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Classification and Projective Metric Realizations of Tile-Transitive Triangle Tilings

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Abstract. The *equivariance classification* of tile-transitive triangle tilings ($\mathcal{T}$, Γ) will be given by D-*symbols* (to honor of B. N. Delone (Delaunay) – M. S. Delaney – A. W. M. Dress) and we shall have 13 non-isomorphic D-*diagrams/graphs*. Each of them describes an infinite series depending on the possible D-*matrices*, i.e., the rotation orders of the vertex classes of Γ-equivalence. If we have 1, 2, 3 vertex classes, then so-called D-morphisms distinguish the representant tilings ($\mathcal{T}$, Γ=Aut $\mathcal{T}$) for Family 1, 2, 3, respectively. Each representant series has the maximal group Γ=Aut $\mathcal{T}$ as a triangle reflection group, thus we have easy criterion to decide the metric realizability on the sphere $\mathbf{S}^2$, in the Euclidean plane $\mathbf{E}^2$ and in the Bolyai-Lobachevskyan hyperbolic plane $\mathbf{H}^2$ just by the angle sum of the reflection triangle [7]. We shall show further modified realizabilities in the other two projective metric geometries as in the Minkowski plane $\mathbf{M}^2$ and in the Galilean (isotropic) plane $\mathbf{G}^2$ as well. To this the projective sphere $\mathcal{PS}^2$ uniformly models the above planes, endowed by a specific polarity (line→point mapping for orthogonality of lines). This polarity is called also projective metric by the classical analogies, however in a generalized sense here. In this paper we discuss the triangle reflecion groups for the maximal representant families, where the reflection lines of triangles are non-isotropic, i.e., a reflection line is not incident to its pole. The other cases will be discussed elsewhere. Computer helps us in realizing these tilings on the screen (see also [1]). The results are summarized in Table 1 and in Figure-series 4, moreover, in Theorems 2.1 and 6.1.

Key Words: triangle tiling, projective metrics, D-symbol

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1. An introductory example

We illustrate all our concepts and machineries on an example. This will be the tiling series $(\mathcal{T}, \Gamma_8\,(3u^+))$. In Fig. 1 and in Table 2 we indicate only the group series $\Gamma_8\,(3u^+)$, the 8^{th} member in Family 1.

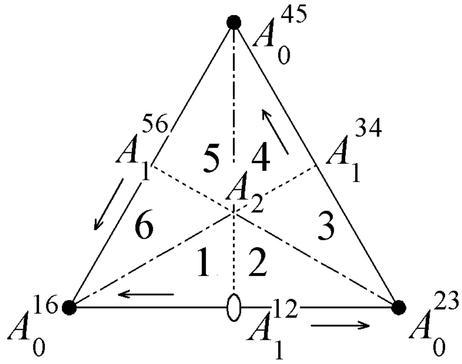


Figure 1: Tiling $(\mathcal{T}, \Gamma_8\,(3u^+))$ as typical example in Family 1

1.1. D-symbol

At $\Gamma_8\,(3u^+)$ a fundamental triangle $\mathcal{F}$ is depicted, where the formal barycentric subdivision by a 2-(dimensional tile) centre A_2, the 1-(edge) centres $A_1^{12}, A_1^{34}, A_1^{56}$ and the 0-(vertex) centres $A_0^{16}, A_0^{23}, A_0^{45}$ are partly indicated, together with the barycentric triangles $C_1 \leftrightarrow 1, \dots, C_6 \leftrightarrow 6$. For these triangles (simplices, in general) we introduce special side lines and adjacency operations: σ_0 is represented by dotted lines, σ_1 by dashed lines, and continuous lines refer to σ_2. ($I = \{0, 1, 2\}$ is the index set, $I = \{0, 1, \dots, d\}$ in general d dimensions, as follows). The operations

$$\sigma_0\colon\ (1,2)(3,4)(5,6) \qquad \sigma_1\colon\ (1,6)(2,3)(4,5)$$

as involutive permutations just describe the fundamental triangle $\mathcal{F}$ by its barycentric triangles. But the operation

$$\sigma_2\colon\ (1,2)(3,5)(4,6)$$

describes the action of any group in the series $\Gamma_8(3u^+)$ by the side pairings

$$\mathcal{I}\left(r\colon\ A_1^{12}A_0^{16} \leftrightarrow A_1^{12}A_0^{23},\ g\colon\ A_0^{23}A_0^{45} \to A_0^{45}A_0^{16},\ g^{-1}\colon\ A_0^{45}A_0^{16} \to A_0^{23}A_0^{45}\right)$$

as the generators of Γ. We imagine a combinatorial tiling $(\mathcal{T}, \Gamma)$ by the Γ-orbit of $\mathcal{F}$ and the Γ-orbits

$$D_x := C_x^{\Gamma} := (C_x^{\gamma}\colon\ \gamma \in \Gamma)\,,\ x = 1, \dots, 6$$

of the barycentric simplices C_x from the whole Γ-invariant barycentric subdivision $\mathcal{C}$ of $(\mathcal{T}, \Gamma)$.

The six simplex orbits form the D-set of $(\mathcal{T}, \Gamma)$:

$$\mathcal{D} = (D_1 \leftrightarrow 1, \dots, D_6 \leftrightarrow 6)$$

with the adjacency operations as free Coxeter group actions

$$\Sigma_I := \left(\sigma_0,\ \sigma_1,\ \sigma_2 - 1 = \sigma_0^2 = \sigma_1^2 = \sigma_2^2\right) \text{ on } \mathcal{C} \text{ and also on } \mathcal{D}$$

since it holds, as associativity laws:

$$(\sigma_i C_x)^\gamma = \sigma_i \left(C_x^\gamma\right) =: \sigma_i C_x^\gamma \quad \text{and, e.g., } (\sigma_j\sigma_i)\left(C_x^{\gamma\delta}\right) = \sigma_j \left(\sigma_i C_x^\gamma\right)^\delta$$

by definitions for any $\sigma_i, \sigma_j \in \Sigma_I$ and $\gamma, \delta \in \Gamma$.

Thus we obtain the D-*diagram/graph* $(\Sigma_I, \mathcal{D}) := \mathcal{D}$ describing also the topological surface $\mathcal{F}$ with the side pairing $\mathcal{I}(r,g)$ at the picture. The symmetric matrix function

$$r_{ij}\colon\ \mathcal{D} \to \mathbb{N}_{I\times I},\ D \mapsto r_{ij}(D);$$

$$r_{ij}(D) = \min\{r : (\sigma_j\sigma_i)^r D = D\} \quad i,j \in I = \{0,1,2\}$$

can be deduced from the adjacencies as follows

$$r_{ij}(D_1) = r_{ij}(D_2) = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & (3) \\ 1 & (3) & 1 \end{pmatrix}$$

$$r_{ij}(D_3) = r_{ij}(D_4) = r_{ij}(D_5) = r_{ij}(D_6) = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & (3) \\ 2 & (3) & 1 \end{pmatrix}.$$

Here, e.g., the (3) — in extra parantheses — refers to the coefficient 3 of u in the group symbol $\Gamma_8(3u^+)$ and refers to the triangle sequence (not illustrated here)

$$D_1 \overset{\sigma_2}{\to} D_2 \overset{\sigma_1}{\to} D_3 \overset{\sigma_2}{\to} D_5 \overset{\sigma_1}{\to} D_4 \overset{\sigma_2}{\to} D_6 \overset{\sigma_1}{\to} D_1$$

around the vertex A_0^{16} in the combinatorial tiling $(\mathcal{T}, \Gamma)$ just being fixed now by the so-called rotation order u^+ as a free natural parameter $(1 \le u)$. Here u^+ refers to the fact that the stabilizer of A_0^{16} is indeed a rotation subgroup of Γ (else dihedral subgroup occurs and then $+$ is missing).

For the concise formulation we introduce the so-called D-*matrix function*

$$m_{ij}\colon\ \mathcal{D} \to \mathbf{N}_{I\times I},\ D \mapsto m_{ij}(D);$$

$$m_{ij}(D) = \min\{m : (\sigma_j\sigma_i)^m C = C,\ C \in D \in \mathcal{D}\} \quad i,j \in I = \{0,1,2\}.$$

Now this will be

$$m_{ij}(D) = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3u \\ 2 & 3u & 1 \end{pmatrix},$$

constant for any $D \in \mathcal{D}$.

Moreover, we introduce the matrix function of rotation orders

$$v_{ij}(D) := m_{ij}(D)/r_{ij}(D) \in \mathbb{N}.$$

E.g., $v_{12}(D) = u$ is fixed for any $D \in \mathcal{D}$ so any $C \in \mathcal{C}$; and especially, $v_{02}(D_1) = v_{02}(D_2) = 2$ determines a half-turn centre at A_1^{12}.

Definition 1.1.1 *The* D-*diagram* $(\Sigma_I, \mathcal{D})$ *and the* D-*matrix function* $m_{ij}:\ \mathcal{D} \to \mathbb{N}_{I\times I}$, *together is called* D-*symbol, in general, if the following requirements are fulfilled:*

$$(\sigma_j\sigma_i)^{m_{ij}(D)}(D) = D,$$

thus in particular $m_{ii}(D) = 1, m_{ij}(D) = m_{ji}(D) = m_{ji}(\sigma_i D)$;

$$m_{ij}(D) = 2 \quad \text{if } 1 < j - i$$
$$m_{ij}(D) \ge 3 \quad \text{if } 1 = j - i$$

for any $D \in \mathcal{D}$ *and* $i < j \in I$.

We might allow $m_{01}(D) = 2$ if we allow digons (2-gons) as tile, or $m_{12}(D) = 2$ if vertex of valence 2 is permitted. Also $m_{01}(D) = \infty$ would be a case, if ∞-gon occur, moreover, $m_{12}(D) = \infty$ would also be possible if vertex of valence ∞ is allowed.

We extend our investigations to this last case, too. Then $m_{12}(D) = \infty = v_{ij}(D)$ means that infinitely many triangles meet at the vertex of any $C \in D$ in the tiling $(\mathcal{T}, \Gamma)$. Then the first relation of the definition does not hold (or with $m_{ij}(D) = 0$ holds trivially). Thus the extension $\mathbb{N}_\infty = \mathbb{N} \cup \infty$ (or $\mathbb{N}_0 = \mathbb{N} \cup 0$) plays some role in the general theory as we shall see later on ([5], [11]).

Namely, $(\mathcal{T}, \Gamma_8(3\infty^+))$ will be a hyperbolic tiling, where all the vertices lie at the absolute of $\mathbf{H}^2$. Any vertex is surrounded by infinitely many triangle tiles of $(\mathcal{T}, \Gamma_8(3\infty^+))$.

1.2. Equivariant tilings for a D-symbol

Our tiling $(\mathcal{T}, \Gamma_8(3u^+))$ to the corresponding D-symbol $(\mathcal{D}, m_{ij})$ is realizable for $u = 2$ in the Euclidean plane $\mathbf{E}^2$ with an equilateral fundamental triangle $\mathcal{F}$. One angle of $\mathcal{F}$ can be chosen with great freedom. This will be the situation on the sphere $\mathbf{S}^2$ for $u = 1$ and in $\mathbf{H}^2$ for $u \geq 3$.

Thus, we shall have a free parameter which characterizes different tilings with the same (isomorphic) D-diagram. The matrix function can still be different. But it is important, when these matrix functions are also the same.

Definition 1.2.1 *Two tilings* $(\mathcal{T}, \Gamma)$ *and* $(\mathcal{T}', \Gamma')$ *are called equivariant if there exists a combinatorial bijection* ϕ *that maps* $\mathcal{T}$ *onto* $\mathcal{T}'$*, preserving all incidences of tiles, edges, vertices, respectively, so that*

$$\Gamma' = \phi^{-1}\Gamma\phi.$$

The main observation of A. W. M. Dress [5] roughly expresses the connection, now in dimension $d = 2$.

Theorem 1.1 *To any "good" D-symbol* $(\mathcal{D}, m_{ij})$ *there exists a tiling* $(\mathcal{T}, \Gamma)$*, uniquely up to equivariance, either on the sphere* $\mathbf{S}^2$*, or in the Euclidean plane* $\mathbf{E}^2$*, or in the hyperbolic plane* $\mathbf{H}^2$ *if the so-called curvature of D-symbol*

$$\kappa(\mathcal{D}, m_{ij}) := \sum_{D \in \mathcal{D}} \left(\frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} - \frac{1}{2} \right)$$

is positive, equal to zero or negative, respectively.

Non-good D-symbols may occur on the sphere in four types where the stabilizer subgroup in Γ at opposite points of $\mathbf{S}^2$ (dictated by $(\mathcal{D}, m_{ij})$) would have different rotation orders. Then a geometric tiling obviously does not exist (although one could imagine non-good orbifolds, e.g., the "tear drop" as a sphere with a singular cone point of $\frac{2\pi}{p}$ angle neighbourhood, $2 \leq p \in \mathbb{N}$ (or $\mathbb{N}_\infty$, as well)).

We remark that for dimensions $3 \leq d$ an analogous theorem in not known and the problem seems to be very difficult [9, 2, 10, 8, 11].

In our example $(\mathcal{T}, \Gamma_8(3u^+)$ the curvature will be

$$\kappa = 6\left(\frac{1}{3} + \frac{1}{3u} - \frac{1}{2}\right) = \frac{1}{u}(2 - u).$$

We see, e.g., that $u = 2$ yields $\kappa = 0$, i.e., a Euclidean tiling, indeed.

1.3. Projective metric planes for realization

As a framework of our machinery we introduce the *projective sphere* $\mathcal{PS}^2(\mathbf{V}^3, \boldsymbol{V}_3, \mathbb{R})$. Here the basis vectors $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ will span the vertices of a fundamental triangle $\mathcal{F}$, $A_0 := A_0^{16}$, $A_1 := A_0^{23}$, $A_2 := A_0^{45}$. The vectors $\mathbf{x} = x^0\mathbf{a_0} + x^1\mathbf{a}_1 + x^2\mathbf{a}_2 = x^i\mathbf{a}_i$ in $\mathbf{V}^3$ will span the points of $\mathcal{PS}^2$ that models an appropriate metric plane from among the sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{E}^2$, the hyperbolic plane $\mathbf{H}^2$, the Minkowski plane $\mathbf{M}^2$ and the Galilean (isotropic) plane $\mathbf{G}^2$, all will be introduced later in more details. Vectors $\mathbf{x}$ and $c\mathbf{x}$ span the same point $X(\mathbf{x})$ of $\mathcal{PS}^2$ if $0 < c \in \mathbb{R}$. The null vector $\mathbf{0}$ does not represent any point.

The dual basis $\boldsymbol{a}^0, \boldsymbol{a}^1, \boldsymbol{a}^2$ in the dual form-space $\boldsymbol{V}_3$ with

$$\mathbf{a_i}\boldsymbol{a}^j = \delta_i^j \quad \text{the Kronecker delta}$$

will describe the side lines $\boldsymbol{a}^0(A_1, A_2)$, $\boldsymbol{a}^1(A_2, A_0)$, $\boldsymbol{a}^2(A_0, A_1)$ of our fundamental triangle $\mathcal{F}$. Forms $\boldsymbol{u} = \boldsymbol{a}^0u_0 + \boldsymbol{a}^1u_1 + \boldsymbol{a}^2u_2 = \boldsymbol{a}^ju_j \in \boldsymbol{V}_3$ will describe the lines of $\mathcal{PS}^2$. Forms $\boldsymbol{u}$ and $\boldsymbol{u}\frac{1}{c}$ characterize the same line, moreover half-sphere $u(\boldsymbol{u})$ of $\mathcal{PS}^2$ if $0 < c \in \mathbb{R}$. The null form $\mathbf{0}$ is excluded, too. The incidence of a point $X(\mathbf{x})$ and a line $u(\boldsymbol{u})$ is defined by

$$0 = \mathbf{x}\boldsymbol{u} = \left(x^i\mathbf{a}_i\right)\left(\boldsymbol{a}^ju_j\right) = x^i\delta_i^ju_j = x^iu_i$$

and $0 < \mathbf{x}\boldsymbol{u}$ stands for point $\mathbf{x}$ of the half-sphere $\boldsymbol{u}$.

The *projective plane* $\mathcal{P}^2(\mathbf{V}^3, \boldsymbol{V}_3, \mathbb{R})$ can be defined by identifying the opposite points (rays) $\mathbf{x}$ and $-\mathbf{x}$ and opposite half-spheres $\boldsymbol{u}$ and $-\boldsymbol{u}$ of $\mathcal{PS}^2$.

The *affine plane* is $\mathcal{A}^2 = \mathcal{P}^2 \setminus i_\infty$ where an ideal line of infinity is distinguished and left out together with its ideal points. The Cartesian coordinate triangle $E_0(\mathbf{e}_0)$, $E_1^\infty(\mathbf{e}_1)$, $E_2^\infty(\mathbf{e}_2)$ is canonized then by an origin $E_0(\mathbf{e}_0)$ as proper point of $\mathcal{A}^2$ and two ideal points $E_1^\infty(\mathbf{e}_1)$ and $E_2^\infty(\mathbf{e}_2)$, i.e., $i_\infty(\boldsymbol{e}^0)$. Then a unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2)$ fixes also the affine coordinates of a proper point

$$X\left(x^0, x^1, x^2\right) \sim \left(1,\ x := \frac{x^1}{x^0},\ y := \frac{x^2}{x^0}\right), \quad x^0 \neq 0.$$

Ideal points as affine point pairs (y^0, y^1, y^2), (z^0, z^1, z^2) or vectors in the plane $\mathbf{e}_1, \mathbf{e}_2$ will be described as

$$U\left(0, u^1, u^2\right) \sim \overrightarrow{YZ}\left(0,\ \frac{z^1}{z^0} - \frac{y^1}{y^0},\ \frac{z^2}{z^0} - \frac{y^2}{y^0}\right), \quad y^0 \neq 0 \neq z^0,$$

indicate. For a unified introduction to the projective metric planes $\mathbf{S}^2$, $\mathbf{E}^2$, $\mathbf{H}^2$, $\mathbf{G}^2$ and $\mathbf{M}^2$ in $\mathcal{PS}^2$ we define the *line pencil*, with holding point $(\boldsymbol{u}_*)$ as *pole*, orthogonal to the line $\boldsymbol{u}$, through a *projective polarity* given by $(_*)$ as a symmetric linear mapping

$$\begin{aligned}(_*)\quad & \boldsymbol{V}_3 \to \mathbf{V}^3,\ \boldsymbol{u} \mapsto \boldsymbol{u}_* \ \text{ by a matrix } \Pi^{ij} = \Pi^{ji},\\ & \text{as } \boldsymbol{V}_3 \ni \boldsymbol{e}^i \mapsto \boldsymbol{e}^i_* := \mathbf{e}^i := \Pi^{ij}\mathbf{e_j} \in \mathbf{V}^3 \text{ describes, } i,j = 0,1,2\,,\\ & u \sim \boldsymbol{u} = \boldsymbol{e}^iu_i \mapsto U \sim \boldsymbol{u}_* = \left(\boldsymbol{e}^iu_i\right)_* = u_i\boldsymbol{e}^i_* = u_i\Pi^{ij}\mathbf{e}_j.\end{aligned}$$

All these can be expressed in any (e.g., the canonical) basis $\mathbf{e}_j$, $j = 0, 1, 2$ of $\mathbf{V}^3$ and in its dual basis $\boldsymbol{e}^i$, $i = 0, 1, 2)$ of $\boldsymbol{V}_3$ where

$$\mathbf{e}_j\boldsymbol{e}^i = \delta_j^i \ \text{ is the Kronecker delta.}$$

The *symmetric bilinear form* or *scalar product*

$$\langle\, , \rangle\colon\ \boldsymbol{V}_3 \times \boldsymbol{V}_3 \to \mathbb{R},\ \langle \boldsymbol{u}, \boldsymbol{v}\rangle := \boldsymbol{u}_*\boldsymbol{v} = \boldsymbol{v}_*\boldsymbol{u} = \langle \boldsymbol{v}, \boldsymbol{u}\rangle$$

leads to an equivalent introduction. A polarity (or bilinear form) changes its matrix by

$$\Pi^{i'j'} := \left(\boldsymbol{e}^{i'}_*, \boldsymbol{e}^{j'}\right) = \left(\boldsymbol{e}^{i} e^{i'}_{i}\right)_* \left(\boldsymbol{e}^{j} e^{j'}_{j}\right) = \left(e^{i'}_{i}\Pi^{ir}\mathbf{e}_r\right)\left(\boldsymbol{e}^{j} e^{j'}_{j}\right) = e^{i'}_{i}\Pi^{ir}\delta^{j}_{r}e^{j'}_{j} = e^{i'}_{i}\Pi^{ij}e^{j'}_{j},$$

where the Einstein and Schouten index conventions were used. We remark that degenerated polarity or bilinear form is also allowed. Every polarity has a diagonal form in an appropriate dual basis pair, then its signature — by Sylvester's inertial law — can be defined, too:

- The **spherical plane $\mathbf{S}^2$** has its canonical polarity $\boldsymbol{e}^i_* = \Pi^{ij}\mathbf{e}_j$ by

$$({}_*)\colon\ \left(\boldsymbol{e}^0\ \boldsymbol{e}^1\ \boldsymbol{e}^2\right) \mapsto \begin{pmatrix} \boldsymbol{e}^0_* \\ \boldsymbol{e}^1_* \\ \boldsymbol{e}^2_* \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$$

of signature $(+ + +)$. Now the polarity $(*)$ has an inverse polarity

$$\begin{aligned} ({}^*)\colon\ & \mathbf{V}^3 \to \boldsymbol{V}_3 \text{ with matrix } \pi_{ij} = \pi_{ji}, \text{ in general, as} \\ & \mathbf{V}^3 \ni \mathbf{e_j} \ \mapsto\ \mathbf{e}^*_j = \boldsymbol{e}^i\pi_{ij} \in V_3 \text{ describes, with } \pi_{ij}\Pi^{jk} = \delta^k_i. \end{aligned}$$

Then a point $(\mathbf{x})$ as pole has its unique polar $((\mathbf{x}^*) = x)$ by

$$\mathbf{V}^3 \ni \mathbf{x} := x^j\mathbf{e}_j \ \mapsto\ \mathbf{x}^* = \mathbf{e}^*_j x^j = \boldsymbol{e}^i\pi_{ij}x^j =: \boldsymbol{e}^i x_i =: \boldsymbol{x} \in \boldsymbol{V}_3.$$

The spherical polarity has its characteristic property that no pole coincides with its polar, thus for any polar $\boldsymbol{u} = \boldsymbol{e}^i u_i \in \boldsymbol{V}_3$

$$0 < rr =: \boldsymbol{u}_*\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{u}\rangle = u_i\Pi^{ij}u_j = u_0u_0 + u_1u_1 + u_2u_2$$

holds. With $\boldsymbol{u}_* := \mathbf{u} = u^j\mathbf{e}_j$ as pole and the scalar product in $\mathbf{V}^3$ — which can be defined by $({}^*)$, as for $x^i\mathbf{e}_i = \mathbf{x}$, $y^j\mathbf{e}_j = \mathbf{y}$,

$$\langle \mathbf{x}, \mathbf{y}\rangle := \mathbf{x}\mathbf{y}^* = \left(x^i\mathbf{e}_i\right)\left(\mathbf{e}^*_j y^j\right) = x^i\pi_{ij}y^j$$

shows — the equation last but one can be written in the form

$$rr = \mathbf{u}\mathbf{u}^* = \langle \mathbf{u}, \mathbf{u}\rangle = u^i\pi_{ij}u^j = u^iu_i \text{ or now}$$

$$rr = u^0u_0 + u^1u_1 + u^2u_2 = u^0u^0 + u^1u^1 + u^2u^2.$$

Thus $\mathbf{S}^2$ can be modelled in the Euclidean space $\mathbf{E}^3\,(O, \mathbf{V}^3, \boldsymbol{V}_3)$ above by a usual sphere of radius $r > 0$.

We also remark that a polarity $({}_*)$ can be equivalently defined by another matrix $\overline{\Pi}^{ij}$, if

$$\overline{\Pi}^{ij} = c \cdot \Pi^{ij} = \Pi^{ij} \cdot c$$

with $0 < c \in \mathbb{R}$ for $\mathcal{PS}^2$ or $c \in \mathbb{R} \setminus \{0\}$ for $\mathcal{P}^2$. Based on this observation we can distinguish first 8, but finally only 5 projective metric planes, already mentioned above.

- For the **Euclidean plane** $\mathbf{E}^2$ the polarity $({}_*)$ and the scalar product will be degenerated and of signature $(0++)$, given by

$$\begin{pmatrix} \boldsymbol{e}_*^0 \\ \boldsymbol{e}_*^1 \\ \boldsymbol{e}_*^2 \end{pmatrix} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Thus any line $u = e^i u_i$ has its pole $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ as non-incident ideal point on the ideal line

$$i_\infty\left(e^0\right), \text{ i.e., with equation } x^0 = 0 \text{ for its points } \mathbf{x} = x^i\mathbf{e}_i.$$

- For the **hyperbolic plane** $\mathbf{H}^2$ the polarity $({}_*)$ and the scalar product will be indefinite of signature $(-++)$, given by

$$\begin{pmatrix} \boldsymbol{e}_*^0 \\ \boldsymbol{e}_*^1 \\ \boldsymbol{e}_*^2 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

The polarity is obviously invertible, again. The formula

$$\boldsymbol{u}_*\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \mathbf{u}\mathbf{u}^* = \langle \mathbf{u}, \mathbf{u} \rangle$$

describes the polars $u\,(\boldsymbol{u})$ and their poles $U(\mathbf{u})$ in the following way:

 - if it is positive then u is proper line, U is exterior point,
 - if negative then u is exterior line, U is proper point,
 - if equals zero then u touches the absolute quadratics (circle) in the absolute point U.

That means the proper points $\mathbf{u} = u^j\mathbf{e}_j$ and exterior lines $u = \boldsymbol{e}^i u_i$ satisfy

$$\begin{aligned} 0 > -kk = \langle \mathbf{u}, \mathbf{u} \rangle = \langle u, u \rangle = u^i \pi_{ij} u^j = u_i \Pi^{ij} u_j = \\ = -u^0u^0 + u^1u^1 + u^2u^2 = -u_0u_0 + u_1u_1 + u_2u_2. \end{aligned}$$

This characterizes just the points of a two parted hyperboloid in $\mathbf{E}^3\,(O, \mathbf{V}^3, \boldsymbol{V}_3)$ as a pseudosphere (of imaginary radius k). The absolute points $y^i\mathbf{e}_i = \mathbf{y} \in \mathbf{V}^3$ satisfy the equation

$$0 = \langle \mathbf{y}, \mathbf{y} \rangle = y^i \pi_{ij} y^j = -y^0y^0 + y^1y^1 + y^2y^2.$$

It has some sense to introduce outer or exterior hyperbolic plane $\overline{\mathbf{H}}^2 = \mathcal{PS} \setminus \mathcal{D}\mathbf{H}^2$, i.e., the complement of double hyperbolic plane on the projective sphere (see, e.g., Fig. 4 2. b) $\mathbf{H}^2 ii$)). This will be understood into generalized hyperbolic plane $\mathbf{H}^2$ modelled on $\mathcal{PS}^2$ endowed with projective metric polarity of signature $(-++)$ above.

- For the **Minkowski plane** $\mathbf{M}^2$ the polarity $({}_*)$ and the scalar product will be degenerated again and of signature $(0-+)$, given by

$$\begin{pmatrix} \boldsymbol{e}_*^0 \\ \boldsymbol{e}_*^1 \\ \boldsymbol{e}_*^2 \end{pmatrix} = \begin{pmatrix} 0 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Thus the parallel lines $\boldsymbol{e}^0u_0 + \boldsymbol{e}^1(+1) + \boldsymbol{e}^2(+1)$ all coincide with their ideal pole $I_1^\infty\,(0\,\mathbf{e}_0 + (-1)\,\mathbf{e}_1 + (+1)\,\mathbf{e}_2)$, and so do the parallel lines $\boldsymbol{e}^0v_0 + \boldsymbol{e}^1\,(-1) + \boldsymbol{e}^2\,(+1)$ with their ideal pole $I_2^\infty\,(0\,\mathbf{e}_0 + (+1)\,\mathbf{e}_1 + (+1)\,\mathbf{e}_2)$.

We say that the absolute line quadratics of $\mathbf{M}^2$ degenerates

$$0 = -u_1 u_1 + u_2 u_2 = (-u_1 + u_2)(u_1 + u_2)$$

into two line pencils with holding points $I_1^\infty(0,-1,1)$ and $I_2^\infty(0,1,1)$, respectively. These lines are the light-lines of the special relativity. Timelike vectors (the endpoints as events can be achieved from the starting point (event) by real motion) $\mathbf{x}$ has coordinates $(0,x^1,x^2)$ with $0 > -x^1x^1 + x^2x^2$. Spacelike vectors (whose endpoints can be events with the same time coordinate) $\mathbf{y}\,(0,y^1,y^2)$ satisfy $0 < -y^1y^1 + y^2y^2$.

- For the **Galilean** (isotropic) **plane** $\mathbf{G}^2$ the polarity $(_*)$ and scalar product will be doubly degenerated of signature (00+), given by

$$\begin{pmatrix} \boldsymbol{e}_*^0 \\ \boldsymbol{e}_*^1 \\ \boldsymbol{e}_*^2 \end{pmatrix} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Any line $\boldsymbol{u} = \boldsymbol{e}^i u_i$ has its pole $\mathbf{u} = u_2\mathbf{e}_2$, i.e., the unique absolute point $U^\infty(\mathbf{e}_2)$ in this $\mathcal{P}^2$-model. We say that the absolute line quadratics of $\mathbf{G}^2$ doubly degenetates:

$$0 = u_2 u_2$$

to the line pencil with ideal carrier point $U^\infty(0,0,1)$.
Any of these lines $u = e^0u_0 + e^1u_1$, i.e., with equation $0 = x^0u_0 + x^1u_1$, or in affine coordinates $0 = u_0 + xu_1$ characterizes points (x,y) with the same Galilean time coordinate x. Then we have unified or absolute *Galilean world time*.

2. Triangle reflection groups on the projective metric sphere

As we have reported by [1], [7] and illustrated in Fig. 1 and in Table 1, we can metrically realize all the 13 parametrized tiling series, if we realize the 3 triangle reflection groups, describing the 3 representant maximal tilings ($\mathcal{T}$, $\Gamma =$ Aut Γ) for families 1, 2, 3, respectively.

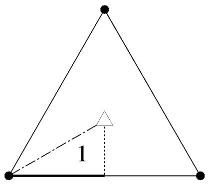
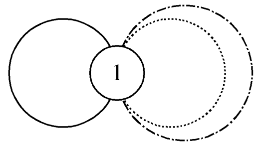
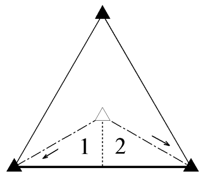
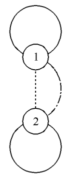
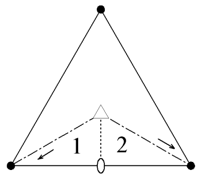
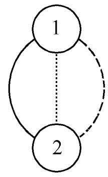
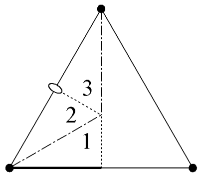
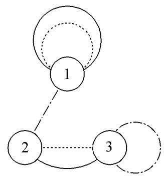
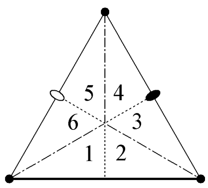
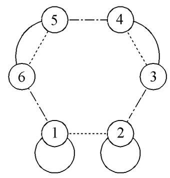
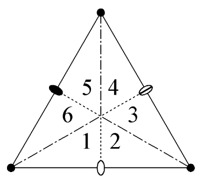
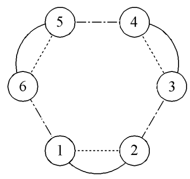
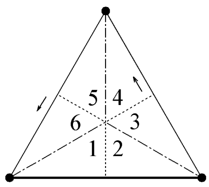
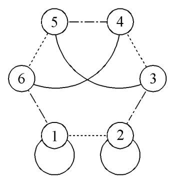
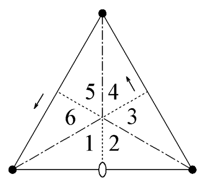
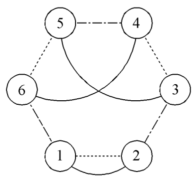
Here we only mention the concept of tiling *family* in the analogy of Definition 1.2.1, now by surjective (homo)morphism of $\mathcal{D}$-morphic images, where $\Gamma' = \mathrm{Aut}\,\mathcal{T}'$ (see, e.g., [11]), and (homo)morphism leaves all the adjacencies Σ_I and the matrix function m_{ij} invariant.

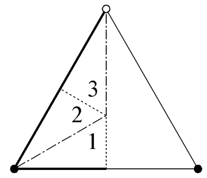
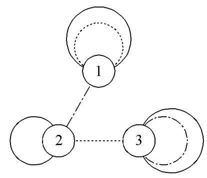
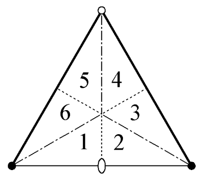
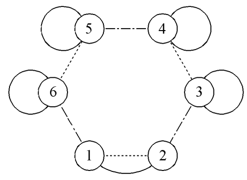
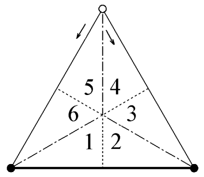
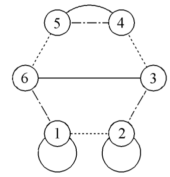
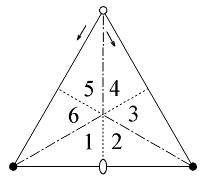
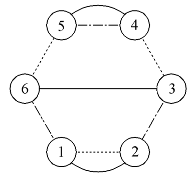
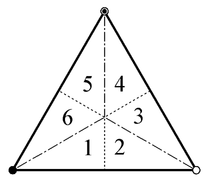
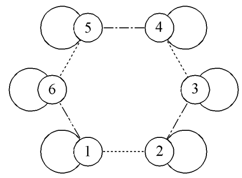
For Familiy 1 with 8 series the group is ${}_6^{3m}\Gamma(u)$, $3 \leq u \in \mathbb{N}^\infty$ where all the vertices are Γ-equivalent. Then the tile stabilizer (indicated by $3m$) is generated by 2 line reflections, meeting with angle $\frac{\pi}{3}$ in the centre of the triangle. Another angle of the barycentric triangle is rectangle, the characteristic third angle is $\frac{\pi}{u}$. We see that the adjacency parameter $u = 3$ yields a spherical barycentric triangle and the regular tetrahedron tiling on the sphere $\mathbf{S}^2$, consisting of 4 triangle tiles. Similarly $u = 4, 5$ lead to $\mathbf{S}^2$-tilings; the octahedron and icosahedron tilings, respectively.

The parameter $u = 6$, however, leads to the regular triangle tiling ($\mathcal{T}$, ${}_6^{3m}\Gamma(6) = \mathbf{p6m} = \mathbf{236}$) in the Euclidean plane $\mathbf{E^2}$ where $u = 6$ triangles meet in any vertex of $\mathcal{T}$.

We can see in the Cayley-Klein model (e.g.) of the Bolyai-Lobachevskian hyperbolic plane $\mathbf{H^2}$, that $7 \leq u$ — by a barycentric triangle of $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{u}$ angles — lead to hyperbolic triangle tilings. Moreover, $u = \infty$ has also meaning. Then the barycentric triangle has angles $\frac{\pi}{3}$, $\frac{\pi}{2}$ as before, but at the last vertex we have either hyperbolic parallel sides (of angle 0), or sides meeting in outer point of the CK-model. Then these sides have a common perpendicular line and an arbitrary distance along this line.

Table 1: Table for the 13 triangle tiling series in 3 Families

Family 1		
${}^{3m}_{6}\Gamma\,(u)$ $3 \le u \in \mathbb{N}_\infty$		
${}^{3}_{3}\Gamma_1\,(2u)$ $2 \le u \in \mathbb{N}_\infty$		
${}^{3}_{3}\Gamma_2\,(u^+)$ $3 \le u \in \mathbb{N}_\infty$		
${}^{m}_{2}\Gamma_2\,(3u)$ $1 \le u \in \mathbb{N}_\infty$		
$\Gamma_3\,(6u)$ $1 \le u \in \mathbb{N}_\infty$		
$\Gamma_4\,(3u^+)$ $1 \le u \in \mathbb{N}_\infty$		
$\Gamma_6\,(6u)$ $1 \le u \in \mathbb{N}_\infty$		
$\Gamma_8\,(3u^+)$ $1 \le u \in \mathbb{N}_\infty$		

Family 2		
${}^{m}_{2}\Gamma_1\,(2u;v)$ $2 \le u;\ 3 \le u;\ 2u \ne v$		
$\Gamma_2\,(4u;2v)$ $1 \le u;\ 2 \le u;\ 4u \ne 2v$		
$\Gamma_5\,(4u;v^+)$ $1 \le u;\ 3 \le u;\ 4u \ne v$		
$\Gamma_7\,(2u^+;v^+)$ $2 \le u;\ 3 \le u;\ 2u \ne v$		
Family 3		
$\Gamma_1\,(2u;2v;2w)$ $2 \le u < v < w$		

All these $u = \infty$ cases can be realized, as we shall see. Our example triangle tilings ($\mathcal{T}$, $\Gamma_8\,(3u)$) $u \ge 1$ can also be realized by so-called symmetry breaking of the former tiling ($\mathcal{T}$, ${}^{3m}_{6}\Gamma\,(\overline{u} = 3u)$). However, the metric realization is not unique. The possible quasi-conforme deformations can also be described, not detailed more here.

In Family 2 we have 4 series with two adjacency parameters u and v for the two Γ-equivalence classes of triangle vertices. The representant maximal tiling ($\mathcal{T}$, ${}^{m}_{2}\Gamma\,(2u;v)$) has a reflectional fundamental domain consisting of 3 barycentric triangles amounting a reflection triangle, now with "angles" $\frac{\pi}{2}$, $\frac{\pi}{u}$, $\frac{\pi}{v}$; $2 \le u \in \mathbb{N}^\infty$, $3 \le v \in \mathbb{N}^\infty$ ($2u \ne v$) in a generalized sense.

For example, ($\mathcal{T}$, ${}^{m}_{2}\Gamma\,(2 \cdot 3; 6)$), i.e., $u = 3$, $v = 6$ lead to a Euclidean tiling, again with regular triangles, but with marks on them to break the complete automorphism group that would be as above ($\mathcal{T}$, ${}^{3m}_{6}\Gamma\,(\overline{u} = 2 \cdot 3)$). In this sense the triangle tiling ($\mathcal{T}$, ${}^{m}_{2}\Gamma\,(2 \cdot 3; 6)$) belongs to Family 1; so as in all cases $2u = v$. That means $2u \ne v$ is a proper condition for Family 2. E.g., $u = 2$, $v = 3$ lead to an $\mathbf{S^2}$-tiling, properly lying in Family 2. We could introduce $u = \infty$ or $v = \infty$ in $\mathbf{H}^2$ in an appropriate way.

The Family 3 is represented by a reflection triangle and group $\Gamma_1\,(2u, 2v, 2w)$ $2 \le u < v < w$. Equal parameters yield richer automorphism group now. E.g., $w = \infty$ can have multiple meaning, again. Dihedral angles $\frac{\pi}{u}$, $\frac{\pi}{v}$, $\frac{\pi}{w}$ are classical for a reflection triangle.

For summarizing the triangle tiling classification ([7], [1]) by D-symbol isomorphism, and D-morphism we formulate

Theorem 2.1 *We have 13 equivariance class series of plane triangle tilings up to isomorphism of D-diagrams, parametrized by D-matrix functions as our Table 1 shows. These are ordered into 3 Families by D-morphism, depending on equivalence of the triangle vertices.*

Of course, "dihedral angles" above should have also a combinatorial meaning, namely, the order (periodicity) of the product of reflections in the two sides meeting at the vertex considered. The rotation of a finite order is only a special case. Parallel sides in $\mathbf{E}^2$ or $\mathbf{H}^2$ yield a product of infinite order, so as other cases as well in $\mathbf{H}^2$, $\mathbf{M}^2$, $\mathbf{G}^2$, suprisingly may be.

3. Groups generated by 3 involutive central axial collineations

Let us introduce 3 line reflections ${}^0M$, ${}^1M$, ${}^2M$ in the coordinate lines a^0, a^1, a^2, respectively, as involutive "central axial collineations" of $\mathcal{PS}^2$ onto itself. Each of them, e.g., ${}^0M$ is linear mapping of points and lines, preserving incidence, by the dual basis pair:

$$ {}^0M\colon \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} \to \begin{pmatrix} -1 & n & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} =: {}^0M_i^j \mathbf{a}_j $$

or consequently by

$$ \left(\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2\right) \to \left(\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2\right) \begin{pmatrix} -1 & n & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1} $$

and similarly, with the matrices only

$$ {}^1M\colon \begin{pmatrix} 1 & 0 & 0 \\ r & -1 & q \\ 0 & 0 & 1 \end{pmatrix}, \quad {}^2M\colon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & t & -1 \end{pmatrix}. $$

Indeed, we look at ${}^0M$, in formula (3.1), e.g., that the line of $(\mathbf{a}_1) =: A_1$ and $(\mathbf{a}_2) =: A_2$, i.e., $a^0 = (\boldsymbol{a}^0)$, is pointwise fixed (Fig. 2). The product ${}^0M\,{}^0M$, by row-column multiplication,

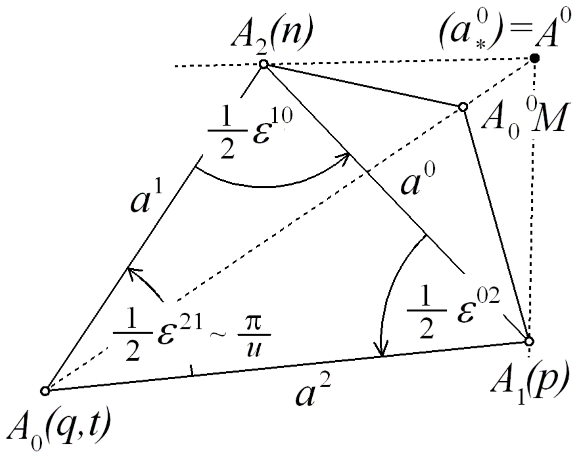


Figure 2: Line reflection ${}^0M$ in axis a^0 with centre A^0

is the identity matrix $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. This involutivity guarantees at the same time that the mapping ${}^0M$ preserves incidence of points and lines. We see, however, that $(\boldsymbol{a}^0)$ is mapped onto $(-\boldsymbol{a}^0)$, i.e., the half-planes (half-spheres) of a^0 changes under ${}^0M$. Looking for fixed lines $(\boldsymbol{a}^i f_i)$ we get by the linearity of ${}^0M$

$$\begin{aligned}(\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2)\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} \to\ & (\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2)\begin{pmatrix} -1 & n & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \\ & = (\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2)\begin{pmatrix} -f_0 + nf_1 + pf_2 \\ f_1 \\ f_2 \end{pmatrix}, \\ & \text{i.e., } 0 = -2f_0 + nf_1 + pf_2. \end{aligned} \tag{3.2}$$

This means, fixed lines coincide with the point

$$A^0 = (-2\mathbf{a}_0 + n\mathbf{a}_1 + p\mathbf{a}_2) \sim \left(-1,\ \frac{n}{2},\ \frac{p}{2}\right)$$

and so with $\left(1, -\frac{n}{2}, -\frac{p}{2}\right)$, too. The point pair above is the "centre pair" of the mapping ${}^0M$ (it is a central axial collineation of $\mathcal{PS}^2$, indeed). We see that ${}^0M$ changes the two centres, and it commutes with the central inversion $\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ of $\mathcal{PS}^2$, in general. This just means that ${}^0M$ is a projective collineation of $\mathcal{P}^2$, as well.

All these can be visualized in the "double" affine or Euclidean plane where an ideal line $i = (\boldsymbol{i} = \boldsymbol{e}^0)$ at infinity and a Cartesian (orthonormal) homogeneous coordinate system

$$E_0\left(\mathbf{e}_0\right),\ E_1^\infty\left(\mathbf{e}_1\right),\ E_2^\infty\left(\mathbf{e}_2\right) \sim e^0\left(\boldsymbol{e}^0\right),\ e^1\left(\boldsymbol{e}^1\right),\ e^2\left(\boldsymbol{e}^2\right) \tag{3.3}$$

is distinguished for the computer screen (see Section 1.3). *Later on, by stereographic projection (and with a 4-dimensional machinery), we could give also conforme pictures where the lines will be Euclidean circles or lines, and only one ideal point ∞ at infinity will be distinguished* ([6, 1]).

Of course, the matrix of ${}^0M$ is determined up to a positive constant factor for $\mathcal{PS}^2$, moreover, a basis vector change, e.g.,

$$\mathbf{a}_0 \to \mathbf{a}_{0'} = c \cdot \mathbf{a}_0, \quad \mathbf{a}_1 \to \mathbf{a}_{1'} = \mathbf{a}_1, \quad \mathbf{a}_2 \to \mathbf{a}_{2'} = \mathbf{a}_2$$

for any $c \in \mathbb{R} \setminus \{0\}$, yields the same mappings ${}^0M$, ${}^1M$, ${}^2M$ with other matrices

$$\begin{aligned} {}^0M\colon & \begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \end{pmatrix} \to \begin{pmatrix} -1 & n \cdot c & p \cdot c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \end{pmatrix}, \\ {}^1M\colon & \begin{pmatrix} 1 & 0 & 0 \\ \frac{r}{c} & -1 & q \\ 0 & 0 & 1 \end{pmatrix}, \quad {}^2M\colon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{s}{c} & t & -1 \end{pmatrix} \end{aligned} \tag{3.4}$$

This is just the projective freedom, mentioned as advantage of our treatment in this paper. Thus we can achieve some equalities for some parameters, while we change the unit point (and *unit line as well) of our coordinate triangle.* Of course, a general coordinate transformation by an inverse matrix pair $(e^i_{i'})$, $\left(E^j_{j'}\right) = (e^i_{i'})^{-1}$, i.e., by

$$\begin{pmatrix} \mathbf{e}_{0'} \\ \mathbf{e}_{1'} \\ \mathbf{e}_{2'} \end{pmatrix} = \begin{pmatrix} e^0_{0'} & e^1_{0'} & e^2_{0'} \\ e^0_{1'} & e^1_{1'} & e^2_{1'} \\ e^0_{2'} & e^1_{2'} & e^2_{2'} \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{e}^{0'} & \boldsymbol{e}^{1'} & \boldsymbol{e}^{2'} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}^0 & \boldsymbol{a}^1 & \boldsymbol{a}^2 \end{pmatrix} \begin{pmatrix} E^{0'}_0 & E^{1'}_0 & E^{2'}_0 \\ E^{0'}_1 & E^{1'}_1 & E^{2'}_1 \\ E^{0'}_2 & E^{1'}_2 & E^{2'}_2 \end{pmatrix}$$

yields a new matrix $\left({}^zM^{k'}_{i'}\right)$ instead of $\left({}^zM^j_i\right)$, $z = 0, 1, 2$, by conjugacy:

$$\begin{aligned} \mathbf{a}_i &\to \left({}^zM^j_i\right)\mathbf{a}_j \Rightarrow e^i_{i'}\mathbf{a}_i \to e^i_{i'}\left({}^zM^j_i\right)E^{k'}_i\mathbf{e}_{k'} \Rightarrow \mathbf{e}_{i'} \to e^i_{i'}\left({}^zM^j_i\right)E^{k'}_i\mathbf{e}_{k'} \Rightarrow \\ &\Rightarrow \left({}^zM^{k'}_{i'}\right) = e^i_{i'}\left({}^zM^j_i\right)E^{k'}_i \end{aligned} \tag{3.5}$$

In our computer screen the basis $\{\mathbf{e}_{0'}, \mathbf{e}_{1'}, \mathbf{e}_{2'}\}$ is fixed, $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$ can be chosen by $\mathbf{a}_i = E^{k'}_i\mathbf{e}_{k'}$. Thus everything can be visualized by our computer.

4. Invariant polarity for the triangle group

We examine whether a line → point assignement exists by a symmetric linear mapping

$$\begin{aligned} &({}_*):\ \ (\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2) \to \begin{pmatrix} \boldsymbol{a}^0_* \\ \boldsymbol{a}^1_* \\ \boldsymbol{a}^2_* \end{pmatrix} =: \begin{pmatrix} \mathbf{a}^0 \\ \mathbf{a}^1 \\ \mathbf{a}^2 \end{pmatrix} = \begin{pmatrix} \Pi^{00} & \Pi^{01} & \Pi^{02} \\ \Pi^{01} & \Pi^{11} & \Pi^{12} \\ \Pi^{02} & \Pi^{12} & \Pi^{22} \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} \\ &\text{i.e.,}\ \ \boldsymbol{a}^i \to \boldsymbol{a}^i_* = \mathbf{a}^i = \Pi^{ij}\mathbf{a}_j,\ \Pi^{ij} = \Pi^{ji} \end{aligned} \tag{4.1}$$

which is invariant under each reflection ${}^zM$, $z = 0, 1, 2$, in formulas (3.1). As we have seen earlier, $({}_*)$ equivalently defines a symmetric $\mathbb{R}$-valued bilinear form, i.e., scalar product as follows

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}_* \boldsymbol{v} \rangle = \langle \boldsymbol{v}_* \boldsymbol{u} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle \in \mathbb{R} \tag{4.2}$$

and the orthogonality of lines

$$u \perp v \text{ by } \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0,\ (\boldsymbol{u}_*)\, I\, (\boldsymbol{v}),\ (\boldsymbol{v}_*)\, I\, (\boldsymbol{u}),\ \text{etc.}$$

So we get $\mathcal{PS}^2\ (\mathbb{R}, \langle\,,\rangle)$ as a projective metric sphere, depending on the signature of $\langle\,,\rangle$ as indicated in Section 1.3.

We have to determine the polarity Π^{ij} just by the matrices $\left({}^zM^j_i\right)$, $z = 0, 1, 2$.

We further require that every reflection line does not coincide with its pole, i.e., every line a^i is not isotropic. Moreover, we require that our group generated by ${}^0M$, ${}^1M$, ${}^2M$ acts discretely (i.e., without gap and overlap) with the triangle fundamental domain $A_0A_1A_2\,(A)$ in a domain of $\mathcal{PS}^2$ as in a model of $\mathbf{S}^2$, $\mathbf{H}^2$, $\mathbf{E}^2$, $\mathbf{M}^2$, $\mathbf{G}^2$, as reported in Section 1.3.

Let us consider ${}^zM$, $z = 0, 1, 2$, with its matrix $\left({}^zM^j_i\right)$ in (3.1). By our requirements above we get a general formula

$$\begin{aligned} \langle \boldsymbol{a}^i, \boldsymbol{a}^j \rangle &= (\boldsymbol{a}^i_*\boldsymbol{a}^j) = \left(\Pi^{ik}\mathbf{a}_k\boldsymbol{a}^j\right) = \Pi^{ik}\delta^j_k = \Pi^{ij} \sim \langle \boldsymbol{a}^r\,({}^zM^i_r)\,, \boldsymbol{a}^s\,({}^zM^j_s) \rangle = \\ &= [(\mathbf{a}^r\,({}^zM^i_r))_*\,\boldsymbol{a}^s\,({}^zM^j_s)] = \left[{}^zM^i_r\Pi^{rk}\mathbf{a}_k\boldsymbol{a}^s\,({}^zM^j_s)\right] \\ &= ({}^zM^i_r)\,\Pi^{rk}\delta^s_k\,({}^zM^j_s) = ({}^zM^i_r)\,\Pi^{rs}\,({}^zM^j_s) = {}^zc\,\Pi^{ij} \end{aligned} \tag{4.3}$$

as homogeneous equations for the six components of Π^{ij} in (4.1). In the last term ${}^{z}c$ expresses the projective (multiplicative) freedom, ${}^{z}c$ $(z = 0, 1, 2)$ denotes three independent constants in principle. By the involutivity of ${}^{z}M$, however, it follows ${}^{z}c = \pm 1$. Consider concretely ${}^{0}M$, first, in (3.1)

$$\begin{aligned} {}^{0}c \cdot \Pi^{00} &= (-1)\,\Pi^{00}\,(-1) \\ {}^{0}c \cdot \Pi^{01} &= (-1)\,\Pi^{00}\,(n) + (-1)\,\Pi^{01}\,(1) \\ {}^{0}c \cdot \Pi^{02} &= (-1)\,\Pi^{00}\,(p) + (-1)\,\Pi^{02}\,(1) \\ {}^{0}c \cdot \Pi^{11} &= (n)\,\Pi^{00}\,(n) + 2\,(n)\,\Pi^{01}\,(1) + (1)\,\Pi^{11}\,(1) \\ {}^{0}c \cdot \Pi^{12} &= (n)\,\Pi^{00}\,(p) + (1)\,\Pi^{01}\,(p) + (n)\,\Pi^{02}\,(1) + (1)\,\Pi^{12}\,(1) \\ {}^{0}c \cdot \Pi^{22} &= (p)\,\Pi^{00}\,(p) + 2\,(p)\,\Pi^{02}\,(1) + (1)\,\Pi^{22}\,(1) \end{aligned} \tag{4.4}$$

Finally we shall have 4 cases, by logical symmetry, for $({}^{0}c, {}^{1}c, {}^{2}c) = (1, 1, 1), (1, 1, -1), (-1, -1, 1), (-1, -1, -1)$. *The complete discussion will be left to a forthcoming work.*

Now, our requirement for nonisotropic reflection line yields that only $({}^{0}c, {}^{1}c, {}^{2}c) = (1, 1, 1)$ is to be considered. Namely, ${}^{0}c = 1$ gives in (4.4) only two essential equations

$${}^{0}c = 1: \;\; \Pi^{01} + \frac{n}{2}\Pi^{00} = 0, \quad \Pi^{02} + \frac{p}{2}\Pi^{00} = 0. \tag{4.5}$$

For ${}^{0}c = -1$ (4.4) will be a little bit complicated:

$${}^{0}c = -1: \;\; \Pi^{00} = 0, \;\; \Pi^{11} = -n\Pi^{01} \;\; \Pi^{12} = -\frac{p}{2}\Pi^{01} - \frac{n}{2}\Pi^{02}, \;\; \Pi^{22} = -p\Pi^{02}. \tag{4.6}$$

In (4.6) $0 = \Pi^{00} = \langle \boldsymbol{a}^0, \boldsymbol{a}^0 \rangle$ means that the reflection line a^0 would be isotropic that is excluded in classical geometries, therefore in our present requirements, as well. Thus only $({}^{0}c, {}^{1}c, {}^{2}c) = (1, 1, 1)$ has to be discussed in the following. Similarly to (4.5) we get 6 homogeneous linear equations for Π^{01}, Π^{02}, Π^{12}, Π^{00}, Π^{11}, Π^{22} as follows in short form

$$\begin{aligned} \Pi^{01} &= -\tfrac{n}{2}\Pi^{00} = -\tfrac{r}{2}\Pi^{11} \\ \Pi^{02} &= -\tfrac{p}{2}\Pi^{00} = -\tfrac{s}{2}\Pi^{22} \\ \Pi^{12} &= -\tfrac{q}{2}\Pi^{11} = -\tfrac{t}{2}\Pi^{22}. \end{aligned} \tag{4.7}$$

The usual 6×6 determinant criterion guarantees non-trivial solution, iff

$$nqs - prt = 0. \tag{4.8}$$

We know by (3.4) that with appropriate basis vector change, e.g., $|n| = |r|$ and $|p| = |s|$ can be assumed if they are not zeros. Then from (4.8)

$$\frac{q}{t} = \frac{r}{n} \cdot \frac{p}{s} = \pm 1, \;\text{ i.e. }\; |q| = |t|$$

follows if neither q nor t is zero.

Thus we shall have the following strategy. If, e.g., $r = 0$ but $n \neq 0$, then $\Pi^{00} = 0$ follows by (4.7). Then a^0 would be isotropic again, already excluded.

First, we discuss by (4.7) and (4.8) the extra cases **1) a), b), c)**, which will not lead to discrete action with triangle $A_0A_1A_2$.

$$\textbf{1) a)}\quad \begin{cases} r=-n<0 & \Pi^{01}=-\frac{n}{2}\Pi^{00}=\frac{n}{2}\Pi^{11} \\ s=-p<0 & \Pi^{02}=-\frac{p}{2}\Pi^{00}=\frac{p}{2}\Pi^{22} \\ t=q\neq 0 & \Pi^{12}=-\frac{q}{2}\Pi^{11}=-\frac{q}{2}\Pi^{22} \end{cases}$$
$$\textbf{1) b–c)}\quad \begin{cases} r=-n<0 & \Pi^{01}=-\frac{n}{2}\Pi^{00}=\frac{n}{2}\Pi^{11} \\ s=p=0 & \Pi^{02}=0 \\ q>0, t=\pm q & \Pi^{12}=-\frac{q}{2}\Pi^{11} \quad \Pi^{22}=\pm\Pi^{11} \end{cases} \tag{4.9}$$

The other main cases **2)** — including some zero subcases as well:

$$\textbf{2)}\quad \begin{cases} r=n\geq 0 & \Pi^{01}=-\frac{n}{2}\Pi^{00}=-\frac{n}{2}\Pi^{11} \\ s=p\geq 0 & \Pi^{02}=-\frac{p}{2}\Pi^{00}=\frac{p}{2}\Pi^{22} \\ t=q & \Pi^{12}=-\frac{q}{2}\Pi^{11}=-\frac{q}{2}\Pi^{22} \end{cases} \tag{4.10}$$

will be discussed by the products ${}^1M\,{}^0M$, ${}^0M\,{}^2M$, ${}^2M\,{}^1M$ and by the signature of Π^{ij}. Of course, the parameters n, p, q can be varied too, also equal to zero. As naturally expected for, two polarities given by Π^{ij} and $-\Pi^{ij}$ will not be distinguished. The signature of Π^{ij} or the signes in the square sum of the line quadratics $\xi_i\Pi^{ij}\xi_j$ will determine our five geometries and the corresponding discrete triangle tilings in these geometries. Then our aim will be completed.

5. Triangle reflection groups with non-discrete actions

We recall, first by (4.9) and (4.10), moreover by (3.1), (3.4) the extra cases and the main cases, i.e., in short form

$$\begin{array}{ll} \textbf{1)}\ \Pi^{11}\begin{pmatrix} -1 & \frac{n}{2} & \frac{p}{2} \\ \frac{n}{2} & 1 & -\frac{q}{2} \\ \frac{p}{2} & -\frac{q}{2} & \pm 1 \end{pmatrix} & \begin{array}{c} r=-n<0 \\ s=-p\leq 0 \\ q>0,\ t=\pm q \end{array} \\ \textbf{2)}\ \Pi^{00}\begin{pmatrix} 1 & -\frac{n}{2} & -\frac{p}{2} \\ -\frac{n}{2} & 1 & -\frac{q}{2} \\ -\frac{p}{2} & -\frac{q}{2} & 1 \end{pmatrix} & \begin{array}{c} r=n\geq 0 \\ s=p\geq 0 \\ t=q \end{array} \end{array} \tag{5.1}$$

respectively, as polarities $(_*)$ in (4.1), each assingnes a centre $A^i=(\boldsymbol{a}^i_*)$ as pole to the axis line a^i of the central axial collineation ${}^iM$, $i=0,1,2$. The negative polarity with $(-\Pi^{ij})$ just assignes the opposite centre in each ${}^iM$. *This is why we do not distinguish* (Π^{ij}) *and* $(-\Pi^{ij})$, *and this motivates our treatment that orthogonality, polarity and reflection are correlated concepts in metric geometries, now through* $\mathcal{PS}^2$.

We examine the product ${}^1M\,{}^0M$ *of reflections around the vertex* A_2, *as typical example first, for guaranteeing discrete action there in the sense of universal covering (Poincaré theorem)* (Fig. 2 and 3). By row-column multiplication it holds

$${}^1M\,{}^0M:\ \begin{pmatrix} 1 & 0 & 0 \\ r & -1 & q \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & n & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -1 & n & p \\ -r & rn-1 & rp+q \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

For the eigen-value eigen-vector problem of ${}^1M\,{}^0M$ we consider

$$0 = \det \begin{pmatrix} -1-\lambda & n & p \\ -r & rn-1-\lambda & rp+q \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\left[\lambda^2 - \lambda\,(rn-2)+1\right]$$

whose roots are

$$ {}^2\lambda = 1, \quad {}^0\lambda, {}^1\lambda = \frac{1}{2}\left[rn-2 \pm \sqrt{(rn-2)^2-4}\right]. \tag{5.3}$$

We discuss this for the fixed lines and fixed points of ${}^1M\,{}^0M$ as usual, depending on r and n. As we know $r=\pm n$ and $n>0$ can be assumed now.

For the fixed lines $(u_0,u_1,u_2)^{\mathrm{T}}$ of ${}^1M\,{}^0M$ through A_2 we get $(-1-\lambda)\,u_0 + nu_1 = 0$, $u_2=0$ with $\lambda = {}^0\lambda$ or ${}^1\lambda$ above in (5.3).

From this

$$\frac{u_1}{u_0} = \frac{1+\lambda}{n} = {}^{+}\!/\!_{-}\frac{n}{2} \pm \sqrt{\frac{nn}{4}{}^{-}\!/\!_{+}1} \text{ for } r = {}^{+}\!/\!_{-}n \neq 0 \text{ and } \lambda = {}^0\lambda, {}^1\lambda \tag{5.4}$$

as two independent alternatives. The same occurs if we look for isotropic lines through A_2:

$$\begin{gathered} 0 = \langle \boldsymbol{a}^0u_0 + \boldsymbol{a}^1u_1,\ \boldsymbol{a}^0u_0 + \boldsymbol{a}^1u_1\rangle = \Pi^{00}u_0u_0 + 2\Pi^{01}u_0u_1 + \Pi^{11}u_1u_1 \\ \text{with } \Pi^{00} = {}^{+}\!/\!_{-}1,\ \Pi^{01} = {}^{-}\!/\!_{+}\tfrac{n}{2},\ \Pi^{11}=1 \end{gathered} \tag{5.4a}$$

as quadratic equation for $\frac{u_1}{u_0}$ by the alternative polarities in (5.1):

$$\frac{u_1}{u_0} = \frac{1}{\Pi^{00}}\left(-\Pi^{01} \pm \sqrt{\Pi^{01}\Pi^{01} - \Pi^{00}\Pi^{11}}\right) = {}^{+}\!/\!_{-}\frac{n}{2} \pm \sqrt{\frac{nn}{2}{}^{-}\!/\!_{+}1}. \tag{5.4b}$$

We know that for lines in the angular domain (a^0a^1) of $A_0A_1A_2$ the quotient $\frac{u_1}{u_0}$ is negative. This case has to be excluded for the discrete action of $({}^1M\,{}^0M)^x$, $x\in\mathbb{Z}$ (integers), since the isotropic fixed lines are the accumulation lines of images of $A_0A_1A_2$ if $x\to+\infty$ and $x\to-\infty$, respectively.

1) We choose the extra case $r=-n\neq 0$, say $n>0$, when $|rn-2|>2$. Thus we have 3 fixed lines and 3 fixed points of ${}^1M\,{}^0M$ and the extra polarity in (4.9) and (5.1) where for the quotient in (5.4) and in (5.4b) holds

$$\frac{u_1}{u_0} = -\frac{n}{2} \pm \sqrt{\frac{nn}{4}+1} \text{ with } n>0, \tag{5.4c}$$

that means for one of isotropic fixed lines the quotient is negative.

We remark at the same time to the main case **2)** of (5.1), i.e., now $r=n>2$, that the quotient in (5.4) and in (5.4b) will be

$$\frac{u_1}{u_0} = \frac{n}{2} \pm \sqrt{\frac{nn}{4}-1} \tag{5.4d}$$

both are positive. Then the discrete action of $({}^1M\,{}^0M)^x$, $x\in\mathbb{Z}$, with triangle $A_0A_1A_2$ in the angular domain of isotropic lines, will be guaranteed.

This would not be the case if $r=n<-2$. Thus in the main cases **2. d), e)** of (5.1) $t=q<-2$ will not be allowed for the discrete action of $({}^2M\,{}^1M)^x$, $x\in\mathbb{Z}$.

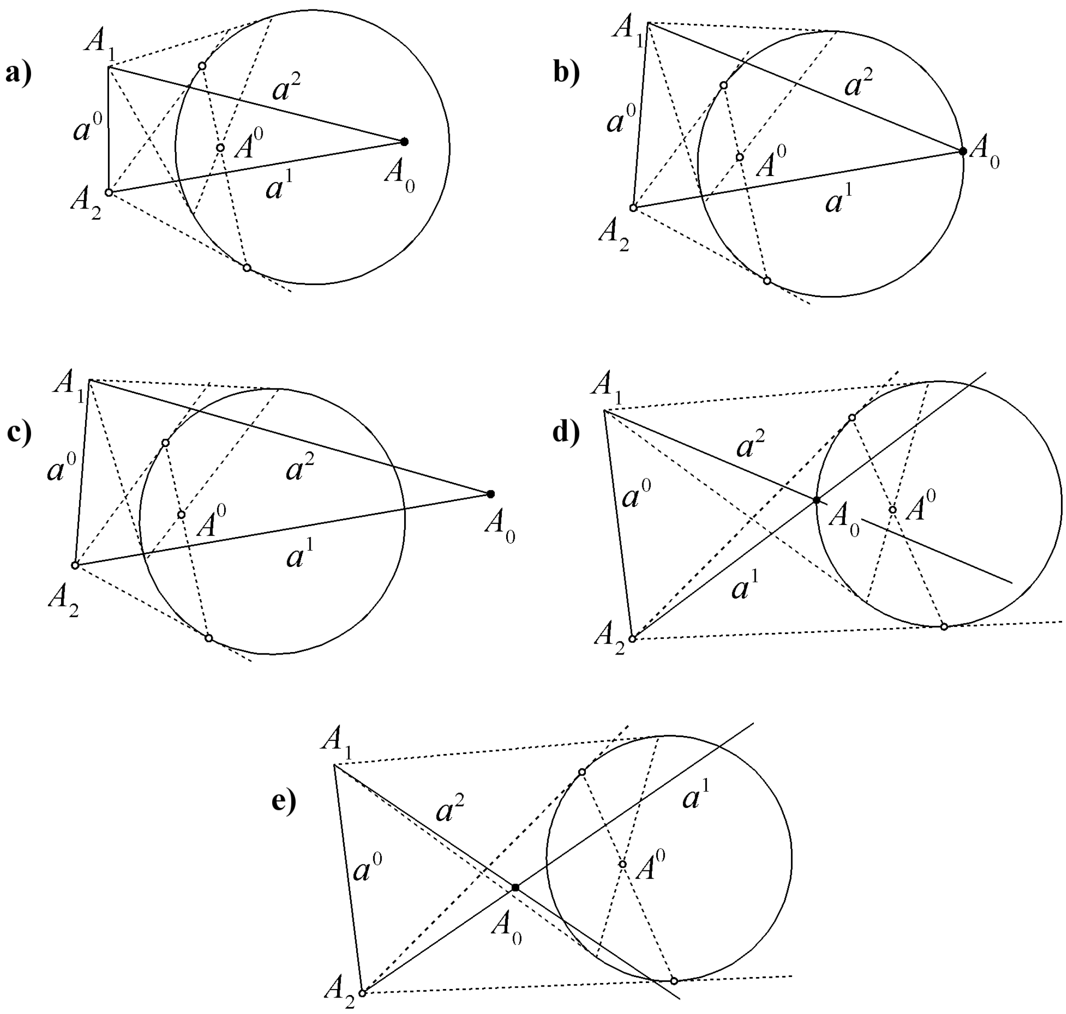


Figure 3: Non-discrete triangle tilings in $\mathbf{H}^2$

For extra case **1)** in Fig. 3 we indicate the hyperbolic possibilities where the isotropic lines with their incident poles just form the absolute conic (circle) of $\mathbf{H}^2$ (Section 1.3) with appropriate + sign of Π^{11} in (5.1) (there would be 7 cases at all in Fig. 3).

Write

$$\begin{aligned} rn - 2 &= -2\cosh d^{10}, \quad \text{so} \\ {}^0\lambda &= -\cosh d^{10} + \sinh d^{10} = -\exp\left(-d^{10}\right) \\ {}^1\lambda &= -\cosh d^{10} - \sinh d^{10} = -\exp\left(d^{10}\right) \end{aligned}$$

We only mention here that $d^{10} > 0$ above is related to the distance of the pole $A^0 = (\boldsymbol{a}^0_*)$ from the proper line a^1, where the distance segment lies on the polar line of vertex A_2. The reflection ${}^0M$ in a^0 changes the absolute conic of $\mathbf{H}^2$ with its opposite conic on $\mathcal{PS}^2$. If we combine ${}^0M$ with the central inversion of $\mathcal{PS}^2$, then we obtain a line reflection of $\mathbf{V}^3$ (or of $\boldsymbol{V}_3$) which realizes a point reflection in $A^0 = (\boldsymbol{a}^0_*)$ in the $\mathcal{P}^2$-model of $\mathbf{H}^2$.

Thus in extra cases **1) a–c)** we see that ${}^1M{}^0M$ does not act discretely with the triangle domain $A_0A_1A_2$.

The quadratic form $\xi_i\Pi^{ij}\xi_j$ for lines $(\boldsymbol{a}^i\xi_i)$ in the extra case **1) a)** of (4.9), (5.1) will be as follows (the alternative $\pm$ signs refers to **1) b–c)** where $s=p=0$)

$$\begin{aligned}\xi_i\Pi^{ij}\xi_j =&\Pi^{11}\left[\left(\xi_1+\frac{n}{2}\xi_0-\frac{q}{2}\xi_2\right)-\left(1+\frac{nn}{4}\right)\xi_0\xi_0+\left(\frac{nq}{2}+p\right)\xi_0\xi_2+\right.\\ &\left.+\left(\pm 1-\frac{qq}{4}\right)\xi_2\xi_2\right]=\Pi^{11}\left\{\left(\xi_1+\frac{n}{2}\xi_0-\frac{q}{2}\xi_2\right)^2-\right.\\ &-\left(1+\frac{nn}{4}\right)\left[\xi_0-\frac{\frac{nq}{4}+\frac{p}{2}}{1+\frac{nn}{4}}\xi_2\right]^2+\xi_2\xi_2\frac{1}{1+\frac{nn}{4}}\cdot\\ &\left.\cdot\left[\pm\left(1+\frac{nn}{4}\right)+\frac{pp}{4}-\frac{qq}{4}+\frac{npq}{4}\right]\right\}.\end{aligned}\tag{5.5}$$

Thus we see that the signature is $(+--)$ or $(+-+)$, i.e. hyperbolic if the last term is not zero. If it is (also in cases **1) b–c)**), i.e.,

$$\pm\left(1+\frac{nn}{4}\right)+\frac{pp}{4}-\frac{qq}{4}+\frac{npq}{4}=0,\tag{5.6}$$

then we get $\mathbf{M}^2$-signature $(+-0)$ for the Minkowski plane (Section 1.3), and the line quadratics in (5.5) will be the product of two linear forms. Thus the equation from (5.5):

$$\begin{aligned}0=\xi_i\Pi^{ij}\xi_j =&\Pi^{11}\left[\xi_1+\left(\frac{n}{2}+\sqrt{\frac{1+nn}{4}}\right)\xi_0-\left(\frac{q}{2}+\frac{\frac{nq}{4}+\frac{p}{2}}{\sqrt{1+\frac{nn}{4}}}\right)\xi_2\right]\cdot\\ &\cdot\left[\xi_1+\left(\frac{n}{2}-\sqrt{1+\frac{nn}{4}}\right)\xi_0-\left(\frac{q}{2}-\frac{\frac{nq}{4}+\frac{p}{2}}{\sqrt{1+\frac{nn}{2}}}\right)\xi_2\right]\end{aligned}\tag{5.7}$$

provides the two absolute points I_1, I_2 and their joint isotropic ideal line for the Minkowski plane $\mathbf{M}^2$. The coordinates of I_1 and I_2 can be read off from the coefficients of ξ_0, ξ_1, ξ_2 in (5.7). These determine the "light directions" of special relativity. Of course $q=t$ can be expressed from (5.6) in case **1) a)**.

1. b–c) The zero case $s=p=0$ with $r=-n<0$ allows us $q=\pm t$ with $\Pi^{22}=\pm\Pi^{11}$, where the double signs refer to each other. $\mathbf{H}^2$-cases occur as formerly. The Minkowski plane $\mathbf{M}^2$ occurs only for upper signs as it can be seen at (5.6).

Summarizing in extra cases 1) a)–b)–c) our triangle group generated by ${}^0M$, ${}^1M$, ${}^2M$ does not act discretely with its domain $A_0A_1A_2$.

6. The main cases for discrete action of triangle groups

By logical symmetry we consider case **2)** in (5.1), recall again here

$$\textbf{2)}\ \Pi^{00}\begin{pmatrix}1 & -\frac{n}{2} & -\frac{p}{2}\\ -\frac{n}{2} & 1 & -\frac{q}{2}\\ -\frac{p}{2} & -\frac{q}{2} & 1\end{pmatrix}\qquad \begin{matrix}r=n\geq 0\\ s=p\geq 0\\ t=q\end{matrix}\tag{6.1}$$

from (4.10). Write here also a critical product matrix:

$${}^1M\,{}^0M\colon\ \begin{pmatrix}-1 & n & p\\ -r & rn-1 & rp+q\\ 0 & 0 & 1\end{pmatrix}\qquad -2+rn=\begin{cases}2\cos\varepsilon^{10}=4\cos^2\frac{\varepsilon^{10}}{2}-2\\ 2\\ 2\cosh d^{10}=4\cosh^2\frac{d^{10}}{2}-2\end{cases}\tag{6.2}$$

We recall, the eigen-value eigen-vector problem from (5.3), but now $r = n \geq 0$ leads to essential simplification. Similarly come other pairs, further simplified:

$$\begin{array}{ll} {}^0M\,{}^2M\colon \begin{pmatrix} -1+ps & n+pt & -p \\ 0 & 1 & 0 \\ s & t & -1 \end{pmatrix} & p = s = \begin{cases} 2\cos\frac{\varepsilon^{02}}{2} = 2\cos\frac{\pi}{v} \\ 2 \\ 2\cosh\frac{d^{02}}{2} \end{cases} \Big\} v = \infty \\ {}^2M\,{}^1M\colon \begin{pmatrix} 1 & 0 & 0 \\ r & -1 & q \\ s+tr & -t & tq-1 \end{pmatrix} & t = q = \begin{cases} 2\cos\frac{\varepsilon^{21}}{2} = 2\cos\frac{\pi}{u} \\ \pm 2 \\ \pm 2\cosh\frac{d^{21}}{2} \end{cases} \Big\} u = \infty \end{array} \tag{6.3}$$

By the well-known theory, we shall see that for the finite action of ${}^1M\,{}^0M$ it holds

$$-2 + rn = 2\cos\frac{k2\pi}{w} \quad \text{with} \quad 2 \leq w \in \mathbb{N},\ w > k \in \mathbb{N},\ (k, w) = 1\,,$$

i.e., k and w are coprimes. However, then ${}^1M\,{}^0M$ generates action with k-times overlap by $A_0A_1A_2$ around A_2. So only $k = 1$ is convenient. Then it follows in (6.2)

$$r = n = 2\cos\frac{\varepsilon^{10}}{2} = 2\cos\frac{\pi}{w} \geq 0. \tag{6.4}$$

For infinite action $w = \infty$ with one fixed line through A_2 it holds

$$r = n = 2; \quad \text{furthermore} \quad r = n = 2\cosh\frac{d^{10}}{2} \tag{6.5}$$

if we have two fixed lines through A_2, by (5.3), and (3.4).

Namely, in (5.3) $|rn - 2| < 2$ yields conjugate complex eigen-values, i.e., ${}^1M\,{}^0M$ is a "combinatorial rotation about A_2 through an angle ε^{10}". We require that the group generated by ${}^0M$ and ${}^1M$ tiles the neighbourhood of A_2 with triangle $A_0A_1A_2$ discretely, i.e., without gap and overlap. Then

$$rn - 2 = 2\cos\varepsilon^{10} = 2\cos\frac{2\pi}{w} \geq 0, \text{ i.e.,}$$
$$rn = 2\cos\frac{2\pi}{w} + 2 = 4\cos^2\frac{\pi}{w} \quad 2 \leq w \in \mathbb{N}.$$

These will be in other analogous cases of (6.3), as we claimed. In case $rn = 4$, i.e., $r = n = 2$ or $r = n = -2$, ${}^0\lambda = {}^1\lambda = 1$ in (5.3). E.g., for the fixed lines $(\boldsymbol{a}^j u_j)$ of ${}^1M\,{}^0M$ it holds

$$\begin{pmatrix} -2 & 2 & p \\ -2 & 2 & 2p+q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ i.e.,} \quad \begin{array}{l} -2u_0 + 2u_1 + pu_2 = 0 \\ -2u_0 + 2u_1 + (2p+q)\,u_2 = 0 \end{array} \tag{6.6}$$

Summarizing: If $q = -p$, the fixed lines of ${}^1M\,{}^0M$ go through the point $(-2, 2, p)$ (and its opposite, of course). The fixed points lie on the line $(1, 1, 0)^{\mathrm{T}}$. That means, ${}^1M\,{}^0M$ is an *elation* (or *dilation*; projective variant of a Euclidean translation). Else $(1, 1, 0)^{\mathrm{T}}$, out of triangle $A_0A_1A_2$ through A_2, is the only isotropic fixed line and $(0, 0, 1)$, $(0, 0, -1)$ are the only fixed points.

In case $r = n = -2$ we similarly get analogous subcases: if $q = p$, the fixed lines go through $(-2, 2, p)$ and $(2, 2, -p)$; the fixed points lie in the line $(-1, 1, 0)$. ${}^1M\,{}^0M$ is an elation. Else

$(1, -1, 0)^{\mathrm{T}}$, partly in $A_0A_1A_2$, is the only isotropic fixed line; $(0, 0, 1)$ and its opposite are the only fixed points.

Case $|rn - 2| > 2$ leads to $|rn - 2| = 2\cosh d^{10}$, but now $rn > 4$ leads to simpler cases as in (5.4). These were indicated in (6.3).

The equation (3.4) permits only one negative parameter from n, p, q, if 0 does not occur among them, else parameters can be non-negative.

The signature of quadratics by (6.1)

$$\begin{aligned} &\xi_i\Pi^{ij}\xi_j = \Pi^{00}\left\{\xi_0\xi_0 - n\xi_0\xi_1 - p\xi_0\xi_2 + \xi_1\xi_1 - q\xi_1\xi_2 + \xi_2\xi_2\right\} \\ &\text{with parameters} \quad n \geq 0,\ p \geq 0,\ q \in \mathbb{R} \quad \text{as above} \end{aligned} \tag{6.7}$$

will systematically be discussed in the following.

2) a) $(n, p, q) = (0, 0, 0)$.

$$\begin{pmatrix} \Pi^{00} & 0 & 0 \\ 0 & \Pi^{11} & 0 \\ 0 & 0 & \Pi^{22} \end{pmatrix} \qquad \begin{array}{ll} (+\,+\,+): & \mathbf{S}^2 \text{ octahedron tiling} \\ (+\,+\,-): & \mathbf{H}^2 \end{array}$$

is allowed by arbitrary (may be with different signs) non-zero elements Π^{00}, Π^{11}, Π^{22}. We have two essential possibilities. By basis vector change $(1, 1, 1)$ and $(1, 1, -1)$ can be achieved. By Section 2 the above case $\mathbf{2aS}^2$ can be ordered into Family 1, $u = 4$ is the adjacency parameter. $\mathbf{2aH}^2$ belongs to Family 2, but $2u = 4$, $v = 4$ are the adjacency parameters. In Fig. 4 $\Pi^{00} < 0$. Description is taken in the double Euclidean plane, the other part could visualize the opposite conic with the opposite triangle, etc.

2) b) $(n, p, q) = (0, 0, q > 0)$, Family 2

$$\begin{pmatrix} \Pi^{00} & 0 & 0 \\ 0 & \Pi^{11} & -\frac{q}{2}\Pi^{11} \\ 0 & -\frac{q}{2}\Pi^{11} & \Pi^{11} \end{pmatrix}$$

$\xi_i\Pi^{ij}\xi_j = \Pi^{00}\xi_0\xi_0 + \Pi^{11}\left[\left(\xi_1 - \frac{q}{2}\xi_2\right)^2 + \left(1 - \frac{qq}{4}\right)\xi_2\xi_2\right]$

signs of Π^{00}, Π^{11} and $q > 0$ by (6.3) determine the cases

$\mathbf{S}^2\,(+\,+\,+) \quad 0 < \Pi^{00} = 1 = \Pi^{11},\ 0 < \dfrac{q}{2} = \cos\dfrac{\pi}{u} < 1 \quad 3 \leq u \in \mathbb{N}.$

$\mathbf{H}^2\,(-\,+\,+)$

i) $\Pi^{00} = -1$, $\Pi^{11} = 1$; $0 < \frac{q}{2} = \cos\frac{\pi}{u} < 1 \quad 3 \leq u \in \mathbb{N}$.

ii) $\Pi^{00} = +1$, $\Pi^{11} = -1$; $2 < \frac{q}{2} = \cosh\frac{d^{21}}{2}$.

$$\xi_i\Pi^{ij}\xi_j = +\xi_0\xi_0 - \left(\xi_1 - \cosh\frac{d^{21}}{2}\xi_2\right)^2 + \left(\cosh^2\frac{d^{21}}{2} - 1\right)\xi_2\xi_2\,.$$

The images of $A_0A_1A_2$ lie in $\overline{\mathbf{H}}^2 = \mathcal{PS}^2 \setminus \mathcal{D}\mathbf{H}^2$, determined by the isotropic lines through A_0. $\mathcal{D}\mathbf{H}^2$ refers to the double conic domain of $\mathcal{PS}^2$. The pole A^1 of a^1 is in $\mathbf{H}^2$, so as A^2. The distance of A^2A^1 is just $\frac{d^{21}}{2}$ above.

$$\mathbf{E}^2\,(+\,+\,0) \quad \Pi^{00} = 1 = \Pi^{11};\ q = 2 \quad (u = \infty)$$

$$\xi_i\Pi^{ij}\xi_j = \xi_0\xi_0 + (\xi_1 - \xi_2)^2\,.$$

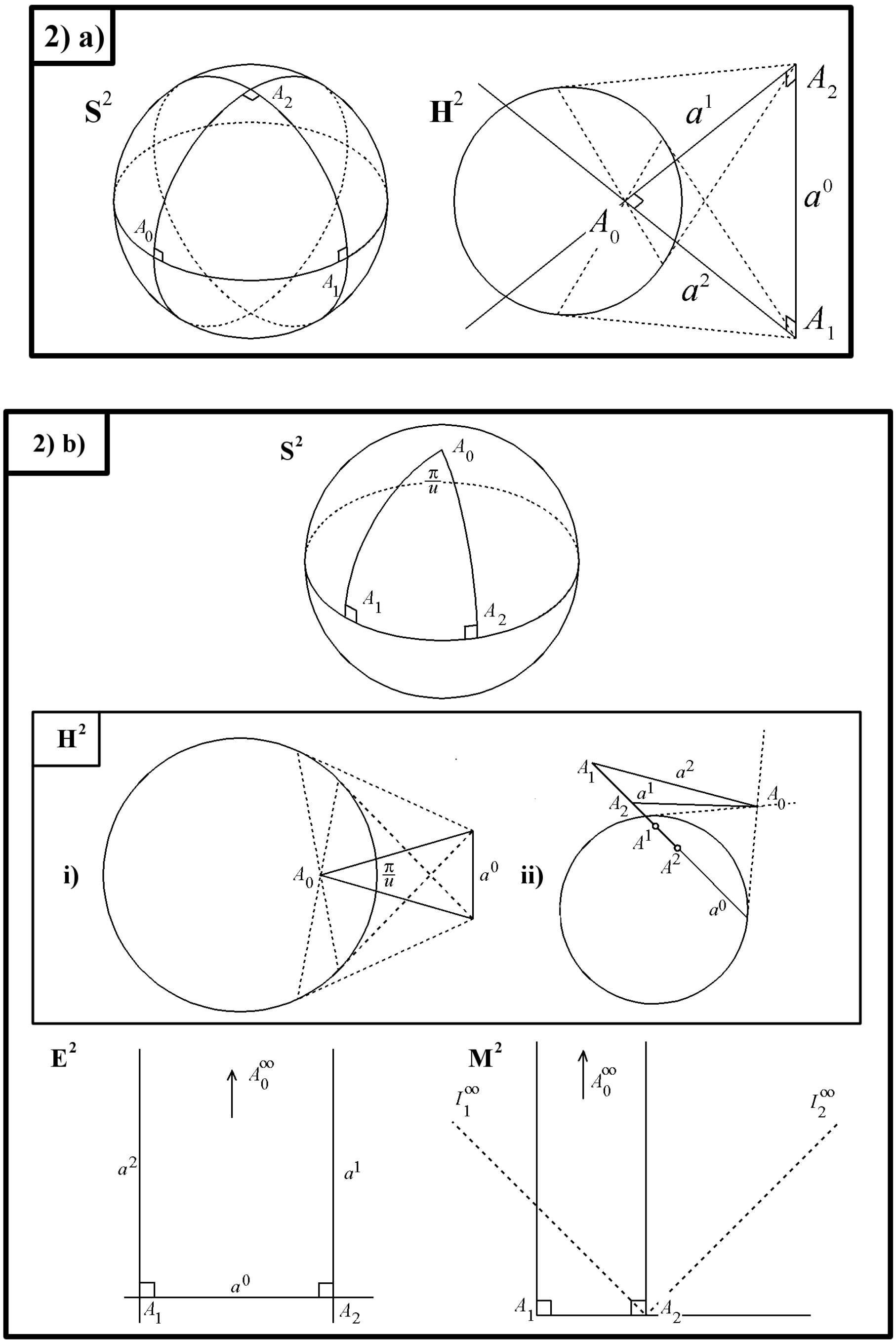


Figure 4, part 1

In Fig. 4 A_0^∞ is ideal point, the pole of $a^0 = A_1A_2$. The ideal line is just the axis of translation ${}^2M\,{}^1M$.

$$\mathbf{M}^2\,(+-0)\,\Pi^{00} = 1,\ \Pi^{11} = -1;\ q = 2$$

$$\xi_i \Pi^{ij} \xi_j = \xi_0\xi_0 - (\xi_1 - \xi_2)^2 = (\xi_0 + \xi_1 - \xi_2)\,(\xi_0 - \xi_1 + \xi_2) = 0$$

provides the absolute "light points" $I_1\,(1, 1, -1)$, $I_2\,(1, -1, 1)$ and their opposites from the coefficients of the linear forms in the product above. $I_1I_2 = i$ is the ideal line of Minkowski plane $\mathbf{M}^2$.

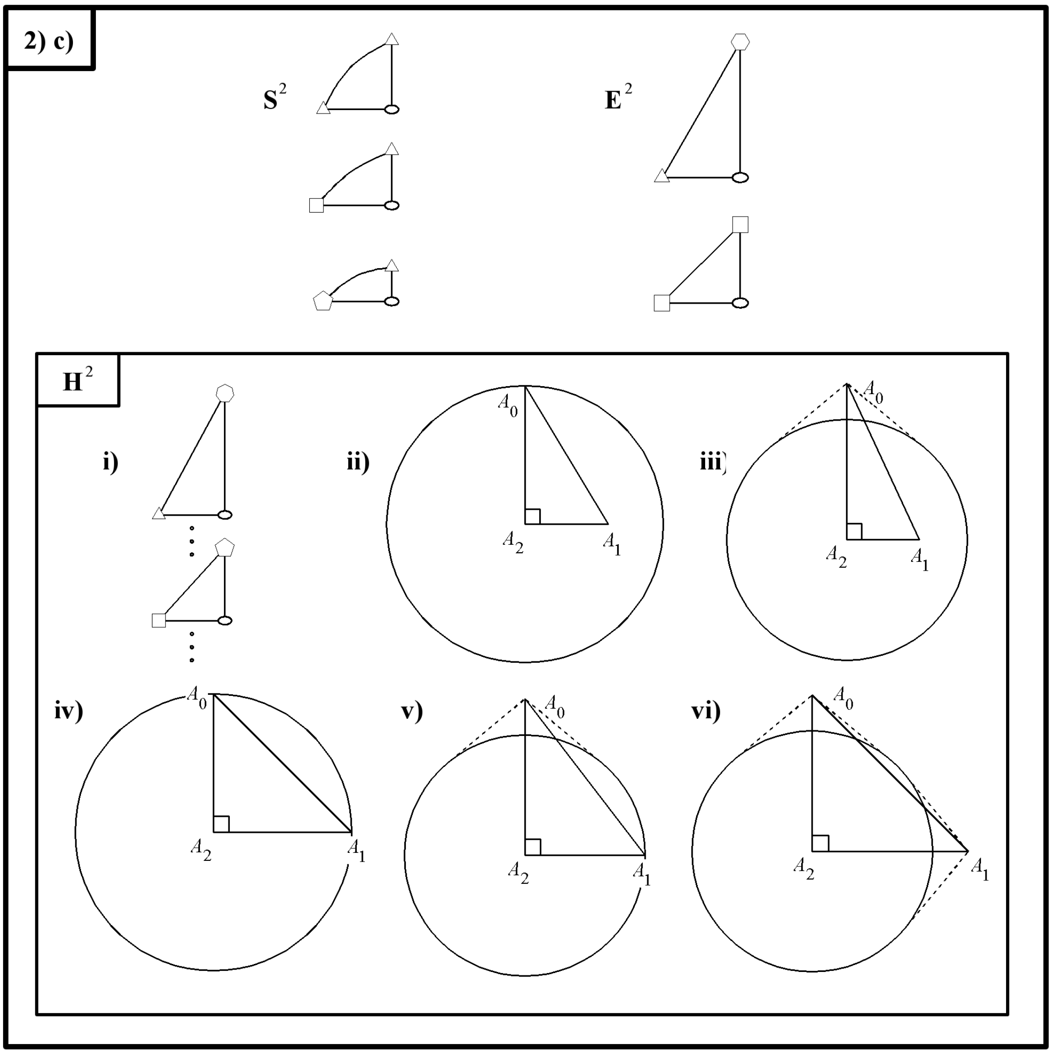


Figure 4, part 2

2) c) $(n,p,q)=(0,p>0,q>0)$ Family 2 and Family 3

$$\Pi^{00}\begin{pmatrix} 1 & 0 & -\frac{p}{2} \\ 0 & 1 & -\frac{q}{2} \\ -\frac{p}{2} & -\frac{q}{2} & 1 \end{pmatrix}$$

$$\xi_i\Pi^{ij}\xi_j=\Pi^{00}\left\{\left(\xi_0-\frac{p}{2}\xi_2\right)^2+\left(\xi_1-\frac{q}{2}\xi_2\right)^2+\left(1-\frac{pp}{4}-\frac{qq}{4}\right)\xi_2\xi_2\right\}.$$

$\Pi^{00}=1$; p, q provide the subcases by

$$1-\frac{pp}{4}-\frac{qq}{4}\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}0\left\{\begin{matrix} \mathbf{S}^2 \\ \mathbf{E}^2 \\ \mathbf{H}^2 \end{matrix}\right. \qquad \frac{p}{2}=\cos\frac{\pi}{v},\ \frac{q}{2}=\cos\frac{\pi}{u}.$$

$\mathbf{S}^2:\ (+++)\quad (u,v)=(3,3)\,,\,(3,4)\,,\,(3,5)$
$\mathbf{E}^2:\ (++0)\quad (u,v)=(3,6)\,,\,(4,4)$
$\mathbf{H}^2:\ (++-)$

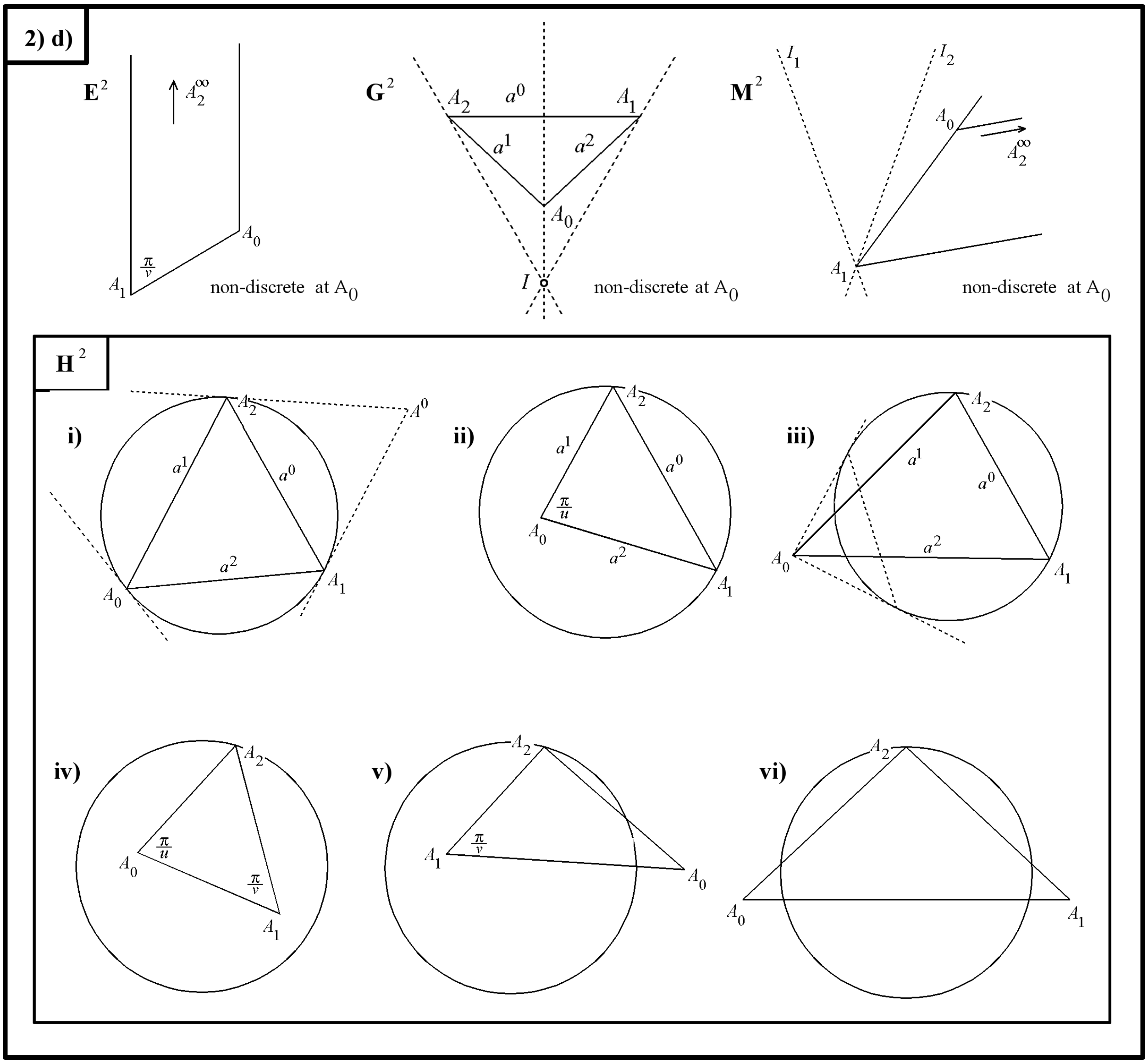


Figure 4, part 3

i) $(u,v) = (3,7)\,,\,(3,8)\,,\,\ldots,\,(4,5)\,,\,(4,6)\,,\,\ldots,\,(5,5)\,,\,(5,6)\ldots$

ii) $\dfrac{p}{2} = \cos\dfrac{\pi}{v},\ \dfrac{q}{2} = 1,\ 3 \leq v \in \mathbb{N}$

iii) $\dfrac{p}{2} = \cos\dfrac{\pi}{v},\ 1 < \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2},\ 3 \leq v \in \mathbb{N}$

iv) $\dfrac{p}{2} = \dfrac{q}{2} = 1$

v) $\dfrac{p}{2} = 1 < \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$

vi) $1 < \dfrac{p}{2} = \cosh\dfrac{d^{02}}{2},\ \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$

Fig. 4 indicates the typical cases with obvious denotations.

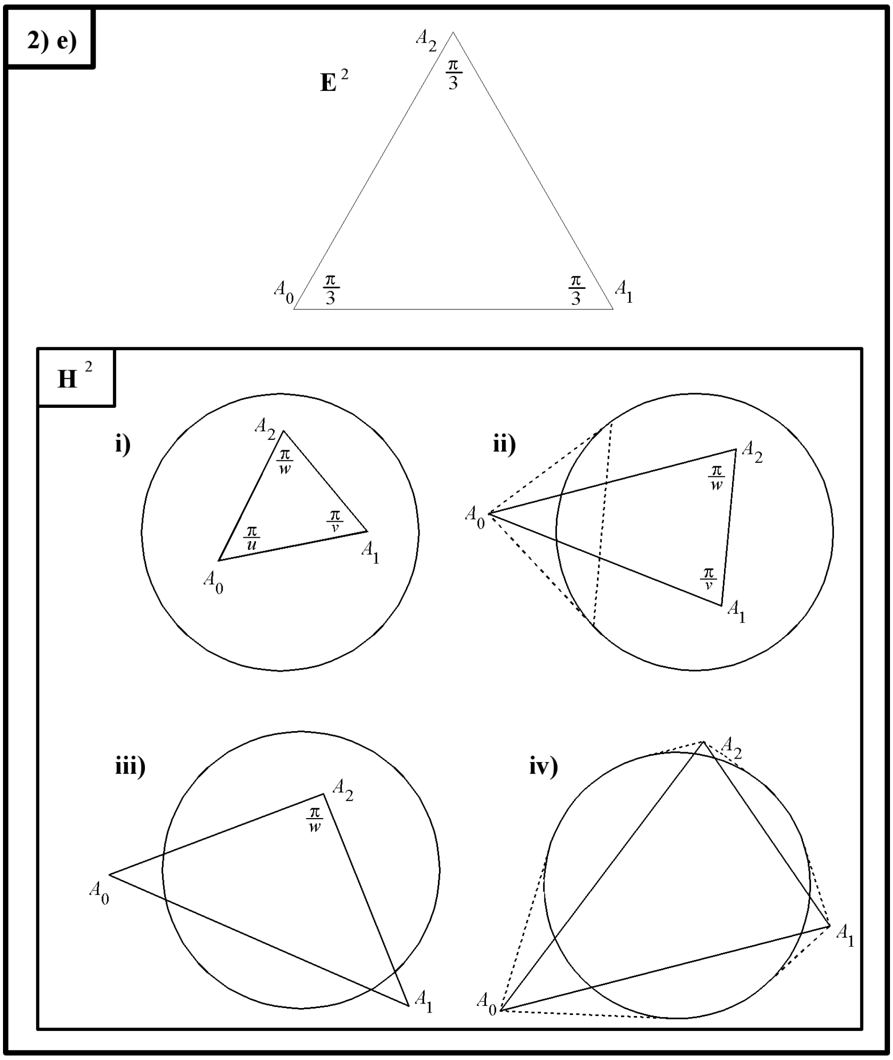


Figure 4: Results for discrete triangle reflection tiling series (with non-isotropic reflection lines); $\mathbf{S}^2$: 3 series, $\mathbf{E}^2$: 3 series, $\mathbf{H}^2$: 19 series, $\mathbf{M}^2$: 1 series, $\mathbf{G}^2$: 0 series

2) d) $(n > 0, p > 0, q \neq 0)$ two of them are positive

$$\Pi^{00} \begin{pmatrix} 1 & -\frac{n}{2} & -\frac{p}{2} \\ -\frac{n}{2} & 1 & -\frac{q}{2} \\ -\frac{p}{2} & -\frac{q}{2} & 1 \end{pmatrix}$$

$$\xi_i \Pi^{ij} \xi_j = \Pi^{00} \left\{ \left(\xi_0 - \frac{n}{2}\xi_1 - \frac{p}{2}\xi_2\right)^2 + \left(1 - \frac{nn}{4}\right)\xi_1\xi_1 - \right.$$
$$\left. - \left(q + \frac{np}{2}\right)\xi_1\xi_2 + \left(1 - \frac{pp}{4}\right)\xi_2\xi_2 \right\}$$

$\Pi^{00} = 1$. We examine the specific cases first.

$\mathbf{G}^2$ (Galilean plane) $(+\,0\,0)$: $n = 2 = p$, $q = -\frac{np}{2} = -2$, $0 = (\xi_0 - \xi_1 - \xi_2)^2$ provide the isotropic line pencil through the unique absolute point pair $I\,(1, -1, -1)$ and its opposite $I'\,(-1, 1, 1)$ by the coefficients.
On the polarity matrix (Π^{ij}) we see that I is the common pole of $a^0 a^1 a^2$, and centre of every reflection ${}^zM$, $z = 0, 1, 2$. Each product ${}^1M\,{}^0M$, ${}^0M\,{}^2M$, ${}^2M\,{}^1M$ will be elation

with the common centre I and with isotropic axes IA_2, IA_1, IA_0, respectively as the summary on ${}^1M\,{}^0M$ for $n = \pm 2$ described in this section.
We can see that the images of triangle $A_0A_1A_2$ non-discretely tiles at A_0 the projective spherical digon $IA_1I'A_2I$ under our triangle group. A_0, A_1, A_2 and their images have infinitely many triangles in their neighbourhoods, respectively. Visualizing this tiling seems to be a very interesting and attractive task.
Let $n = 2 = p$ and $q \neq -\frac{np}{2} = -2$ then we get $\mathbf{H}^2$ cases, as

$$\begin{aligned}\xi_i\Pi^{ij}\xi_j = &(\xi_0 - \xi_1 - \xi_2)^2 - (q+2)\,\xi_1\xi_2 = (\xi_0 - \xi_1 - \xi_2)^2 + (q+2)\cdot \\ &\cdot \left[\left(\xi_1 - \frac{1}{4}\xi_2\right)^2 - \left(\xi_1 + \frac{1}{4}\xi_2\right)^2\right]\end{aligned}$$

has signature

$\mathbf{H}^2$ $(++-)$:

i) $n = p = 2 = q$, Family 1
$A_0A_1A_2$ has ideal vertices in the absolute conic of $\mathbf{H}^2$

ii) $n = p = 2$, $\dfrac{q}{2} = \cos\dfrac{\pi}{u} > 0$; $\frac{q}{2} < 0$ does not lead to discrete action, Family 2

iii) $n = p = 2$, $1 < \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$, Family 2
$\frac{q}{2} < -1$ will be discussed at the end.

iv) $n = 2$, $\dfrac{p}{2} = \cos\dfrac{\pi}{v}$, $\dfrac{q}{2} = \cos\dfrac{\pi}{u}$, Family 2, 3

v) $n = 2$, $\dfrac{p}{2} = \cos\dfrac{\pi}{v}$, $1 < \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$, Family 3

vi) $n = 2$, $1 < \dfrac{p}{2} = \cosh\dfrac{d^{02}}{2} \leq \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$, Family 2, 3

To above $n = 2$, $p \neq 2$ subcases iv)–vi) the quadratic form conveniently is

$$\xi_i\Pi^{ij}\xi_j = \left(\xi_0 - \xi_1 - \frac{p}{2}\xi_2\right)^2 + \left(1 - \frac{pp}{2}\right)\left\{\left[\xi_2 - \frac{\frac{p}{2} + \frac{q}{2}}{1 - \frac{pp}{4}}\xi_1\right]^2 - \xi_1\xi_1\frac{\left(\frac{p}{2} + \frac{q}{2}\right)^2}{1 - \frac{pp}{4}}\right\}.$$

If here $-q = p > 0$; that means either $1 > \frac{p}{2} = \cos\frac{\pi}{v}$, but $\frac{q}{2} = \cos\left(\pi - \frac{\pi}{v}\right)$ with signature $(++0)$ *will not be convenient for discrete triangle tilings in* $\mathbf{E}^2$. Or $1 < \frac{p}{2} = \cosh\frac{d^{02}}{2} = -\frac{q}{2}$ leads to signature $(+-0)$ of $\mathbf{M}^2$

$$\begin{aligned}\mathbf{M}^2\colon &\left[\xi_0 - \xi_1 - \left(\cosh\frac{d^{02}}{2} + \sinh\frac{d^{02}}{2}\right)\xi_2\right]\cdot\left[\xi_0 - \xi_1 - \left(\cosh\frac{d^{02}}{2} - \sinh\frac{d^{02}}{2}\right)\xi_2\right] = \\ &= \left[\xi_0 - \xi_1 - \exp\left(\frac{d^{02}}{2}\right)\xi_2\right]\left[\xi_0 - \xi_1 - \exp\left(-\frac{d^{02}}{2}\right)\xi_2\right]\end{aligned}$$

leads to absolute points $I_1\left(1, -1, -\exp\left(\frac{d^{02}}{2}\right)\right)$ and opposite, and
$I_2\left(1, -1, -\exp\left(-\frac{d^{02}}{2}\right)\right)$ and opposite.
We have again a triangle tiling with overlaps, not convenient for us (Fig. 4 $\mathbf{2dM}^2$).

2) e) Less specific tilings

$0 < n \neq 2$, $0 < p \neq 2$, $0 < |q| \neq 2$; $\Pi^{00} = 1$

$$\Pi^{00} \begin{pmatrix} 1 & -\frac{n}{2} & -\frac{p}{2} \\ -\frac{n}{2} & 1 & -\frac{q}{2} \\ -\frac{p}{2} & -\frac{q}{2} & 1 \end{pmatrix}$$

$$\xi_1 \Pi^{ij} \xi_j = \Big(\xi_0 - \frac{n}{2}\xi_1 - \frac{p}{2}\xi_2\Big)^2 + \Big(1 - \frac{nn}{4}\Big) \cdot \left[\xi_1 - \frac{\frac{q}{2} + \frac{np}{4}}{1 - \frac{nn}{4}}\xi_2\right]^2 + \frac{1}{1 - \frac{nn}{4}} \cdot$$
$$\cdot \left[1 - \frac{nn}{4} - \frac{pp}{4} - \frac{qq}{4} - \frac{npq}{4}\right] \xi_2 \xi_2$$

$\mathbf{E}^2$: We know that $\frac{n}{2} = \frac{p}{2} = \frac{q}{2} = \cos\frac{\pi}{3} = \frac{1}{2}$ yields just zero determinant term at the end, i.e., signature $(++0)$, $\mathbf{S}^2$-tiling does not exist at this stage, since $\frac{\pi}{2}$ rectangle is excluded. We have $(++-)$ signature, if the above determinant $\left[1 - \frac{nn}{2} - \frac{pp}{4} - \frac{qq}{4} - \frac{npq}{4}\right] < 0$.

$\mathbf{H}^2$ $(++-)$:

i) $0 < \dfrac{n}{2} = \cos\dfrac{\pi}{w} \leq \dfrac{p}{2} = \cos\dfrac{\pi}{v} \leq \dfrac{q}{2} = \cos\dfrac{\pi}{u}$ yields negative determinant if

$$\frac{\pi}{w} + \frac{\pi}{v} + \frac{\pi}{u} < \pi \text{ (Family 1, 2, 3)}$$

This well-known fact can be derived explicitely, as well. This holds for the following cases, too.

ii) $0 < \dfrac{n}{2} = \cos\dfrac{\pi}{w} \leq \dfrac{p}{2} = \cos\dfrac{\pi}{v}$, $1 < \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$ (Family 2)

iii) $0 < \dfrac{n}{2} = \cos\dfrac{\pi}{w}$, $1 < \dfrac{p}{2} = \cosh\dfrac{d^{02}}{2} \leq \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$ (Family 2)

iv) $1 < \dfrac{n}{2} = \cosh\dfrac{d^{10}}{2} \leq \dfrac{p}{2} = \cosh\dfrac{d^{02}}{2} \leq \dfrac{q}{2} = \cosh\dfrac{d^{21}}{2}$.

Family 1, 2, 3 can occur each with corresponding parameters.

It may be suprising that in the main cases **2) d) e)** only $q = t > 0$ is allowed for discrete action with triangle $A_0A_1A_2$. The logical symmetry with n, p, q allow us to discuss only that $0 < n$, $0 < p$, but $q = t < 2$ yield non-discrete action with $A_0A_1A_2$. Namely, $({}^2M\,{}^1M)^x$, $x \in \mathbb{Z}$, do so at vertex A_0, since the isotropic fixed lines through A_0 then intersect the triangle $A_0A_1A_2$. This was just our remark at formula (5.4d), where the fixed isotropic lines were discussed. Thus we obtain the $\mathbf{H}^2$ cases of **2) d), e)** with non-negative n, p, q in formula (6.1).

At the end we formulate our results of Sections 3–6 in

Theorem 6.1 *The group generated by three involutive central axial collineations, briefly reflections* ${}^0M$, ${}^1M$, ${}^2M$ *in (3.1), acts discretely with its fundamental triangle* $A_0A_1A_2$ *in a domain of the projective-sphere* $\mathcal{PS}^2$, *as generalized models of the planes* $\mathbf{S}^2$, $\mathbf{E}^2$, $\mathbf{H}^2$, $\mathbf{M}^2$, $\mathbf{G}^2$ *as described in the Figure series 4:*

by 3 parametrized series for $\mathbf{S}^2$*; 3 series for* $\mathbf{E}^2$*; 19 series for* $\mathbf{H}^2$*; 1 series for* $\mathbf{M}^2$*, 0 (zero) series for* $\mathbf{G}^2$

All these series and the corresponding metric planes are parametrized by non-negative real $n = r$, $p = s$, $q = t$ *of (3.1) and characterized by projective polarity or scalar product in (6.1) as we described in Section 6 in details.*

In these paper we assumed in addition that the triangle lines of $A_0A_1A_2$ *are non-isotropic.*

The further cases where isotropic triangle line (coinciding with its pole), furthermore, non-discrete (with overlapping triangle images) action are allowed will be classified in a forthcoming work.

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