Journal for Geometry and Graphics
Volume 12 (2008), No. 1, 11–21.

A Method for Establishing Trigonometric Inequalities Involving the Cotangents of the Angles of a Triangle

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Abstract. In this paper, we find relations that define the cotangents of the angles of a general triangle, and we use these relations to describe a method for optimizing symmetric functions in the cotangents of such angles and for establishing trigonometric inequalities that involve such functions.

Keywords: Geometric inequalities, extrema, trigonometric inequalities, trigonometric identities, cotangents, Brocard angle, discriminant

MSC 2007: 51M16, 52B10

1. Introduction

In [9], the problem of finding the equifacial tetrahedron for which the sum of the contents of the corner angles is maximal was solved by reducing it to maximizing a certain symmetric rational function in $\cos A$, $\cos B$, and $\cos C$, where A, B, and C are the angles of an arbitrary acute triangle; see the introduction of [7] for details. This led the author of [7] to look for a systematic method for optimizing symmetric functions in $\cos A$, $\cos B$, and $\cos C$, where A, B, and C are the angles of an acute, or general, triangle. The method he obtained is described in Section 4 of [7], and some of the natural questions raised by that method are discussed in the last section of that paper. An immediate issue is to investigate and record methods that work for the remaining trigonometric functions. Clearly, it is sufficient to consider the sines and cotangents. As mentioned in [7], the sines would offer more challenge since the algebraic relation among the sines of the angles A, B, and C of a general triangle ABC is of degree 4, in contrast to degree 2 for the cosines. Explicitly, if

$$u = \cos A, \quad v = \cos B, \quad w = \cos C, \quad U = \sin A, \quad V = \sin B, \quad W = \sin C,$$

then the algebraic relations that govern the triples (u, v, w) and (U, V, W) are given by

$$\begin{aligned} H_1 &= u^2 + v^2 + w^2 + 2uvw - 1 = 0 \\ H_2 &= (U^4 + V^4 + W^4) - 2(U^2V^2 + V^2W^2 + W^2U^2) + 4U^2V^2W^2 = 0; \end{aligned}$$

ISSN 1433-8157/\$ 2.50

see [6]. On the other hand, the relation among the cotangents is given by the simple and elegant formula[1]

$$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1. \tag{1}$$

This follows from the identity

$$\cot(A + B + C) = \frac{(\cot A + \cot B + \cot C) - \cot A \cot B \cot C}{1 - (\cot A \cot B + \cot B \cot C + \cot C \cot A)} \tag{2}$$

which is easy to prove. Usually, (2) appears in the equivalent form

$$\tan(A + B + C) = \frac{(\tan A + \tan B + \tan C) - \tan A \tan B \tan C}{1 - (\tan A \tan B + \tan B \tan C + \tan C \tan A)}$$

and generalizes elegantly to any number of angles; see [4, Formulas 676 (p. 165) and 763 (p. 175)] and [11, § 125, p. 132].

Using this relation and the methods in [7], we obtain a method for optimizing certain symmetric functions in $\cot A$, $\cot B$, and $\cot C$. Although the treatment is similar to that given in [7] for the cosines, we feel that the details and results are worth recording for ease of reference.

2. Relations that define the cotangents of the angles of a triangle

Theorem 1 below characterizes triples of real numbers that can serve as the cotangents of the angles of a triangle. For the proof, we use the following simple lemma.

Lemma 1 *Let a, b, and c be real numbers such that $ab + bc + ca = 1$. Then*

$$(a + b + c)^2 \;\geq\; 3. \tag{3}$$

If $a + b + c > 0$, then

$$abc \;\leq\; \frac{1}{3\sqrt{3}}. \tag{4}$$

Proof: It follows from

$$2(a + b + c)^2 = (a - b)^2 + (b - c)^2 + (c - a)^2 + 6(ab + bc + ca) \tag{5}$$

that

$$(a + b + c)^2 \;\geq\; 3(ab + bc + ca). \tag{6}$$

Replacing a, b, and c by bc, ca, and ab, we obtain

$$(ab + bc + ca)^2 \;\geq\; 3abc(a + b + c). \tag{7}$$

The desired inequalities follow immediately from (6) and (7). □

[1] The equivalent formula $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ is one of the trigonometric *gems* listed in [12, Appendix 3, pp. 220–221]. According to the author Eli Maor, this and similar formulas *would appeal to anyone's sense of beauty.*

Theorem 1 *Let α, β, and γ be real numbers and let*

$$u = \alpha + \beta + \gamma, \quad v = \alpha\beta + \beta\gamma + \gamma\alpha, \quad w = \alpha\beta\gamma. \tag{8}$$

Then the following are equivalent.

(i) *There exists a triangle ABC such that $(\alpha, \beta, \gamma) = (\cot A, \cot B, \cot C)$,*

(ii) $v = 1$, $u \geq \sqrt{3}$, *and* $w \leq \dfrac{1}{3\sqrt{3}}$,

(iii) $v = 1$ *and* $u > -\sqrt{3}$.

The triangle ABC is acute, right, or obtuse according as w is greater than, equal to, or less than 0.

Proof: Suppose that (i) holds, and let u, v, and w be as defined in (8). To prove (ii), it is enough, in view of Lemma 1, to show that $v = 1$ and $u > 0$. That $v = 1$ follows from (2) by rewriting it as

$$\cot(A + B + C) = \frac{u - w}{1 - v}. \tag{9}$$

To prove that $u > 0$, suppose that $A \leq B \leq C$ and thus A and B are acute and $\cot A$ and $\cot B$ are positive. Then

$$\begin{aligned} \cot A + \cot B + \cot C &= \cot A + \cot B - \cot(A + B) \\ &= \cot A + \cot B + \frac{1 - \cot A \cot B}{\cot A + \cot B} \\ &= \frac{1 + \cot A \cot B + \cot^2 A + \cot^2 B}{\cot A + \cot B} > 0\,, \end{aligned} \tag{10}$$

as desired. Thus we have proved that (i) implies (ii). Since (ii) implies (iii) trivially, it remains to prove that (iii) implies (i).

Suppose (iii) holds. By (3), the assumption $u > -\sqrt{3}$ implies that $u > 0$ (in fact, $u \geq \sqrt{3}$). Let $A = \cot^{-1}\alpha$, $B = \cot^{-1}\beta$, and $C = \cot^{-1}\gamma$. We need only show that $A + B + C = \pi$.

It follows from (9) and the assumption $v = 1$ that $A + B + C$ is a multiple of π. Since each of A, B, and C lies in the open interval $(0, \pi)$, it follows that their sum is either π or 2π. Suppose that $A+B+C = 2\pi$, and assume, without loss of generality, that $A \geq B \geq C$. Then A and B are obtuse, and $\cot A + \cot B + \cot C < 0$ by (10). This contradicts the assumption that $u > 0$, and leads to $A + B + C = \pi$, as desired. □

Remark 1. The implication (i) $\Longrightarrow$ (ii) in Theorem 1 is not new; see for example [15]. A form of the implication (iii) $\Longrightarrow$ (i) has appeared as Problem 21 (page 66) of [2], where the condition on u is unintentionally omitted. One of the authors of [2] wrote to one of us [1] that the problem was intended to say that α, β, and γ (and hence u) are positive. In this regard, we should mention that the assumption $u > -\sqrt{3}$ in Theorem 1 (iii) is irredundant. To see this, take

$$(\alpha, \beta, \gamma) = \left(-\sqrt{3},\ -\sqrt{3},\ \frac{1}{\sqrt{3}}\right).$$

Although $v = 1$, the corresponding angles (A, B, C), being nothing but $(150^\circ, 150^\circ, 60^\circ)$, do not add to 180°.

Remark 2. The inequality $u \geq \sqrt{3}$ has a special significance in Brocardian geometry. We recall that for any triangle ABC, there exist unique points P_1 and P_2 inside ABC, known as the Brocard points, such that

$$\angle P_1AB = \angle P_1BC = \angle P_1CA = \omega_1, \text{ say, } \angle P_2AC = \angle P_2BA = \angle P_2CB = \omega_2, \text{ say.}$$

The angles ω_1 and ω_2 turn out to be equal and their common value ω, known as the Brocard angle, satisfies the elegant relation $\cot\omega = \cot A + \cot B + \cot C$; see [17], [8, p. 102], or [10, Problem 2 (p. 62) and Problem 2 (p. 76)]. Thus the inequality $u \geq \sqrt{3}$ simply says that the Brocard angle ω of any triangle cannot exceed 30°.

Remark 3. The equivalent of the inequality $u \geq \sqrt{3}$ in terms of the tangents takes the form

$$\begin{aligned} \infty > \tan A + \tan B + \tan C &\geq 3\sqrt{3} \qquad \text{if } ABC \text{ is acute} \qquad (11)\\ -\infty < \tan A + \tan B + \tan C &< 0 \qquad \text{if } ABC \text{ is obtuse.} \qquad (12) \end{aligned}$$

This form, although less elegant than its cotangent twin

$$\sqrt{3} \leq \cot A + \cot B + \cot C < \infty \tag{13}$$

is listed among the *trigonometric gems* in [12, Appendix 3, pp. 220–221]. Listed there also is the equivalent of (1), namely

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

The contrast between formulas for the cotangents and their equivalents in terms of the tangents may be an indication of the advantage of using the former. We should mention here that the inequalities (11), (12), and (13) are to be interpreted in the sharp sense that $a < f < b$ stands for $\inf f = a$ and $\sup f = b$.

Now let u, v, and w be given real numbers and let α, β, and γ be the zeros of the cubic $f(T) := T^3 - uT^2 + vT - w$. Thus α, β, and γ are completely defined by (8). It is well known [13, Theorem 4.32, p. 239] that α, β, and γ are real if and only if the discriminant of $f(T)$ is non-negative, i.e., if and only if

$$\Delta = -27w^2 + 18uvw + u^2v^2 - 4u^3w - 4v^3 \geq 0. \tag{14}$$

If we further assume that $v = 1$, then (14) simplifies into

$$\Delta = -27w^2 + 18wu + u^2 - 4u^3w - 4 \geq 0. \tag{15}$$

Solving for w, we re-write (15) in the equivalent form

$$\left.\begin{aligned} f_1(u) \leq 27w \leq f_2(u), \quad \text{where}\\ f_1(u) = 9u - 2u^3 - 2(u^2-3)^{3/2},\ f_2(u) = 9u - 2u^3 + 2(u^2-3)^{3/2}. \end{aligned}\right\} \tag{16}$$

From this, we see that u lies in $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$. Fig. 1 below is a sketch in the (u, w)-plane of the region Ω defined by $f_1(u) \leq 27w \leq f_2(u)$ where $u \in [\sqrt{3}, \infty)$. The sketch is rather rough, but it shows that $f_1(u)$ is decreasing and concave down, that $f_2(u)$ is decreasing and concave up, and that

$$f_1\left(\sqrt{3}\right) = f_2\left(\sqrt{3}\right) = \frac{1}{3\sqrt{3}}, \quad \lim_{u\to\infty} f_1(u) = -\infty, \quad \lim_{u\to\infty} f_2(u) = 0.$$

It also shows that if $u \geq \sqrt{3}$, then $w \leq 1/3\sqrt{3}$, in agreement with Lemma 1. We summarize this in Theorem 2.

Theorem 2 *Let $f_1(u)$ and $f_2(u)$ be as defined in (16) and let Ω be the region in the (u,w)-plane defined by $u \geq \sqrt{3}$ and any of the equivalent conditions (15) and (16). Let u, v, and w be real numbers.*
Then the zeros of the cubic $T^3 - uT^2 + vT - w$ (are real and) qualify as the cotangents of the angles of a triangle if and only if $v = 1$ and (u,w) lies in Ω. The boundary of Ω consisting of the curves $w = f_1(u)$ and $w = f_2(u)$ corresponds to isosceles triangles. Acute, right, and obtuse triangles correspond to $w > 0$, $w = 0$, and $w < 0$, respectively.

We now give convenient parametrizations for the curves $w = f_1(u)$ and $w = f_2(u)$ corresponding to isosceles triangles. If ABC is isosceles, say $(A,B,C) = (\theta,\theta,\pi - 2\theta)$, and if $t = \cot\theta$, then

$$\cot C = -\cot 2\theta = \frac{1-t^2}{2t},$$

and therefore

$$u = \frac{1+3t^2}{2t}, \quad w = \frac{t(1-t^2)}{2}, \quad t > 0. \tag{17}$$

Also,

$$\text{the isosceles triangle in (17) is } \left\{ \begin{array}{c} \text{acute} \\ \text{right} \\ \text{obtuse} \end{array} \right\} \iff \left\{ \begin{array}{c} 0 < t < 1 \\ t = 1 \\ t > 1 \end{array} \right\} \tag{18}$$

It is also clear that right triangles are parametrized by

$$w = 0, \ u \in [2, \infty). \tag{19}$$

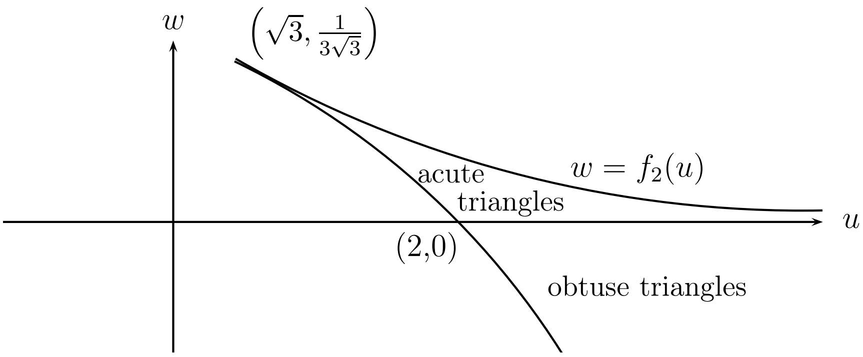

Figure 1:

- The region shown, not drawn to scale, is the part of the (u,w)-plane ABC containing points (u,w) that can be realized by triangles under the correspondence $\cot A + \cot B + \cot C = u$, $\cot A \ \cot B \ \cot C = w$.
- The curves $w = f_1(u)$ and $w = f_2(u)$ correspond to isosceles triangle.
- The regions $w > 0$, $w < 0$, and $w = 0$ correspond to acute, obtuse, and right triangles, respectively.
- The line segment $u = u_0$, $f_1(u) \leq w \leq f_2(u)$ corresponds to triangles with the same Brocard angle $\cot^{-1} u_0$.

3. Summary of the Method

In this section, we describe a systematic method for optimizing certain symmetric functions in the cotangents (and the tangents) of the angles of a general triangle. Letting $\alpha = \cot A$, $\beta = \cot B$, and $\gamma = \cot C$, the given function G is assumed to be symmetric in α, β, and γ. It is also assumed that G can be written as a function G_1 of u, v, and w, where u, v, and w are as defined in (8). This can certainly be done if G is a rational function; see [16, pp. 69–70] and [5, Exercises 37–43, pp. 621–622]. Substituting $v = 1$ in G_1, we obtain a function $H = H(u, w)$ of u and w. Now we describe steps that we follow in order to find the optimum values of H. These steps are based on Theorem 2, (17), (18), and (19).

1. Find the critical points of H in the interior of Ω by solving the system

$$\frac{\partial H}{\partial u} = \frac{\partial H}{\partial w} = 0, \quad \Delta = -27w^2 + 18uw + u^2 - 4wu^3 - 4 > 0.$$

2. Optimize H on isosceles triangles, i.e. optimize

$$H\left(\frac{3t^2+1}{2t}, \frac{t-t^3}{2}\right), \quad t \in (0, \infty).$$

3a. If the optimization is to be done on acute triangles only, then one adds the condition $w > 0$ to the system given in Step 1, restricts t in Step 2 to the interval $(0, 1]$, and adds a third step that consists in optimizing H on right triangles, i.e., optimizing

$$H(u, 0), \quad u \in (2, \infty). \tag{20}$$

3b. For obtuse triangles, one adds the condition $w < 0$ to Step 1, restricts t in Step 2 to the interval $[1, \infty)$, and optimizes H on right triangles as described in (20).

4. Examples

We illustrate the method described above by giving few examples. We take ABC to be our reference triangle and, as before, we let

$$\alpha = \cot A, \quad \beta = \cot B, \quad \gamma = \cot C.$$

We let u, v, and w be as defined in (8). Of course $v = 1$.

We shall use the identities

$$\alpha^2 + \beta^2 + \gamma^2 = u^2 - 2, \tag{21}$$
$$\alpha^3 + \beta^3 + \gamma^3 = u^3 - 3u + 3w. \tag{22}$$

The first follows immediately from $v = 1$. As for the second, one starts with the well known factorization

$$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

and uses $v = 1$ and (21) to obtain

$$\left(\alpha^3 + \beta^3 + \gamma^3\right) - 3w = u(u^2 - 3) = u^3 - 3u.$$

We shall also use the identity

$$\csc A \csc B \csc C \quad = \quad u - w. \tag{23}$$

This is seen as follows:

$$\begin{aligned}
\csc^2 A \csc^2 B \csc^2 C &= \left(1+\alpha^2\right)\left(1+\beta^2\right)\left(1+\gamma^2\right)\\
&= 1+\left(\alpha^2+\beta^2+\gamma^2\right)+\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)+\alpha^2\beta^2\gamma^2\\
&= 1+(u^2-2v)+(v^2-2uw)+w^2\\
&= 1+(u^2-2)+(1-2uw)+w^2\\
&= (u-w)^2.
\end{aligned}$$

Since $u \ge \sqrt{3}$ and $w \le 1/3\sqrt{3}$, we have $u \ge w$ and the rest follows by taking square roots.

Example 1. The inequality

$$\cot A + \cot B + \cot C \le \frac{9}{8}\left(\csc A \csc B \csc C\right)$$

is proved in [3, (2), p. 11]. By (23), this is equivalent to $8u \le 9(u-w)$, i.e., $u \ge 9w$, which is trivial since Ω lies to the right of the line $u = 9w$.

Example 2. The inequality in [3, (26), p. 15] states that

$$\cot A \cot B \cot C \le \frac{8}{27}\sin A \sin B \sin C$$

for acute triangles. Using (23), we reduce this to $27w(u-w) \le 8$. So we take $H(u,w) = wu - w^2$, and we prove that its maximum on acute triangles is $8/27$. Clearly, H has no critical points inside Ω. Also, $H = 0$ on right triangles where $w = 0$. So it remains to consider

$$J(t) = H\left(\frac{3t^2+1}{2t}, \ \frac{t(1-t^2)}{2}\right) = \frac{\left(1-t^2\right)\left(1+t^2\right)^2}{4}$$

for $0 < t \le 1$. Direct calculations show that

$$J'(t) = \frac{t(1+t^2)(1-3t^2)}{2}, \quad 0 < t \le 1.$$

Thus $J(t)$ attains its maximum value at $t = 1/\sqrt{3}$, and the maximum value of J is $8/27$.

Example 3. In [2, Problem 19, p. 76], we are asked to prove that

$$\cot^3 A + \cot^3 B + \cot^3 C + 6\cot A \cot B \cot C \ge \cot A + \cot B + \cot C$$

for acute triangles. By (22), this reduces to

$$(u^3 - 3u + 3w) + 6w \ge u.$$

Thus we let

$$H(u,w) = u^3 - 4u + 9w,$$

and we show that $H \geq 0$ for acute triangles. Obviously, H has no critical points. Also, on right triangles (parametrized by $w = 0$ and $u \in [2, \infty)$), $H = u^3 - 4u$ and increases with u. Thus $H \geq H(0) = 0$ there. So it remains to treat acute isosceles triangles. Thus we consider

$$J(t) = H\left(\frac{3t^2+1}{2t}, \frac{t(1-t^2)}{2}\right) = \frac{(1-t^2)(3t^2-1)^2}{8t^3},$$

and we show that the minimum of $J(t)$ for $0 < t \leq 1$ is 0. This is trivial, since $J(t) \geq 0$ on $(0, 1]$ and since $J(1/\sqrt{3}) = 0$. Note that J attains its minimum at $1/\sqrt{3}$ and at 1. These correspond to the equilateral triangle and to the $(45°, 45°, 90°)$-triangle.

In [2, Problem 5, p. 74], we are asked to prove that if

$$x_n = 2^{n-3}\left(\cos^n A + \cos^n B + \cos^n C\right) + \cos A \cos B \cos C,$$

then

$$x_1 + x_2 + x_3 \geq \frac{3}{2}$$

for all acute triangles. Misreading cos as cot, we ended up with Example 4 below. It would be interesting to produce purely trigonometric proofs of some particular cases of this example. We suggest the cases $r = 11$ and $r = 17$.

Example 4. Let ABC be an acute triangle, and let

$$y_n = 2^{n-3}\left(\cot^n A + \cot^n B + \cot^n C\right) + r \cot A \cot B \cot C,$$

where $r \in \mathbb{R}$. We shall show that if $r \leq 3\sqrt{3} - \frac{7}{4}$, then

$$y_1 + y_2 + y_3 \geq \frac{(7+r)\sqrt{3}+6}{12}, \tag{24}$$

and if $r \geq 3\sqrt{3} - \frac{7}{4}$, then

$$y_1 + y_2 + y_3 \geq \frac{7}{2}. \tag{25}$$

Using (22) and (21), we reduce (24) to

$$4(u^3 - 3u + 3w) + 2(u^2 - 2) + u + 12rw \quad \geq \quad \frac{(7+r)\sqrt{3}+6}{3},$$

i.e.,

$$4u^3 + 2u^2 - 11u + 12(1+r)w \quad \geq \quad \frac{(7+r)\sqrt{3}+18}{3}. \tag{26}$$

Similarly, we reduce (25) to

$$4u^3 + 2u^2 - 11u + 12(1+r)w \quad \geq \quad 18. \tag{27}$$

Now let

$$H(u, w) = 4u^3 + 2u^2 - 11u + Sw, \tag{28}$$

where $S = 12(1+r)$. Thus we are to show that

$$H(u,w) \quad \geq \quad \begin{cases} 6 + \left(1 + \frac{S}{9}\right)\sqrt{3} & \text{if } S \leq 9\left(4\sqrt{3}-1\right) \\ 18 & \text{if } S \geq 9\left(4\sqrt{3}-1\right) \end{cases} \tag{29}$$

for acute isosceles triangles.

Clearly, H has no critical points. Also, for right triangles, $w = 0$ and $H = 4u^3 + 2u^2 - 11u$ is increasing on $u \in [2, \infty)$. Therefore u attains its minimum when $u = 2$. This corresponds to the isosceles $(45°, 45°, 90°)$-triangle. Thus we restrict ourselves to acute isosceles triangles.

Thus we consider

$$J(t) = H\left(\frac{3t^2+1}{2t}, \frac{t(1-t^2)}{2}\right) = \frac{(27-S)t^6 + 9t^5 + (S-6)t^4 + 6t^3 - 2t^2 + t + 1}{2t^3}$$

for $0 < t \leq 1$. Direct calculations show that

$$J'(t) = \frac{(3t^2-1)((27-S)t^4 + 6t^3 + 7t^2 + 2t + 3)}{2t^4} = \frac{(3t^2-1)f(t)}{2t^4},$$

where

$$\begin{aligned} f(t) &= (27-S)t^4 + 6t^3 + 7t^2 + 2t + 3 \\ &= (45-S)t^4 + (1-t)(18t^3 + 12t^2 + 5t + 3). \end{aligned}$$

This shows that if $S \leq 45$, then $f(t) > 0$ on $(0,1)$, and therefore the only zero of J' in (0,1) is $t = r_0 = \sqrt{3}/3$. Also, J' is decreasing on $[0, r_0]$ and increasing on $[r_0, 1]$. Hence J attains its minimum on $(0,1]$ at $t = r_0$, and that minimum is given by

$$J(r_0) = 6 + \left(1 + \frac{S}{9}\right)\sqrt{3}.$$

This proves the first part of (29) for $S \leq 45$.

Thus we restrict our attention to the case $S > 45$. The discriminant in t of the cubic f' is a quadratic in S which is negative for all $S \geq 45$ (actually for all $S \geq 34$). Therefore f' has exactly one real zero, x say. Since $f'(0) = 2 > 0$ and $f'(1) < -38 < 0$, it follows that $0 < x < 1$. Thus f increases on $(0,x)$ and decreases on $(x,1)$, and therefore has a maximum at x. Also $f(0) = 3 > 0$, $f(1) = 45 - S < 0$. So f has a unique zero y in $(0,1)$ that lies in $(x,1)$. To see whether y is greater or smaller than $1/\sqrt{3}$, we use

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{-S + 75 + 12\sqrt{3}}{9}$$

to conclude that

$$y \gtreqless \frac{1}{\sqrt{3}} \iff f\left(\frac{1}{\sqrt{3}}\right) \gtreqless 0 \iff S \lesseqgtr 75 + 12\sqrt{3} \sim 95.78.$$

Thus if $S \leq 75 + 12\sqrt{3}$, then the minimum of J is attained either at $\sqrt{3}/3$ or at 1. But

$$J(1) - J\left(\frac{1}{\sqrt{3}}\right) = 12 - \sqrt{3} - \frac{S\sqrt{3}}{9}.$$

Therefore

$$J(1) \gtreqless J\left(\frac{1}{\sqrt{3}}\right) \iff S \lesseqgtr 9\left(4\sqrt{3}-1\right) \sim 53.35 \tag{30}$$

This shows that if $S \leq 9\left(4\sqrt{3}-1\right)$, then the minimum is attained at $1/\sqrt{3}$, thus completing the proof of the first part of (29).

It also follows from (30) that if $S_0 = 9\left(4\sqrt{3}-1\right)$, and if J_0 is the function obtained from J by putting $S = S_0$, then the minimum of J_0 on $(0,1]$ is attained at both $1/\sqrt{3}$ and at 1, and that the minimum is 18. Letting J_1 be the function J corresponding to $S = S_1$, where $S_1 > S_0$, we see that for $t \in (0,1]$,

$$\begin{aligned} J_1(t) &\geq J_0(t), \text{ because } J_1 - J_0 = (S - S_0)(t^4 - t^6) \geq 0 \text{ on } (0,1] \\ &\geq 18, \text{ because the minimum of } J_0 \text{ is } 18. \end{aligned}$$

Since $J_1(1) = 18$, it follows that the minimum of J_1 on $(0,1]$ is 18 for all $S \geq S_0$. This completes the proof.

It is worth noting that all the functions considered in the examples above attain their extrema at isosceles, and most often at equilateral, triangles. This holds for the examples considered in [7] and, as far as we are aware of, in all the problems that appear in mathematical competitions and in problem columns of various journals. This observation is highlighted in [14]. In view of this, it is desirable to construct interesting inequalities in which the critical triangles are not isosceles. The next example is motivated by this desire.

Example 5. Let θ be any acute angle and let

$$k = \cot\theta + \tan\theta. \tag{31}$$

Thus k can be any real number greater than 2. We shall show that

$$\begin{aligned} -2k(\cot A + \cot B + \cot C) + (\cot^2 A + \cot^2 B + \cot^2 C)& \\ + (\cot A \cot B \cot C)^2 &\geq -k^2 - 2 \end{aligned} \tag{32}$$

for all triangles, and that the left hand side attains its minimum at the right triangle having θ as one of its angles. If $\theta \neq \pi/4$, then our inequality is sharp at a non-isosceles triangle, as desired.

In terms of u and w, the left hand side of (32) is nothing but

$$G = -2ku + (u^2 - 2) + w^2.$$

The gradient of G is given by

$$\nabla G = (-2k + 2u, 2w)\,, \tag{33}$$

and therefore the only critical point of G is $(u_0, w_0) = (k, 0)$. This point lies inside Ω (since $k > 2$) and it corresponds to the triangle whose angles are $(\pi/2, \theta, \pi/2 - \theta)$. Also, the Hessian of G at this point is

$$H = \frac{\partial^2 G}{\partial u^2}\frac{\partial^2 G}{\partial w^2} - \left(\frac{\partial^2 G}{\partial u \partial w}\right)^2 = 4 > 0. \tag{34}$$

Hence G has a local minimum at (u_0, w_0). Since this is the only critical point inside Ω, it follows that G attains its absolute minimum at (u_0, w_0). Finally, $G(u_0, w_0) = -k^2 - 2$, as desired.

Acknowledgement

This work is supported by a grant from Yarmouk University.

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Received October 19, 2007