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An Angle Criterion for Conical Mesh Vertices

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Abstract. A vertex in a mesh surface with planar faces may have the property that offsetting all the face planes incident with the vertex by a constant distance leads to planes which intersect again in a common point. This is equivalent to the property that the planes, consistently oriented via the connectivity of the mesh, are tangent to an oriented cone of revolution. We show that for vertices of valence 4, this conical property is characterized in terms of the interior angles of the faces adjacent to the vertex: *The two sums of opposite angles are equal.* For a convex vertex this angle criterion follows directly from known results in spherical geometry concerning convex spherical quadrilaterals. For other types of vertices, however, the occurrence of non-convex spherical quadrilaterals makes it necessary to exhaustively enumerate and study a number of cases. The present short note resolves this combinatorial difficulty and proves that all conical vertices are characterized by this same angle criterion. This result is especially relevant in the context of modeling with conical meshes.

Key Words: spherical geometry, conical meshes

MSC 2000: 51M04, 51N20, 52B70

1. Introduction

The concept of conical meshes was introduced in [3], thus laying the foundation for significant new research in the geometry of freeform designs realized as steel/glass constructions (see [4] and the textbook [5]). This paper deals with an important detail, namely the proof that the angle-based condition which characterizes conical meshes (Theorem 1 below), is indeed true in all cases. This condition was first discovered in the case of a convex mesh vertex by H. Pottmann.

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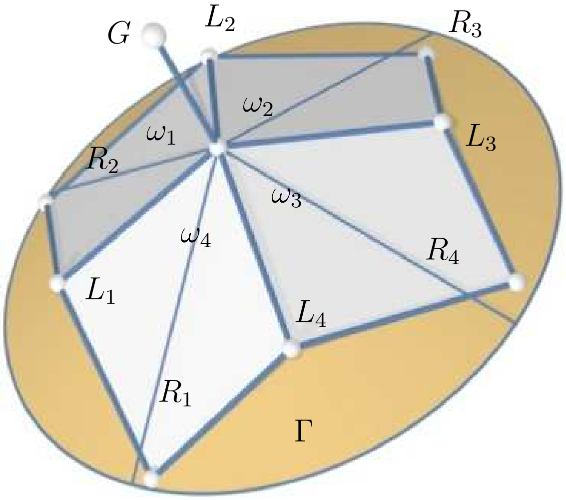

Figure 1: Conical vertex of valence four. The faces touch the common cone Γ with axis G along rulings R_1, R_2, R_3, R_4, and have interior angles $\omega_1, \omega_2, \omega_3, \omega_4$.

2. Conical vertices

In a mesh with planar faces, each face is equipped with a unit normal vector. These normal vectors can be consistently oriented only if the mesh surface is orientable, but anyway a consistent orientation is possible for the faces adjacent to a fixed vertex $\mathbf{v}$. A mesh vertex is said to be *conical* if the oriented planes adjacent to $\mathbf{v}$ are tangent to a common oriented cone of revolution. The axis of this cone can be regarded as a discrete surface normal at the vertex $\mathbf{v}$. For geometry and applications of meshes all of whose vertices are conical, see [3].

Consider a vertex $\mathbf{v}$ of valence 4 as shown in Fig. 1. Let L_i be the edges incident with $\mathbf{v}$, $i = 1, 2, 3, 4$. Let ω_i denote the unsigned angle formed by L_i and L_{i+1} (indices mod 4), $i = 1, 2, 3, 4$. We assume that no face is degenerate, i.e., any two consecutive edges are not parallel, and $\omega_i > 0$. The main result of this note is the following geometric fact.

Theorem 1 *A vertex $\mathbf{v}$ of valence 4 is conical if and only if the sums of opposite angles are equal, i.e., $\omega_1 + \omega_3 = \omega_2 + \omega_4$.*

Before giving the proof of Theorem 1 at the end of the paper, as preparation, we shall first present several results concerning the conical property, the existence of offset meshes of a mesh, and spherical quadrilaterals which have an incircle.

A mesh with planar faces usually does not have an offset mesh consisting of planar faces which are at constance distance from the original ones. That is because planes intersecting in a common point in general lose this property when each of them is moved by a fixed distance. The following result shows that the existence of an offset mesh is equivalent to the property that all vertices of the mesh are conical.

Theorem 2 *Suppose that planes $\epsilon_1, \ldots, \epsilon_k$, $k \geq 4$, with unit normal vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ contain the faces of a mesh which are incident with a common vertex $\mathbf{v}$. Translating each plane ϵ_i in the direction of $\mathbf{n}_i$ by a fixed distance $d \neq 0$ yields its offset plane ϵ_i^d. Then the following statements are equivalent:*

1. *The offset planes ϵ_i^d have a point in common for some $d \neq 0$;*
2. *The offset planes ϵ_i^d have a point in common for all d;*

3. *The planes $\epsilon_1, \ldots, \epsilon_k$ are tangent to a common cone of revolution, including the plane as limit case (the limit case of a straight line does not occur).*
4. *The normal vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$, regarded as points on the unit sphere S^2, satisfy a linear equation $\langle \mathbf{n}_i, \mathbf{x}_0 \rangle = d$, for some $\mathbf{x}_0 \neq 0$ (the case $d = 0$ does not occur).*

Proof: We first show the equivalence of statements 3 and 4. Note that the oriented planes ϵ_i are tangent to a common oriented cone of revolution (including the limit cases of line and plane) if and only if the unit normal vectors $\mathbf{n}_i$ lie on a circle contained in the unit sphere S^2, including the limit case of zero radius. This happens if and only if they satisfy a linear equation $\langle \mathbf{n}_i, \mathbf{x}_0 \rangle = d$ for some $\mathbf{x}_0 \neq 0$.

The case $d = 0$ is a limit case where the cone degenerates into a line, and the circle in question is a great circle. This would imply that all the edges of the mesh emanating from the vertex are parallel, which contradicts the assumption that we work in a mesh. Therefore, the case of $d = 0$ does not occur.

We are now going to show $1 \iff 4$. We choose a coordinate system such that $\mathbf{v} = 0$. The equation of the plane ϵ_i is $\mathbf{x} \in \epsilon_i \iff \langle \mathbf{n}_i, \mathbf{x} \rangle = 0$, where $\mathbf{n}_i$ is the oriented unit normal vector. The offset plane ϵ_i^d has the equation $\langle \mathbf{n}_i, \mathbf{x} \rangle = d$. Clearly, the k planes ϵ_i^d have a common point $\mathbf{x}_0$ if and only if the k normal vectors $\mathbf{n}_i$ satisfies the equation $\langle \mathbf{n}, \mathbf{x}_0 \rangle = d$. Assuming 1, $d \neq 0$ implies $\mathbf{x} \neq 0$, so 4 follows. Conversely, 4 implies that $\mathbf{x}_0 \in \epsilon_i^d$ for all i. For all $\lambda \neq 0$, the equations $\langle \mathbf{n}_i, \lambda \mathbf{x}_0 \rangle = \lambda d$ are equivalent. This shows that $1 \iff 2$. □

3. Convex spherical quadrilaterals

Let S be a sphere centered at a mesh vertex $\mathbf{v}$ of valence 4. Then the four faces incident with $\mathbf{v}$ cut out four circular arcs on S which form a spherical quadrilateral $Q(\mathbf{v})$. We choose units such that S is the unit sphere. Clearly, the vertex $\mathbf{v}$ is convex if and only if $Q(\mathbf{v})$ is a convex spherical quadrilateral. In this connection, the next result relates to the special case of Theorem 1 where the vertex under consideration is convex.

Theorem 3 *Suppose that a spherical convex quadrilateral with consecutive sides $e_1, \ldots, e_4$ has an incircle. Let α_i be the length of the side e_i. Then $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$. Conversely, a convex spherical quadrilateral with the property $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$ has an incircle.*

According to p. 1038 of [6], the first part of Theorem 3 together with its dual version (i.e., a convex spherical quadrilateral has a circumcircle if and only if the two sums of opposite angles are equal) is due to Anders Johan Lexell [2], and the converse is due to M. J. B. Durrande [1]. However, we found it difficult to locate recent references, and so for the sake of completeness we give a proof below. As the proof of Theorem 3 does not refer to properties of the sphere which are different from those of the Euclidean plane, this result is also true in Euclidean geometry, as well known. For brevity, we will often use *quad* for *quadrilateral.*

Proof: We begin with a convex quad that has an incircle. Suppose that the incircle touches the four sides e_i at the points $\mathbf{p}_i \in e_i$, as shown in Fig. 2a. Let $\mathbf{u}_i$ denote the vertex which is the intersection of the sides e_i and e_{i+1} (mod 4). Let $\overline{\mathbf{ab}}$ denote the spherical distance between two points $\mathbf{a}$ and $\mathbf{b}$, which is the angle of the smallest arc of a great circle on S^2 connecting $\mathbf{a}$ and $\mathbf{b}$.

Because the two sides incident with a vertex are tangents of the same incircle, we have

$$\overline{\mathbf{u}_1\mathbf{p}_1} = \overline{\mathbf{u}_1\mathbf{p}_2}, \quad \overline{\mathbf{u}_2\mathbf{p}_2} = \overline{\mathbf{u}_2\mathbf{p}_3}, \quad \overline{\mathbf{u}_3\mathbf{p}_3} = \overline{\mathbf{u}_3\mathbf{p}_4}, \quad \overline{\mathbf{u}_4\mathbf{p}_4} = \overline{\mathbf{u}_4\mathbf{p}_1}.$$

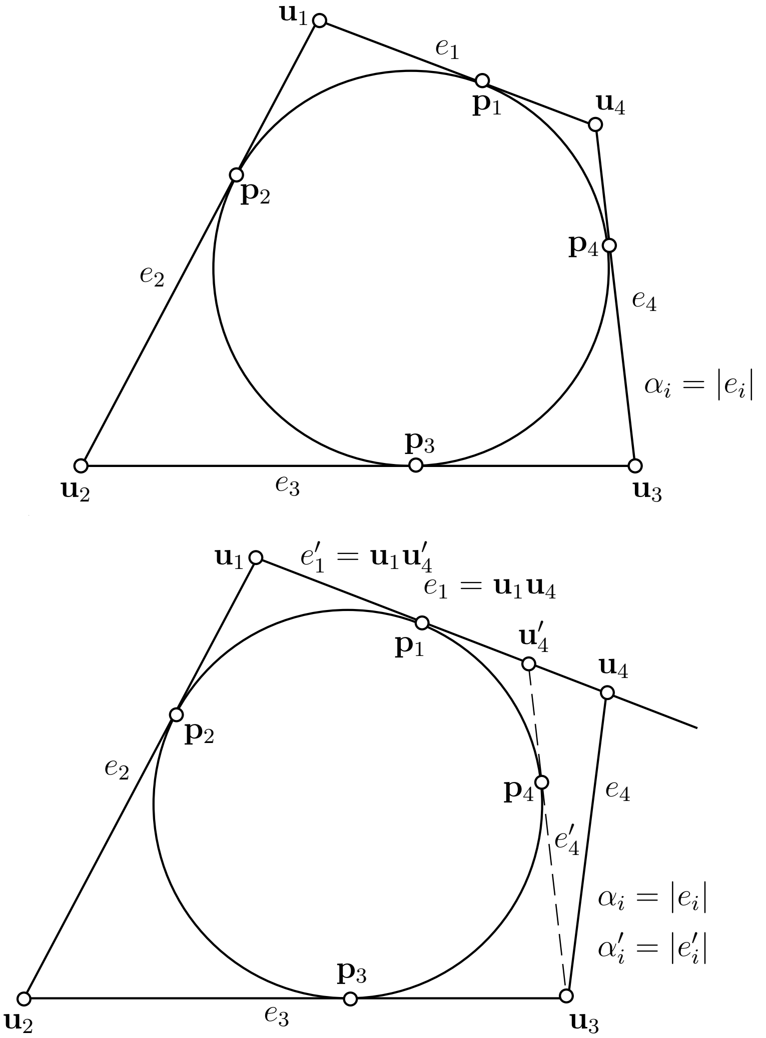

Figure 2: (a) A convex spherical quad having an incircle. This schematic illustration shows great circles as straight lines. (b) A convex spherical quad satisfying the angle criterion.

It follows that

$$\begin{aligned}\alpha_1 + \alpha_3 &= \overline{\mathbf{u}_1\mathbf{p}_1} + \overline{\mathbf{u}_4\mathbf{p}_1} + \overline{\mathbf{u}_2\mathbf{p}_3} + \overline{\mathbf{u}_3\mathbf{p}_3} \\ &= \overline{\mathbf{u}_1\mathbf{p}_2} + \overline{\mathbf{u}_4\mathbf{p}_4} + \overline{\mathbf{u}_2\mathbf{p}_2} + \overline{\mathbf{u}_3\mathbf{p}_4} = \alpha_2 + \alpha_4.\end{aligned}$$

Conversely, suppose that

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4. \tag{1}$$

We shall prove by contradiction that the quad $Q : \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4$ under consideration has an incircle. Assume that Q does not have an incircle. Consider the family of the circles that are contained in the convex quad Q and tangent to e_2 and e_3. Obviously this family either contains a circle, denoted by C, which is tangent to e_1 but not e_4, or a circle tangent to e_4 but not e_1. Without loss of generality, suppose that the former case occurs (see Fig. 2b).

Let $\mathbf{u}'_4$ be the unique point on e_1 between $\mathbf{p}_1$ and $\mathbf{u}_4$ such that the side $e'_4 = \mathbf{u}_3\mathbf{u}'_4$ is tangent to the circle C at $\mathbf{p}_4$. Then, by the first part of the proof, the convex quad $Q' : \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}'_4$ satisfies the angle criterion, i.e.,

$$\alpha'_1 + \alpha_2 = \alpha_3 + \alpha'_4, \tag{2}$$

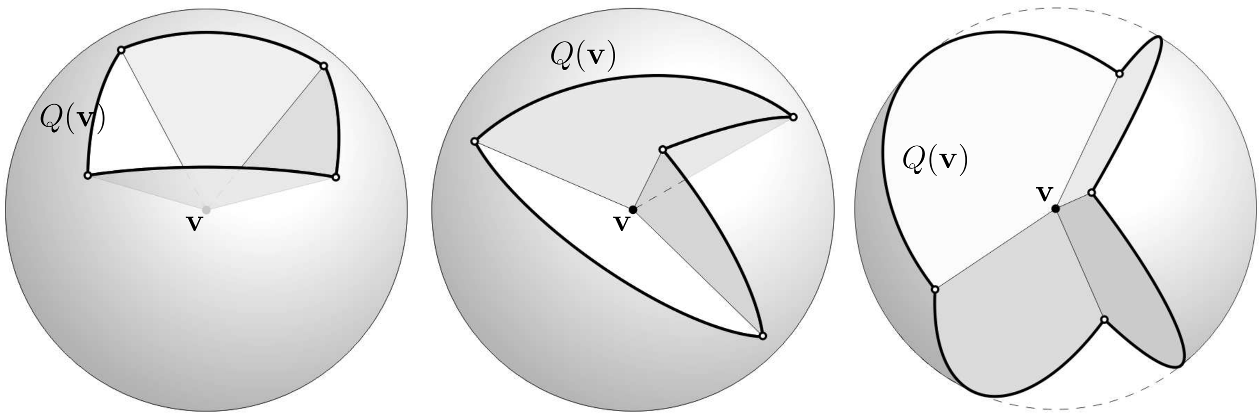

Figure 3: From left: elliptic(upper left), parabolic(upper right), and hyperbolic (lower) vertices, and their associated spherical quadrilaterals $Q(\mathbf{v})$. The vertex $\mathbf{v}$ is the center of the sphere.

where $\alpha_1' = \overline{\mathbf{u}_1\mathbf{u}_4'}$ and $\alpha_4' = \overline{\mathbf{u}_3\mathbf{u}_4'}$. Subtracting Eqn. (2) from Eqn. (1) yields

$$\alpha_1 - \alpha_1' = \alpha_4 - \alpha_4'.$$

It follows that $\alpha_4 = \alpha_4' + \alpha_1 - \alpha_1' = \alpha_4' + \overline{\mathbf{u}_4\mathbf{u}_4'}$. On the other hand, by the triangle inequality, we have $\alpha_4 < \alpha_4' + \overline{\mathbf{u}_4\mathbf{u}_4'}$. This is a contradiction, implying that the quadrilateral Q has an incircle. This completes the proof. $\square$

4. General spherical quadrilaterals

We consider in this section general conical vertices of valence 4, i.e., vertices incident with four planar faces. There are three types of mesh vertices of valence 4, as defined below.

Definition 1 *Consider a mesh vertex* $\mathbf{v}$ *of valence 4 and its associated quadrilateral* $Q(\mathbf{v})$.

1. $\mathbf{v}$ *is an* elliptic vertex *if the vertices of* $Q(\mathbf{v})$ *are contained in a hemisphere, and no vertex is contained in the spherical triangle formed by the other three vertices (note that the interior of a spherical triangle is naturally defined once we restrict ourselves to a hemisphere).*
2. $\mathbf{v}$ *is a* parabolic vertex *if it is not elliptic but the vertices of* $Q(\mathbf{v})$ *are still contained in a hemisphere.*
3. $\mathbf{v}$ *is a* hyperbolic vertex *if the four vertices of* $Q(\mathbf{v})$ *are not contained in any hemisphere.*

Examples of these three types are shown in Fig. 3. Note that $Q(\mathbf{v})$ is convex if and only if $\mathbf{v}$ is elliptic.

In the following we talk about 'oriented planes' ϵ and 'oriented circles' $S \cap \epsilon$. Generally, a two-sided curve in a 2-manifold, or a two-sided surface in a 3-manifold are oriented if one side is distinguished (which can be done e.g. by the choice of a normal vector pointing to that side). Curve segments inherit orientation from their mother curve. In particular, ϵ is oriented by distinguishing one half-space H whose boundary is ϵ. Then $H \cap S$ fills one of the two sides of $S \cap \epsilon$. This gives an orientation to the circle $S \cap \epsilon$. If we map $S \cap H$ together with $S \cap \epsilon$ under subsequent transformations (e.g. stereographic projection), we can pass on the orientation to the transformed circles.

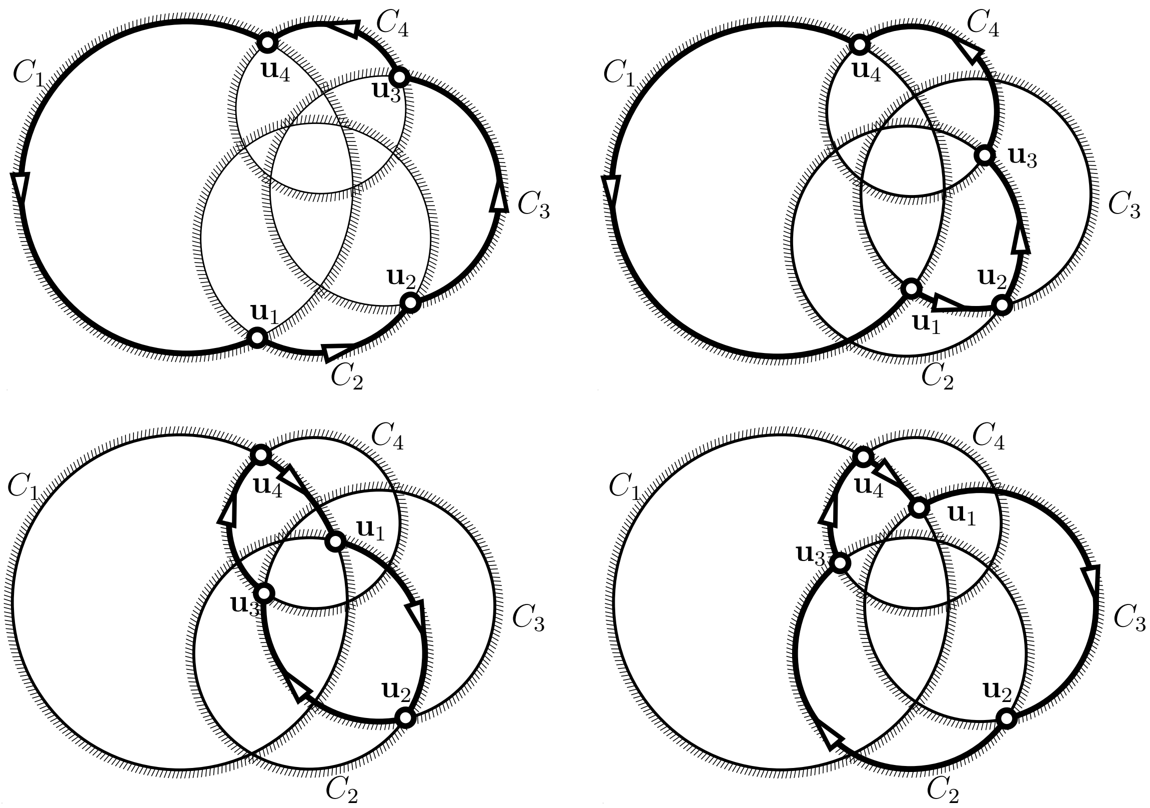

Figure 4: Stereographic images of four oriented great circles on the unit sphere. These four oriented circles correspond to four oriented planes incident with a vertex $\mathbf{v}$. The four figures show 4 of the total 12 admissible quads. Circle orientations are indicated by hatched boundaries.

The four planar faces incident with $\mathbf{v}$ have consistent normal vectors, which give rise to four oriented planes with the same normal vectors. These planes intersect the sphere S in four oriented great circles, denoted by C_i, $i = 1, 2, 3, 4$ These four circles cut each other into a number of oriented circular arcs. The two sides of each arc are distinguished as the outside and the inside, as indicated by the orientation of the plane containing the arc.

Definition 2 *A quadrilateral with sides $e_1, \ldots, e_4$ contained in the oriented great circles $C_1, \ldots, C_4$ is* admissible *if the orientations of the four sides are consistent. This means the following: The quadrilateral decomposes the unit sphere into two connected components. Then it is required that the normal vectors of the four oriented planes containing to $C_1, \ldots, C_4$, when positioned along the sides $e_1, \ldots, e_4$, point consistently towards one of these two components.*

Figure 4 shows some of admissible spherical quads. Here the sphere S is mapped onto the plane via stereographic projection, which maps the four oriented great circles on S to four oriented circles in the plane, indicated by hatched boundaries. The center of projection is chosen not to be on any of the four great circles.

By saying that an admissible spherical quad has an incircle, we mean that the four oriented planes containing the four sides of the quad are tangent to a common oriented cone. Then the following is obvious: The vertex in question is conical $\iff$ the four oriented face planes are tangent to an oriented cone $\iff$ the oriented great circles are tangent to an oriented circle $\iff$ the convex ones among the admissible quads have an incircle $\iff$ the convex

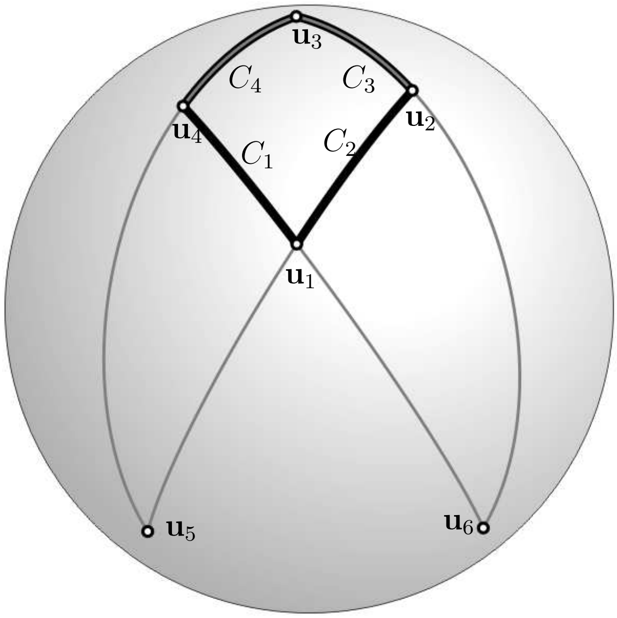

Figure 5: Converting an elliptic configuration $Q_e: \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4$ into a parabolic configuration $Q_{1,p}: \mathbf{u}_1\mathbf{u}_6\mathbf{u}_3\mathbf{u}_5$.

admissible quads satisfy the angle criterion. The last equivalence in this statement follows from Theorem 3.

The next theorem states that any admissible spherical quad has an incircle if and only if it satisfies the angle criterion.

Theorem 4 *For four oriented great circles C_1, C_2, C_3, C_4, in the unit sphere there are in total 12 admissible quadrilaterals, including reflections in the center of the unit sphere. If the four oriented planes which carry $C_1, \dots, C_4$ are tangent to a common oriented cone, then all these 12 quads satisfy the angle criterion, i.e.,*

$$\omega_1 + \omega_3 = \omega_2 + \omega_4,$$

where ω_i is the unsigned length of the i-th side of an admissible quad. Also the reverse implication is true: If any of the 12 admissible quads satisfies the angle criterion, the four oriented planes are tangent to a common oriented cone.

Proof: The four circles C_i have in total 12 pairwise intersection points, since $2 \cdot \binom{4}{2} = 2 \cdot 6 = 12$. Pick one of these 12 intersection points. Without loss of generality, suppose that this point is $\mathbf{u}_4 \in C_1 \cap C_4$. Now we count how many admissible quad contain $\mathbf{u}_4$. In view of the assumption of consistent orientations, there are two ways to choose the arcs of C_1 and C_4 which start at $\mathbf{u}_4$ and are part of an admissible quad. For each of these choices, we have either a quad with sides traversing the four circles in the order $C_1C_2C_3C_4$ or in the order $C_1C_3C_2C_4$ (see Fig. 4). Thus, there are in total 4 quads passing through $\mathbf{u}_4$. Since there are 12 pairwise intersection points among the four circles, we have counted $12 \cdot 4 = 48$ admissible quads. Since each quad has four vertices, it is counted 4 times. So the number of distinct admissible quads is $48/4 = 12$. This proves the first part of the theorem.

Obviously, there is a convex quad among the 12 admissible ones; in fact, there are two, which are reflections of each other. We are going to show that all admissible quads can be

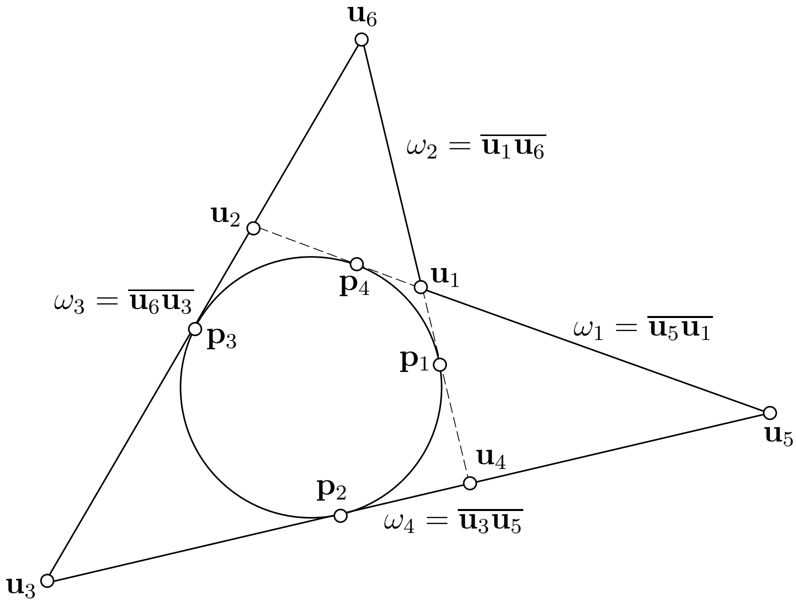

Figure 6: An admissible parabolic quad having an incircle

obtained from a convex admissible quad Q_e by operations which preserve the angle balance in both directions.

Let Q_e be a convex admissible quad with vertices $\mathbf{u}_i$, $i = 1,2,3,4$. Suppose that the sides e_i of Q_e are on the circles C_1, C_2, C_3, C_4 (in this order) and that $\mathbf{u}_i \in C_i \cap C_{i+1}$ (indices modulo 4). Then a parabolic admissible quad $Q_{1,p} : \mathbf{u}_1\mathbf{u}_6\mathbf{u}_3\mathbf{u}_5$ can be derived from Q_e by traversing the circles in the order C_1, C_3, C_4, C_2, thereby making $\mathbf{u}_1$ a concave vertex (see Fig. 5). Similarly, we can derive three other admissible parabolic quads $Q_{2,p}$, $Q_{3,p}$, and $Q_{4,p}$. By reflecting the $Q_{i,p}$ in the center of the sphere, $i = 1,2,3,4$, we obtain in total 8 parabolic quads.

Now we derive hyperbolic quads from Q_e. We replace the vertices $\mathbf{u}_1$ and $\mathbf{u}_3$ of Q_e by their diametrically opposite points $\mathbf{u}_1^*$ and $\mathbf{u}_3^*$ and arrive at the admissible quad $Q_h : \mathbf{u}_1^*\mathbf{u}_2\mathbf{u}_3^*\mathbf{u}_4$ of hyperbolic type. If we flip $\mathbf{u}_2$ and $\mathbf{u}_4$ instead, we get the quad $Q_h^* : \mathbf{u}_1\mathbf{u}_2^*\mathbf{u}_3\mathbf{u}_4^*$, which is the reflection of Q_h. In this way, 2 hyperbolic quads are derived.

Together with Q_e and its reflection Q_e^*, we have obtained 12 admissible quads, which, in view of the total number 12 shown earlier, already exhaust the set of all admissible quads. Hence, we conclude that any admissible quad can be obtained from a convex admissible quad with the above operations.

Next we are going to show that, for any of the nonconvex admissible quads obtained above, the angle criterion characterizes the property that $C_1, \ldots, C_4$ are tangent to a common oriented circle. First consider the case of parabolic admissible quads, using the quad $Q_{1,p}$: $\mathbf{u}_1\mathbf{u}_6\mathbf{u}_3\mathbf{u}_5$ in Fig. 6 for illustration. Denote the lengths of the sides of $Q_{1,p}$ by $\omega_1 = \overline{\mathbf{u}_5\mathbf{u}_1}$, $\omega_2 = \overline{\mathbf{u}_1\mathbf{u}_6}$, $\omega_3 = \overline{\mathbf{u}_6\mathbf{u}_3}$ and $\omega_4 = \overline{\mathbf{u}_3\mathbf{u}_5}$. First suppose that an incircle exists, which means that the convex quad $Q_e : \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4$ has an incircle. Using the fact that the two tangents from a vertex to a circle have equal lengths, we have

$$\begin{aligned}\omega_1 + \omega_3 &= \overline{\mathbf{u}_5\mathbf{u}_1} + \overline{\mathbf{u}_3\mathbf{u}_6} = \overline{\mathbf{p}_4\mathbf{u}_5} - \overline{\mathbf{p}_4\mathbf{u}_1} + \overline{\mathbf{u}_3\mathbf{p}_3} + \overline{\mathbf{p}_3\mathbf{u}_6} = \overline{\mathbf{p}_2\mathbf{u}_5} - \overline{\mathbf{p}_1\mathbf{u}_1} + \overline{\mathbf{u}_3\mathbf{p}_2} + \overline{\mathbf{p}_1\mathbf{u}_6}\\ &= \overline{\mathbf{p}_1\mathbf{u}_6} - \overline{\mathbf{p}_1\mathbf{u}_1} + \overline{\mathbf{u}_3\mathbf{p}_2} + \overline{\mathbf{p}_2\mathbf{u}_5} = \overline{\mathbf{u}_1\mathbf{u}_6} + \overline{\mathbf{u}_3\mathbf{u}_5} = \omega_2 + \omega_4.\end{aligned}$$

It follows that $Q_{1,p}$ satisfies the angle criterion.

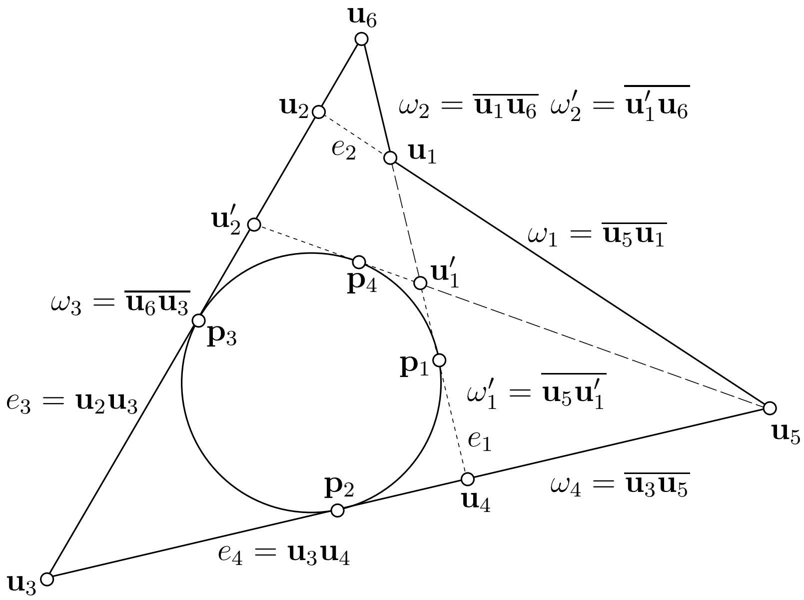

Figure 7: A parabolic admissible quad satisfying the angle criterion

Conversely, suppose that $Q_{1,p}$ satisfies the angle criterion, i.e.,

$$\omega_1 + \omega_3 = \omega_2 + \omega_4. \tag{3}$$

We shall prove by contradiction that $Q_{1,p}$ has an incircle. Assume otherwise, i.e., Q_e has no incircle. Similar to the proof of Theorem 3, consider the family of the circles that are contained in the convex quad Q_e and tangent to $e_3 = \mathbf{u}_2\mathbf{u}_3$ and $e_4 = \mathbf{u}_3\mathbf{u}_4$. Then, there is a circle C in this family that is either tangent to e_1 but not e_2 or tangent to e_2 but not e_1. Without loss of generality, we suppose the former to be the case, as shown in Fig. 7.

Let $\mathbf{u}_1'$ be the unique point on the side $\mathbf{u}_1\mathbf{p}_1$ such that the the great circle containing the side $\mathbf{u}_1'\mathbf{u}_5$ is tangent to the circle C inside the convex quad Q_e. Then the new convex quad $Q_e' : \mathbf{u}_1'\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4$ has an incircle. By the preceding argument, the new parabolic quad $Q_{1,p}' : \mathbf{u}_1'\mathbf{u}_6\mathbf{u}_3\mathbf{u}_5$ satisfies the angle criterion, that is,

$$\omega_1' + \omega_3 = \omega_2' + \omega_4, \tag{4}$$

where $\omega_1' = \overline{\mathbf{u}_5\mathbf{u}_1'}$ and $\omega_2' = \overline{\mathbf{u}_1'\mathbf{u}_6}$. Subtracting Eq. (4) from Eq. (3) yields

$$\omega_1 - \omega_1' = \omega_2 - \omega_2'.$$

It follows that

$$\omega_1' = \omega_1 + \omega_2' - \omega_2 = \omega_1 + \overline{\mathbf{u}_1'\mathbf{u}_1}.$$

On the other hand, by the triangle inequality, we have

$$\omega_1' < \omega_1 + \overline{\mathbf{u}_1'\mathbf{u}_1}.$$

This is a contradiction, implying that $Q_{1,p}$ has an incircle. It follows that the angle criterion characterizes the existence of an incircle also for parabolic quads.

Next we consider the case of hyperbolic quads. We are going to show that $Q_h : \mathbf{u}_1^*\mathbf{u}_2\mathbf{u}_3^*\mathbf{u}_4$, which is constructed from Q_e, has an incircle if and only if Q_h satisfies the angle criterion. This is easier than in the parabolic case, because the side lengths of Q_h are given by $\pi - \alpha_i$, where the α_i are the side lengths of Q_e. Hence, Q_h satisfies the angle criterion if and only if Q_e does, i.e., if and only if an incircle exists. This completes the proof. □

Remark 1: It is possible that a non-admissible quad also enjoys the angle balance, because it might be admissible for a different assignment of orientations, and so the four corresponding planes are tangent to a different oriented cone. This, however, does not diminish the value of Theorem 4 for applications if we consider admissible quads only. This is the case if the quads under consideration come from a consistently oriented quad mesh, like in [3].

Remark 2: In a quad mesh with planar faces which approximates a smooth surface and where almost all vertices have valence four, vertices are typically elliptic or hyperbolic, whereas parabolic vertices occur not as often. This is similar to the distribution of parabolic points in smooth surfaces. The fact that most of the 12 admissible quads discussed in Theorem 4 are parabolic does not contradict this behavior.

Proof of Theorem 1

Proof: The planar faces incident with a mesh vertex $\mathbf{v}$ of valence 4 are consistently oriented such that the spherical quad $Q(\mathbf{v})$ is admissible in the sense of Definition 2. The side lengths of the spherical quad $Q(\mathbf{v})$ are equal to the interior angles of the faces mentioned in the statement of Theorem 1. Hence, the proof follows from Theorem 4. □

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