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Infinitesimally Flexible Skeleta of Cross-Polytopes and Second-Hypersimplices

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Abstract. We study the infinitesimal flexibility of frameworks is R^d with graphs corresponding to the 1-skeleton of a cross-polytope, respectively second-hypersimplex, of dimension d. Both represent generalizations of the classical case of the octahedron: the former in the regular sense, resulting in 'overbraced' linkages for $d \geq 4$, and the latter in the sense of minimally rigid graphs.

Keywords: linkage, rigidity, infinitesimal flexibility, cross-polytope, second-hypersimplex

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1. Introduction

Bricard's paper on flexible octahedra [4] proved to be a most influential study about articulated systems in Euclidean space. As he observed, octahedral structures can be envisaged either as linkages with twelve bars or as hinge structures with six axes. Both perspectives lead to interesting geometric characterizations of infinitesimal flexibility [1, 5]; see also [12, 7]. The remarkable fact is that some of the infinitesimally flexible configurations are actually flexible. Bricard's identification of these possibilities has found a role in Connelly's construction of flexible polyhedral surfaces which are, topologically, embedded spheres [6].

In this paper we pursue two higher dimensional versions of Bennett's description [1] of infinitesimally flexible octahedral linkages.

The first generalization proves, in fact, a conjecture formulated by H. Stachel in [8] which refers to linkages in R^d given by the 1-skeleton of a d-dimensional cross-polytope. Although overconstrained for $d \geq 4$, the main type of infinitesimally flexible configuration retains the flavor of Bennett's result for $d = 3$. We refer to our Theorem 1 for the precise statement. The recourse to a bipartite subgraph explains the paramount role of quadrics [9].

The second generalization considers the octahedron as the three dimensional manifestation of a second-hypersimplex. The 1-skeleton of a d-dimensional second-hypersimplex is then a

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d-minimally rigid graph, as proven in Theorem 4. The paper concludes with Theorem 5 relating infinitesimally flexible configurations to certain hypersurfaces of degree $d-1$.

2. Preliminaries

In this section we recall the definition of d-minimally rigid graphs and review the principle of projective invariance of infinitesimal rigidity. For other details we refer to [3].

Definition. Let $G=(V,E)$ be a graph with $n=|V|\geq d+1$ vertices and $|E|=dn-\binom{d+1}{2}$ edges. Let $L=(\ell_{ij})_{(i,j)\in E},\ \ell_{ij}>0$ be an *edge-length vector.*
The graph G will be called *minimally rigid* in R^d if there is some edge-length vector L, for which (G,L) admits an *infinitesimally rigid realization* in R^d, that is:
there are n points: $p_1,\dots,p_n\in R^d$ with:

$$|p_i-p_j|=\langle p_i-p_j,\, p_i-p_j\rangle^{1/2}=\ell_{ij}\quad\text{for all }\ (i,j)\in E$$

and the differential of the function $(R^d)^n=R^{nd}\to R^{|E|}$ defined by squared distances:

$$x=(x_1,\dots,x_n)\mapsto\delta_E(x)=(|x_i-x_j|^2)_{(i,j)\in E}$$

has maximal rank at $p=(p_1,\dots,p_n)$ i.e.

$$\operatorname{rank}(D\delta_E(p))=|E|=dn-\binom{d+1}{2}$$

In general, a *tangent vector* $v=(v_1,\dots,v_n)\in(R^d)^n=R^{nd}$ which lies in the kernel of

$$D\delta_E(x)\colon\ R^{nd}\to R^{|E|}$$

will be called an *infinitesimal motion* of the realization $G(x)$.

Note that the rank condition at p means that the fiber of δ_E passing through p is locally smooth of dimension $\binom{d+1}{2}$ and therefore obtained by moving the realization p with Euclidean isometries in a neighborhood of the identity.

Indeed, an infinitesimally rigid realization cannot be contained in any proper affine subspace $R^r\subset R^d$ since this would ensure at least:

$$\binom{r+1}{2}+n(d-r)>\binom{d+1}{2}$$

independent infinitesimal motions for the realization $G(p)$: at least $\binom{r+1}{2}$ in the proper subspace, together with $n(d-r)>d(d-r)$ choices in the orthogonal directions. Hence, the isometries fixing $G(p)$ are isolated.

Thus, an infinitesimally rigid realization $G(p)$ has only *trivial infinitesimal motions* (induced from ambiental rigid motions and thereby identifiable with the Lie algebra of the Euclidean group). Locally, (G,L) has no other realization, modulo rigid motions, but $G(p)$: it is *rigid.*

Remark: The proof that, for graphs, minimal rigidity in dimension d is independent of the context (Euclidean or non-Euclidean) in which one defines it, follows from a sequence of elementary lemmas:

(1) shows that the infinitesimal rigidity of a framework $G_n(p)$ in R^d is equivalent to infinitesimal rigidity of a framework $(G_{n+1}(p_0, p)$ in R^{d+1} obtained by introducing a new vertex $p_0 \in R^{d+1}$ away from R^d, and putting new bars to all vertices of $G_n(p)$ i.e. $p_1, \dots, p_n$;

(2) shows that one can move p_i so that $p_i - p_0$ changes to $\lambda_i(p_i - p_0)$, $\lambda_i \neq 0$, and infinitesimal rigidity is still preserved;

(3) shows that, in the spherical case, we preserve infinitesimal rigidity when we take a vertex (with its connections) to its antipode; and

(4) concludes the argument by using the fact that on any bounded region, the metric can be analytically deformed from spherical to hyperbolic.

(1) and (2) above may be found in [10] and prove the projective invariance of infinitesimal rigidity for linkages. In fact, what remains invariant under projective transformations is the dimension of the space of infinitesimal flexes, which is defined as the space of infinitesimal motions modulo trivial infinitesimal motions. A linkage configuration is called *infinitesimally flexible* or *shaky* when it has a non-trivial space of infinitesimal flexes. Accordingly, our results on infinitesimally flexible configurations will have a projective character.

3. The 1-skeleton of the cross-polytope

We define the *cross-polytope* $\diamond_d$ in dimension d by:

$$\diamond_d = \mathrm{Conv}\{\pm e_i \mid i = 1, \dots, d\} \subset R^d$$

Its 1-skeleton is a graph $\mathcal{C}_d$ with $2d$ vertices and $4\binom{d}{2}$ edges, so that for $d \geq 4$ it gives linkages 'overbraced' by:

$$4\binom{d}{2} - \left[2d^2 - \binom{d+1}{2}\right] = 2d(d-1) - \frac{d(3d-1)}{2} = \frac{d(d-3)}{2}$$

This difference gives the excess number of constraints on $2d$ points in R^d when prescribing distances on pairs corresponding to the edges of a cross-polytope, by comparison with the number of constraints required for minimal rigidity.

Our aim will be to generalize a result obtained by BENNETT for infinitesimally flexible octahedra in R^3 [1], in the form already conjectured by STACHEL for $d > 3$ in [8]. Considerations in [11], although focused for the better part on $d = 3$, suggest the same type of extension and principle of proof.

This approach involves a natural decomposition of the edge set of $\mathcal{C}_d$ into a disjoint union of the edge sets of two subgraphs of cross-polytopes (of nearly or exactly half-dimension) and the bipartite graph on their respective vertices:

$$\mathcal{C}_{[\frac{d+1}{2}]}(A),\ \mathcal{C}_{d-[\frac{d+1}{2}]}(B),\ K(A,B)$$

For a precise description, we use the above standard realization of the cross-polytope $\diamond_d$ and consider the orthogonal decomposition of R^d determined by a *choice* of $[(d+1)/2]$ (that is: $d/2$ for even dimension d, or $(d+1)/2$ for odd dimension d) vectors in the standard basis e_i; $i = 1, \dots, d$. The cross-polytope traced in their span is denoted $\diamond_{[(d+1)/2]}(A)$ and the one traced in the orthogonal complement is denoted $\diamond_{d-[(d+1)/2]}(B)$.

A and B should be understood as labels for the vertices of these two cross-polytopes, and when given a realization of $\mathcal{C}_d$ as a framework in R^d, we assume that A and B consist of the corresponding list of (marked) vertex coordinates. The bipartite graph $K(A, B)$ has an edge between any vertex $a \in A$ and $b \in B$. This will match notations in [2] and thereby simplify reference to their results.

Note that the cardinalities are:

$$|A| = |B| = d, \quad \text{for even dimension } d$$

$$|A| = d + 1, \ |B| = d - 1, \quad \text{for odd dimension } d$$

A decomposition of $\mathcal{C}_d$ as described above will be called simply an (A, B) *decomposition.*

Theorem 1 *Consider a realization in R^d of the 1-skeleton $\mathcal{C}_d$ of the cross-polytope $\diamondsuit_d$, with $d \geq 3$. Suppose one can find an (A, B) decomposition such that:*

(0) the affine span $\bar{A}$ of A is a hyperplane in R^d, and the affine span $\bar{B}$ of B is a distinct hyperplane, for even *dimension d; or*

(1) the affine span $\bar{A}$ of A is the whole space R^d, for odd *dimension d.*

Then, the given framework is infinitesimally flexible if and only if there's a quadric passing through all vertices, and containing all edges of $\mathcal{C}_{[\frac{d+1}{2}]}(A)$ and $\mathcal{C}_{d-[\frac{d+1}{2}]}(B)$, but distinct from the rank-two quadric $\bar{A} \cup \bar{B}$ in the even dimensional case.

Proof: The proof relies on a very precise result of Bolker and Roth on the dimension of the space of self-stresses for a bipartite framework [2]. We rephrase it by using the notions of *linear defect* $\delta_1(V)$ and *quadratic defect* $\delta_2(V)$ for a set V of marked points in R^d or, just the same, its projective completion P_d.

Let $V = \{v_1, \ldots, v_n\}$, where the marked points $v_i \in P_d$ need not be all distinct. Let $H(V)$ denote the linear space consisting of all linear forms vanishing on V, and $Q(V)$ the linear space of all quadratic forms vanishing on V, that is:

$$H(V) = \{h(x) \mid h(x) = \sum_{k=0}^{d} h_k x_k, \ h(v) = 0 \ \text{for all} \ v \in V\}$$

$$Q(V) = \{q(x) \mid q(x) = \sum_{i,j} q_{ij} x_i x_j, \ q(v) = 0 \ \text{for all} \ v \in V\}$$

The linear defect of V, with cardinality $|V| = n$ is defined as:

$$\delta_1(V) = \dim H(V) - [(d + 1) - n]$$

and the quadratic defect is defined as:

$$\delta_2(V) = \dim Q(V) - \left[\binom{d+2}{2} - n\right]$$

where dim stands for the dimension of the respective vector space. In other words, the linear and quadratic defects measure to what extent the points in V fail to impose independent conditions on hyperplanes, respectively quadrics, passing through them.

When in R^d, $\delta_1(V)$ is nothing else but the dimension of the space of all affine dependencies amongst the points in V.

Now we can state the main result in [2].

Formula of Bolker and Roth: *Consider a realization in R^d of the bipartite graph $K(A,B)$, with the two sets of vertices A and B seen as marked points in R^d.*
Put $C = (\bar{A} \cap B) \cup (A \cap \bar{B})$, where $\bar{A}$ denotes the affine span of A, and similarly for B.
Then the dimension $\omega = \omega(A,B)$ of the space of self-stresses for the given framework is:

$$\omega(A,B) = \delta_1(A)\delta_1(B) + \delta_2(C) \qquad (BR)$$

Specializing *(BR)* to the particular type of bipartite graph $K(A,B)$ admitted by the assumptions of our statement, we find $\delta_1(A) = 0$ and therefore:

$$\omega(A,B) = \delta_2(C)$$

If we denote by $V = A \bigsqcup B$ the points marking all the vertices of our framework $K(A,B)$ (and also $\mathcal{C}_d$), we have:

$$C \subset V \;\Rightarrow\; \delta_2(C) \leq \delta_2(V)$$

and hence:

$$\dim Q(V) \geq \delta_2(C) + \left[\binom{d+2}{2} - 2d \right] = \delta_2(C) + \binom{d}{2} + 1$$

Let now $f = f(A,B)$ denote the dimension of the space $F(A,B)$ of infinitesimal flexes for the framework $K(A,B)$ i.e. infinitesimal motions modulo rigid infinitesimal motions. Infinitesimal flexes and self-stresses are related by the general formula:

$$f = \omega + \left[d|V| - |E| - \binom{d+1}{2} \right]$$

where for $K(A,B)$ the number of edges $|E| = |A| \cdot |B|$. In our case, according to the parity of d we find:

(0) $f = \omega + [d^2 - \binom{d+1}{2}] = \omega + \binom{d}{2}$ for even d; and
(1) $f = \omega + [d^2 + 1 - \binom{d+1}{2}] = \omega + \binom{d}{2} + 1$ for odd d.

Now, we relate the space $Q(V)$ of quadrics through the vertices of $K(A,B)$ and the space $F(A,B)$ of its infinitesimal flexes. Indeed, as observed in [11], when given a quadric through $V = A \bigsqcup B$, there's a natural way to associate an infinitesimal motion to it.

A simple way to define this map $Q(V) \to F(A,B)$ is to consider the 'projective' reading of our set-up: we consider R^d as the hyperplane $x_0 = 1$ in R^{d+1}, link all vertices of $K(A,B)$ to the origin of R^{d+1} producing the framework $PK(A,B)$.

As already indicated [10], the infinitesimal flexes of $K(A,B)$ in R^d correspond isomorphically with those of $PK(A,B)$ in R^{d+1}. Moreover, the affine quadrics in R^d can be seen as quadratic forms $\langle x, Qx \rangle = x^t Q x$ associated to symmetric $(d+1) \times (d+1)$ matrices $Q = Q^t$. With this interpretation, we have:

Lemma 2 *Let $Q \in Q(V)$ be a quadratic form vanishing on $V = A \bigsqcup B$. Then the following assignment of velocities at the vertices of $PK(A,B)$ defines an infinitesimal motion:*

$$0 \mapsto 0, \quad a \mapsto Qa \text{ for all } a \in A, \quad b \mapsto -Qb \text{ for all } b \in B$$

The induced map: $Q(V) \to F(A,B)$ is surjective.

Proof: The fact that we have an infinitesimal motion is a consequence of the obvious identities:

$$\langle Qv, v\rangle = 0 \text{ for all } v \in V, \ \langle Qa + Qb, a - b\rangle = 0 \text{ for all } a \in A, b \in B$$

For surjectivity, we observe that, under our assumptions, we have already established:

(0) $\dim Q(V) \geq \omega + \binom{d}{2} + 1 = \dim F(A,B) + 1$ for even d; and

(1) $\dim Q(V) \geq \omega + \binom{d}{2} + 1 = \dim F(A,B)$ for odd d.

When we consider the kernel of our map, we see that if Q induces an infinitesimal rigid motion (actually an infinitesimal rotation), we must have:

$$\langle Qa, a'\rangle = 0 \text{ for all } a, a' \in A, \ \langle Qb, b'\rangle = 0 \text{ for all } b, b' \in B$$

This means that the quadric contains the span $\bar{A}$ of A, as well as the span $\bar{B}$ of B. For odd d, this requires $Q = 0$, while for even d it requires $Q = \alpha \cdot \beta^t + \beta \cdot \alpha^t$, with vector α orthogonal to the linear span of A and vector β orthogonal to the linear span of B. Indeed, when non-zero, such a rank two quadric induces the infinitesimal rotation given by the skew-symmetric matrix $\alpha \cdot \beta^t - \beta \cdot \alpha^t$.

Thus, the kernel is zero for odd d and of dimension one for even d, proving not only the claimed surjectivity, but also the more precise relations:

(0) $\dim Q(V) = \omega + \binom{d}{2} + 1 = \dim F(A,B) + 1$ for even d; and

(1) $\dim Q(V) = \omega + \binom{d}{2} + 1 = \dim F(A,B)$ for odd d. □

Now, we can conclude the proof of the theorem:

If the given realization of $\mathcal{C}_d$ has a non-trivial infinitesimal flex, we can use the assumed decomposition (A,B), and find a quadric through the vertices of $\mathcal{C}_d$ which induces, as in the above lemma, the given infinitesimal flex considered on $K(A,B)$.

Since the flex is valid for the edges of the cross-polytopes $\mathcal{C}_{[\frac{d+1}{2}]}(A)$ as well as $\mathcal{C}_{d-[\frac{d+1}{2}]}(B)$, the quadric contains the lines defined by all such edges. In the even dimensional case, it is also obvious that the quadric must be different from $\bar{A} \cup \bar{B}$, which induces a trivial infinitesimal flex.

Conversely, if there is a quadric, other then $\bar{A} \cup \bar{B}$ in the even dimensional case, passing through all vertices and containing the lines defined by all edges in $\mathcal{C}_{[\frac{d+1}{2}]}(A)$ and $\mathcal{C}_{d-[\frac{d+1}{2}]}(B)$, then, the non-trivial infinitesimal flex determined for $K(A,B)$ is in fact good for $\mathcal{C}_d$. □

Remark: In *odd dimension* $d = 2k+1$, an affinely independent A as in the theorem corresponds to a generic realization of $\mathcal{C}_{k+1}$ in $R^{2k+1} \subset P_{2k+1}$. Obviously, any two such generic realizations are *projectively equivalent.*

Using the projective language, we may observe further that the family of quadrics passing through the vertices of a generic realization of $\mathcal{C}_{k+1}$ in P_{k+1} and containing its edges must contain in fact all $P_k \subset P_{2k+1}$ supporting facets of $\diamond_{k+1}$.

There are 2^{k+1} such k-dimensional projective subspaces in P_{2k+1} in a configuration generalizing the case of a *skew-quadrilateral* in P_3 for $k=1$. The pencil of quadric surfaces through the skew-quadrilateral becomes, for arbitrary k, a family of quadrics parametrized by P_k, with any independent subset of $k+1$ quadrics therein intersecting exactly in the given configuration of P_k's (of degree 2^{k+1}).

Indeed, we impose $2(k+1)+4\binom{k+1}{2} = 2(k+1)^2$ conditions when requesting a quadric to pass through the edges of $\mathcal{C}_{k+1}(A)$, and this leaves a linear space of dimension:

$$\binom{2k+3}{2} - 2(k+1)^2 = k+1$$

i.e. a P_k family of quadrics.
A simple way to describe this family relies on a natural basis consisting of rank-two quadrics. Using the notation $a(\pm e_i)$, $i = 1, \ldots, k+1$ for the vertices of the given generic realization of $\diamond_{k+1}$ in P_{2k+1}, we observe that each pair of vertices: $a(e_i)$, $a(-e_i)$ determines a rank-two quadric:

$$q_i = \mathrm{span}[a(e_i), a(\pm e_j),\ j \neq i] \cup \mathrm{span}[a(-e_i), a(\pm e_j),\ j \neq i]$$

This gives $P_k = \mathrm{span}[q_i,\ i = 1, \ldots, k+1]$.

If we use the vertices $a(\pm e_i)$, $i = 1, \ldots, k+1$ as a reference simplex in P_{2k+1}, with $a(e_i)$ related to x_i and $a(-e_i)$ to x_{k+1+i}, then our P_k family of quadrics corresponds to *second diagonal symmetric* $(2k+2) \times (2k+2)$ *matrices.* This standard realization yields:

Proposition 3 *The only projective invariant of a quadric in P_{2k+1} passing through all vertices and edges of a generic realization of $\mathcal{C}_{k+1}$ in $R^{2k+1} \subset P_{2k+1}$ is its rank, which is an even number.*

In other words, given two quadrics of the same (even) rank passing through all vertices and edges of a generic realization of $\mathcal{C}_{k+1}$ in $R^{2k+1} \subset P_{2k+1}$, there is a projective transformation which takes the (image of the) cross-polytope skeleton $\mathcal{C}_{k+1}$ to itself and one quadric onto the other. □

Remark: This discussion shows that in order to produce infinitesimally flexible configurations which satisfy the assumption of our theorem for the odd dimensional case $d = 2k+1$, we may use a projective transformation and identify the realization of $\mathcal{C}_{k+1}(A)$ with the one given by $a(e_i) = e_i$, $a(-e_i) = e_{k+1+i}$; $i = 1, \ldots, k+1$, then choose a symmetric $((2k+2) \times (2k+2)$ matrix with zeroes away from the second diagonal, and then place $\mathcal{C}_k(B)$ with its vertices and edges on this quadric.
If we follow this scenario for the case of a non-degenerate quadric Q, which, according to our proposition, we may choose to have 1 along the second diagonal, we see that placing $\mathcal{C}_k(B)$ with its vertices and edges on $Q_{2k} \subset P_{2k+1}$ amounts to $4\binom{k}{2}$ equations on $(Q_{2k})^{2k}$.
The solution space has dimension (no less than) $4k^2 - 4\binom{k}{2}$. We have a natural action of a $(k+1)$-dimensional torus, corresponding to projective transformations

$$e_i \mapsto \lambda_i e_i, \quad e_{k+1+i} \mapsto \lambda_i^{-1} e_{k+1+i}, \quad i = 1, \ldots, k+1$$

which keep invariant both the fixed realization $\mathcal{C}_{k+1}(A)$ and the fixed quadric through it. This leaves us with a moduli space of dimension (no less than):

$$4k^2 - 4\binom{k}{2} - (k+1) = 2k^2 + k - 1$$

for projective equivalence classes of infinitesimally flexible configurations of cross-polytope skeleta $\mathcal{C}_{2k+1}$ in $R^{2k+1} \subset P_{2k+1}$.
This is perfectly consistent with the fact that there are:

$$d(2d) - [(d+1)^2 - 1] = d(d-2)$$

moduli for projective equivalence classes of realizations of $\mathcal{C}_d$ in $R^d \subset P_d$, and $d(d-3)/2$ overbracing conditions. Indeed, for $d = 2k+1$, we find:

$$d(d-2) - \frac{d(d-3)}{2} = \frac{d(d-1)}{2} = 2k^2 + k$$

Under no overbracing ($d = 3$), the codimension of the infinitesimally flexible configurations is one. Thus, we see consistency with the expectation that overbracing increases the codimension of the infinitesimally flexible locus by the number of (independent) conditions.

4. The 1-skeleton of the second-hypersimplex

We'll prove here the *infinitesimal rigidity* of the framework defined by the *1-skeleton of the second-hypersimplex* $Q_d = \Delta(2, d+1)$ in its standard realization in R^d, and then study infinitesimally flexible configurations.

The standard realization of the *second-hypersimplex* Q_d is:

$$Q_d = \left\{ x \in [0,1]^{d+1} \subset R^{d+1} \;:\; \sum_{i=0}^{d} x_i = 2 \right\} = \mathrm{Conv}(e_i + e_j \;;\; 0 \leq i < j \leq d)$$

or, rescaled by $1/2$, the convex hull of the midpoints of edges in the standard d-(hyper)simplex $\Delta_d = \Delta(1, d+1)$.

The first description makes clear that there are $(d+1)$ facets $x_i = 0$, which are *second-hypersimplices* Q_{d-1}, and $(d+1)$ facets $x_i = 1$, which are *simplices* Δ_{d-1}.

We observe that Q_d has $\binom{d+1}{2}$ vertices and $(d-1)\binom{d+1}{2} = 3\binom{d+1}{3}$ edges, and infinitesimal rigidity amounts to showing that all infinitesimal motions are induced from rigid motions. Since we can use the latter to 'fix' say the simplicial facet $x_0 = 1$, it is equivalent to prove that an infinitesimal motion vector (v_{ij}) with $v_{0i} = 0, 1 \leq i \leq d$ is actually zero.

Our proof will proceed by induction on $d \geq 3$.
For $d = 3$, Q_3 is the *regular octahedron.* It will be proper to see half of its faces as $\Delta(2,3) = \Delta_2$ and the other half as $\Delta(1,3) = \Delta_2$. Of course, in dimension three, infinitesimal rigidity is a consequence of the classical Legendre-Cauchy-Dehn theorem for convex polyhedra, but an explicit argument adapted to our setting can be given as follows:

for an infinitesimal displacement vector v_{ij} applied at vertex $p_{ij} = e_i + e_j$, $1 \leq i < j \leq 3$, we must have:

$$v_{ij} \perp p_{ij} - p_{0i} = e_j - e_0 \text{ and } v_{ij} \perp p_{ij} - p_{0j} = e_i - e_0,$$

Besides, the orthogonal projections $P(v_{ij}) = w_{ij}$ of these vectors on the facet $x_0 = 0$ must represent an infinitesimal motion of that (equilateral) triangle in its plane. However, no such infinitesimal motion other than zero would have its vectors along the three angle bisectors, as is the case with w_{ij}. Thus, v_{ij} must be perpendicular to $x_0 = 0$ as well, and this makes all of them zero. $\square$

For $d > 3$, note that each vertex p_{ij}, with $1 \leq i < j \leq d$ is a vertex of $(d-2) > 1$ facets Q_{d-1} given by $x_k = 0$, for $k \neq 0, i, j$, and all their vertices in the simplex $x_0 = 1$ are fixed. By induction, all these facets correspond to frameworks infinitesimally rigid in their supporting hyperplanes, which means: the orthogonal projections of v_{ij} on all such must be zero. Since we have at least two independent projections, $v_{ij} = 0$. This proves:

Theorem 4 *The linkage given by the edges of the standard second-hypersimplex Q_d in dimension d is infinitesimally rigid, and hence the graph defined by the 1-skeleton of Q_d is d-minimally rigid.*

We shall denote by Γ_d the graph given by the 1-skeleton of the second-hypersimplex $Q_d = \Delta(2, d+1)$.

It may be convenient to think of a framework in R^d which represents Γ_d, for some edge-length vector, as an *assemblage* of $(d+1)$ simplices Δ^i_{d-1}, where any two of them have a common vertex:

$$p_{ij} \in \Delta^i_{d-1} \cap \Delta^j_{d-1}$$

In order to investigate the configurations which would make the framework *infinitesimally flexible*, we'll make use of the equivalences mentioned in (1) and (2) above (which express at the same time the *projective invariance* of the property of infinitesimal rigidity/mobility) and consider instead the framework in R^{d+1} with one vertex at the origin connected with $p_{ij} \neq 0$, $0 \leq i < j \leq d$, with the latter connected according to Γ_d.

Keeping the vertex at the origin fixed, a non-trivial infinitesimal motion amounts to finding a solution *other than* all $A_i = A$ for the system:

$$A_i p_{ij} = A_j p_{ij} \quad 0 \leq i < j \leq d \qquad (S)$$

where A_i denote *skew-symmetric* $(d+1) \times (d+1)$ *matrices*, corresponding to the fact that (with 0 fixed), every d-simplex $(0, \Delta^i_{d-1})$ can move infinitesimally only by such an A_i.

Every pair $i < j$ in (S) gives $(d+1)$ scalar equations which always have the linear dependency resulting from:

$$\langle (A_i - A_j)p_{ij},\ p_{ij} \rangle = 0$$

Let us consider *all vertices but one*, say $x = p_{01}$, fixed (in some generic configuration) and inquire which positions for x correspond to infinitesimal flexibility.

We may put $A_0 = 0$ (fix the d-simplex $(0, \Delta^0_{d-1})$) and look for a non-trivial solution of (S), where, for each pair $i < j$ one 'dependent' scalar equation (see the special considerations for 01 below) has been dropped. Clearly, the determinant of the matrix of coefficients is a homogeneous polynomial of degree d in the coordinates $x_0, \ldots, x_d$ of the "floating" vertex $x = p_{01}$. However, when dropping one equation from the group:

$$-A_1 x = 0$$

say, the k^{th}, we still have a dependency amongst the rest for $x_k = 0$, that is, the determinant has x_k as a factor.

Thus, the true locus for x where there are non-trivial solutions is the locus defined by the vanishing of the *degree* $(d-1)$ homogeneous polynomial obtained after dividing the determinant by its factor x_k.

On the other hand, we may recognize directly some situations where the framework is surely shaky: indeed, when x is in either of the two codimension two subspaces spanned by p_{0k}, $k = 2, \ldots, d$, or p_{1k}, $k = 2, \ldots, d$, we have a degenerate simplex which has a non-trivial self-stress, extending (by zero) to a non-trivial self-stress of the whole framework. Also, when x is in the subspace spanned by p_{0k} and p_{1k}, (for each $k \neq 0, 1$) there's an obvious non-trivial self-stress.

We can reformulate our discussion for the originally intended set-up, namely realizations of Γ_d in R^d, yet retaining the projective character of the matter by conceiving the latter as extended to P_d.

Theorem 5 *Suppose all vertices of* Γ_d *except* $x = p_{01}$ *are assigned (sufficiently general) positions in* $R^d \subset P_d$. *Then, the positions of* x *corresponding to infinitesimally flexible realizations of* Γ_d *are the points of a hypersurface of degree* $(d-1)$ *which contains the two* $(d-2)$*-planes spanned by* p_{0k}, $k = 2, \ldots, d$, *respectively* p_{1k}, $k = 2, \ldots, d$, *and all the lines connecting* p_{0k} *and* p_{1k}, *for* $k \neq 0, 1$.

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