corvey/United Kingdom, Westwerk).

If we use more elements, their relative heights and steepness obviously become even more significant. In case of shape $\mathbf{e}_4 \cap \mathbf{c}_4 \cup \mathbf{a}_8$ of Fig. 12, the slopes of the $\mathbf{e}_4$ and $\mathbf{c}_4$ elements are equal, while the slope of the $\mathbf{a}_8$ element ensures that its ridges exactly meet the valleys formed by the union of the previous two shapes[4].

If we cut off the gables of the previous spire, we get a shape similar to shape $\mathbf{a}_4 \cap \mathbf{c}_4 \cup \mathbf{a}_8$ of Fig. 12. This latter shape (e.g., Patrixbourne/France) has a more straightforward genealogy too: it's simply the union of an $\mathbf{a}_8$ pyramid and the $\mathbf{a}_4 \cap \mathbf{c}_4$ shape of Fig. 10 — which in itself is the intersection of two basic elements.

It is worth noting that — unlike in the figure — the slope of the $\mathbf{a}_4$ and $\mathbf{a}_8$ pyramids might as well be different, which results in a break (a horizontal intersection line) between the two shapes — probably it is the more common solution in the architectural practice. Because of this break, the general appearance of the shape strongly resembles the $\mathbf{a}_8 \cup \mathbf{a}_8'$ shape of Fig. 11 — which may be the reason, why these spire shapes appear together on the square and octagonal towers of the St. Martin Münster in Bonn.

3.4. A special shape

Can we say then, that no other spire shape exists? Obviously no. Firstly, our examination is restricted to spires having square and polygonal bases, but this doesn't mean that we can rule out other (rarely used) base shapes. Secondly, I did not depict every possible combination of the basic shapes — evidently there are variations not shown above (I've even mentioned one in 3.1). It is much harder to find a compound shape that cannot be produced as a combination

[3] It is also true for the $\mathbf{e}_4$ shape.

[4] These considerations are obviously not common in the architectural practice.

of the basic shapes summarised in Fig. 8. I will present one such example — but prior to that, we have to take a look at the roof forms having rectangular bases.

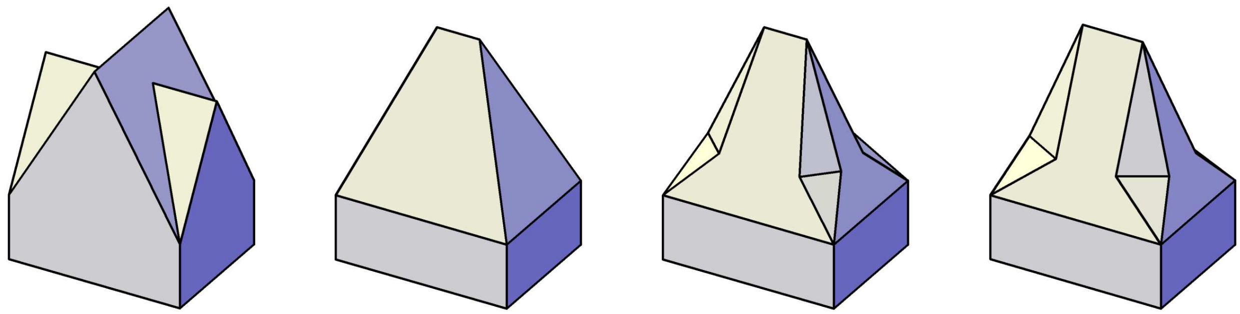


Figure 13: Adaptations of constructions to rectangular base (examples)

The first shape of Fig. 13 shows a possible method for the adaptation of a gabled form to a rectangular base: the height-separation of gable ridges. (The alternative would evidently be the use of different steepness of the sloping faces.) The second shape demonstrates the easiest and probably most aesthetic solution for adapting a pyramid to a rectangular base: simply cut it into two, then insert a horizontal ridge between the two separated apexes. Naturally, we might as well say that this form is a simple hipped roof — but the above logic can be applied to any shape without gables — as the third figure demonstrates it, adapting the $\mathbf{a}_4 \cap \mathbf{c}_4 \cup \mathbf{a}_8$ shape of Fig. 10.

At first glance the fourth shape of Fig. 13 (similar to the gate tower of the Charles Bridge, Prague) shows only insignificant variations compared to the previous shape: the only difference is that the ridges of the upper portion of its frontal sloping face are parallel. However — as it can be seen in Fig. 14 — this small change means that we have to use a hexagonal $\mathbf{a}_6$ pyramid as a base shape to ensure equal horizontal length for the sloping triangular and sideward pentagonal faces at the height of the horizontal breaks. Furthermore, in order to produce the above breaks, we need to change the other two elements of the compound also: instead of the two square-based pyramids of the “regular” $\mathbf{a}_4 \cap \mathbf{c}_4$ shape, this time we need pyramids having rectangular and rhomboidal bases.

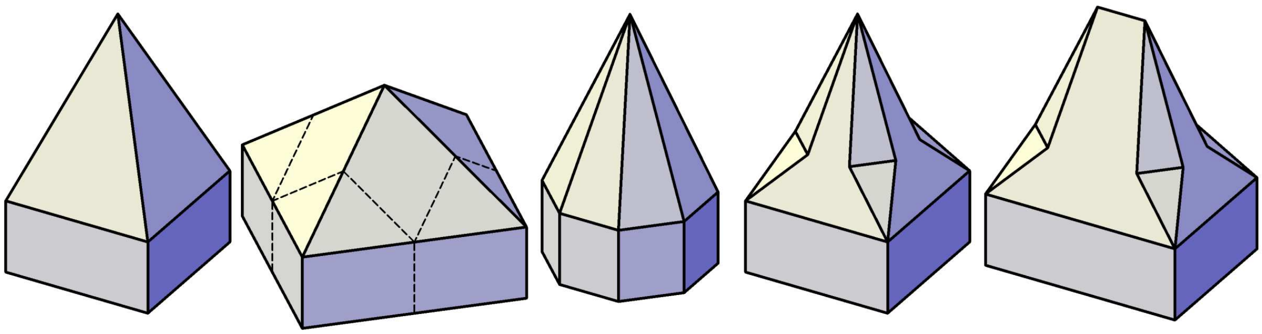
It is true that this shape lies outside the range of the delineated constructional method — however its rotational symmetry (unlike the spires above) is somewhat decreased.

3.5. Spire-shapes that were never built

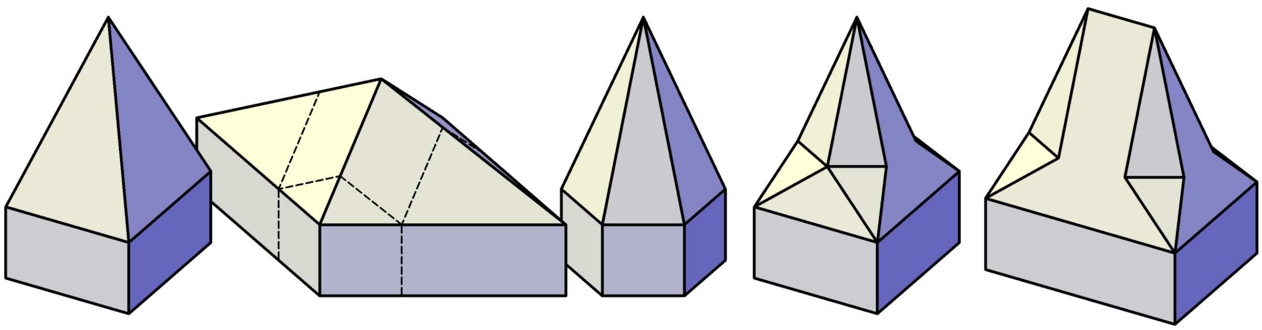
Since we do not (and cannot) have any proofs, only observations, we obviously cannot say with mathematical certainty that no spire has ever been built with a different construction method then the ones we already saw. Moreover, their theoretical possibility is definite, as it is demonstrated by the shapes of Fig. 15.

The octagonal element of shape $\mathbf{a}_4 \cup \mathbf{a}_8$ of Fig. 11 can be seen as the intersection of two $\mathbf{a}_4$ pyramids (one being in rotated position). If we change the steepness of one of these pyramids, it evidently modifies the shape of the intersection. In case of the first shape of Fig. 15 I set the slope so that the pentagonal faces have two parallel ridges.

The second shape of Fig. 15 is an intersection of a $\mathbf{c}_4$ shape and two $\mathbf{a}_4$ pyramids. The rhomboid faces are inherited from the $\mathbf{c}_4$ shape, the squares from the first $\mathbf{a}_4$ pyramid, and the equilateral triangles from the second, less steep one. Since the gable itself is a half hexagon, the edges all have equal length.

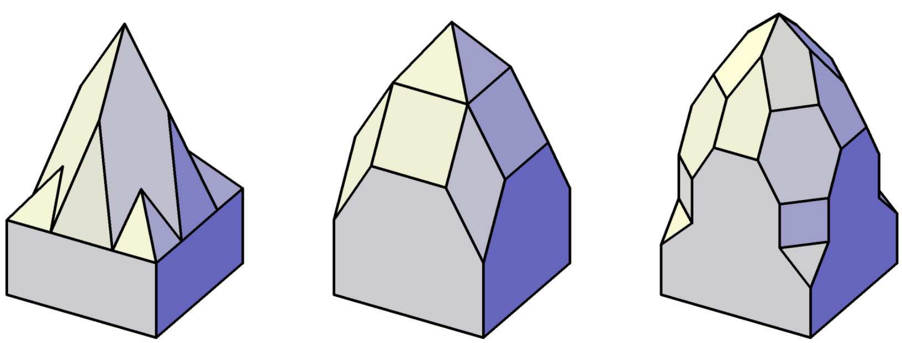


$\mathbf{a}_4$, $\mathbf{a}_4^r$ (or $\mathbf{c}_4$), and $\mathbf{a}_8$ base shapes, their $\mathbf{a}_4 \cap \mathbf{c}_4 \cup \mathbf{a}_8$ compound, and its extrusion



$\mathbf{a}_4^\delta$, $\mathbf{a}_4^\rho$, and $\mathbf{a}_6$ base shapes, their $\mathbf{a}_4^\delta \cap \mathbf{a}_4^\rho \cup \mathbf{a}_6$ compound, and its extrusion

Figure 14: Covering a rectangular base — different elements, similar results



$\mathbf{a}_4 \cap \mathbf{a}_4^r \cup \mathbf{a}_4'$ $\quad$ $\mathbf{c}_4 \cap \mathbf{a}_4 \cap \mathbf{a}_4'$ $\quad$ $\mathbf{s}_4$

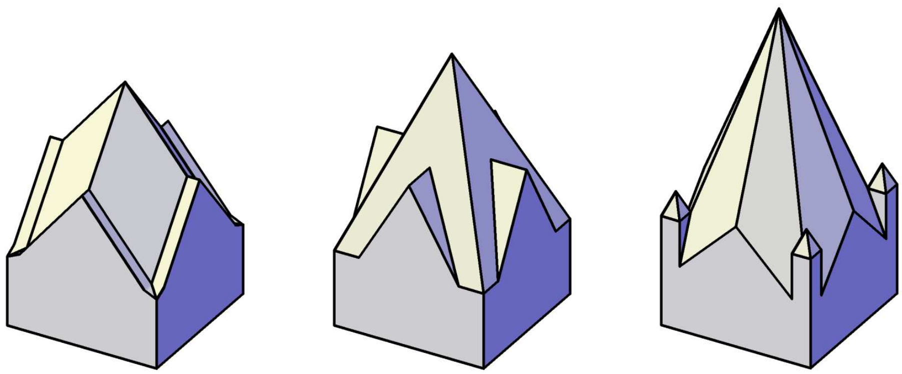
Figure 15: Spire-shapes that were never built

The third shape of Fig. 15 is the octagonal variation of the previous one, demonstrating that — despite of its symmetry — how easy to exceed the complexity limits of a credible medieval spire shape.

In my opinion the first two shapes prove that — though it's far from being easy — it is possible to find compound shapes that are simple, geometric, and aesthetic enough to be conceivably possible, while being completely outside the range of the historical shape set. At the same time they demonstrate, that even these shapes can be dealt with using the above construction methods.

3.6. Variations

Did we see every architecturally relevant form then? Certainly no. Though the domain of used forms seems to be practically closed, architects created a wide variety of shapes using this relatively narrow range of geometrical elements. In addition to the diversity produced by the variations of steepness and relative heights of the elements of compound shapes, architects sometimes deliberately and habitually violated the rules of pure geometrical constructions in order to produce more fascinating shapes. Fig. 16 shows some examples of these "limited irregularities".



$\mathbf{c}_4^v$ $\mathbf{e}_4 \cup \mathbf{a}_4^v$ $\mathbf{b}_4^v$

Figure 16: Architectural variations of geometrical constructions

Perhaps the simplest change is the use of gables as substantive elements (shape $\mathbf{c}_4^v$, e.g., Cathedral of Limburg/Germany). If the sloping faces of the $\mathbf{c}_4$ shape end not at the outer, but at the inner edges of the gables, the overall appearance resembles the $\mathbf{e}_4 \cup \mathbf{c}_4$ shape of Fig. 10, since the gable walls appear to be short gable roofs. The characteristic difference between the two shapes is in the direction of the valleys, which in this case run parallel to the facades.

Another frequently used solution is the divergence of base dimensions of elements. We have already observed the kind of freedom that can be derived from the difference of the base dimensions of the elements (for example in 3.2). The second shape of Fig. 16 ($\mathbf{e}_4 \cup \mathbf{a}_4^v$, e.g., Cathedral of Pécs/Hungary), can be seen as a simple truncation: if we elongate its edges downwards (and outwards), we get the "original" $\mathbf{e}_4 \cup \mathbf{a}_4$ shape of Fig. 10.

The third shape of Fig. 16 also represents this sort of height-irregularity (shape $\mathbf{b}_4^v$, e.g., Cathedral of Paderborn/Germany). The significant difference is that — unlike the $\mathbf{a}_4$ pyramid above — the $\mathbf{b}_4$ shape used here has got only one square horizontal section: its original base. Hence, when it is cut above its normal base, a new element is needed in order to cover the whole plan: a pinnacle. (When we decide the width of the gable, at the same time we set the width of the adjoining pinnacle, and the intersection of the inner vertical edge of the pinnacle and the diagonal ridge of the spire determines the minimum height of the pinnacle too.)

The introduction of the pinnacle does not effects the delineated geometric system since it uses the same shapes as the spires: its most typical form is probably the $\mathbf{e}_4 \cup \mathbf{a}_4$ shape of Fig. 10 — with steeply pointed $\mathbf{a}_4$ pyramids [1] — but we can also find more complicated forms like the $\mathbf{d}_8^*$ shape for example on Groß St. Martin in Cologne/Germany.

This architectural element sometimes appear on towers in some functional role (e.g., on watch-towers), sometimes as a geometrical necessity (like above), but most of the times its presence simply has architectural (mostly aesthetical) aim — it serves as a geometrical compositional element.

Figure 17: **Polyhedral spires** (examples)
Top: Bonn, St. Martin Münster; Corvey, abbey church; Cerisy-la-Forêt, St. Vigor
Middle: Patrixbourne, church; Prague, gate towers; Maria Laach, cloister church
Bottom: Lübeck, Marienkirche and Petrikirche; Limburg, cathedral; Cologne, St. Aposteln

4. Summary

I wouldn't say that the suggested analytic categorization method is appropriate for every theoretically imaginable roof shape, but it is certainly proved to be appropriate for the existing shapes of architectural practice — in fact, its range is rather too wide: we found some shapes that were never built, but none that were built, but cannot be dealt with. In my opinion the examples above convincingly certify that the method discussed in this paper is applicable for depicting, describing and categorizing the spires bounded by polygonal planes.

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Received November 30, 2006; final form June 22, 2007