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Yff Conics

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Abstract. Suppose that a, b, c are algebraic indeterminates and $U = u : v : w$ is a point given in homogeneous trilinear coordinates. The Yff conic of U is defined as the locus of a point $X = x : y : z$ satisfying the equation $f(x, y, z) = f(u, v, w)$, where $f(u, v, w) = (vw + wu + uv)/(u^2 + v^2 + w^2)$. The symbolic substitution $(a, b, c) \to (bc, ca, ab)$ maps the Yff conic of the symmedian point to that of the centroid. This mapping and others are used to find a large number of special points on many Yff conics.

Keywords: ellipse, hyperbola, parabola, symbolic substitution, triangle center, trilinear coordinates, trilinear product, Yff conic

MSC 2007: 51M05

1. Introduction

Suppose that a, b, c are algebraic indeterminates over the field of complex numbers. A *point* P is, briefly speaking, *defined* by homogeneous trilinear coordinates

$$P = p(a, b, c) : q(a, b, c) : r(a, b, c).$$

A precise definition of point is given elsewhere (e.g., [1]–[3]) and need not be repeated here. The same holds for special points known as triangle centers, hundreds of which are indexed in [1]; e.g., the incenter of the reference triangle ABC having vertices

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1$$

is indexed as X_1 and is defined by $X_1 = 1 : 1 : 1$; the centroid, X_2, is defined by $X_2 = bc : ca : ab$, or equivalently, by $X_2 = 1/a : 1/b : 1/c$, and so on. Note that the definitions are algebraic rather than euclidean. For example, the centroid is defined for $(a, b, c) = (2, 3, 6)$ even though no euclidean triangle has sidelengths $2, 3, 6$.

On the other hand, for any values of a, b, c which are sidelengths of a euclidean triangle ABC, the results presented below fit into the subject of traditional triangle geometry. Of particular interest is the use of a nongeometric method in Section 5, called symbolic substitution, which gives geometrically meaningful results.

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Suppose $U = u : v : w$ is a point, and consider the locus of a point $X = x : y : z$ such that

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} = \frac{vw + wu + uv}{u^2 + v^2 + w^2}. \tag{1}$$

Peter YFF — the surname is pronounced "ife" and rhymes with *life* — studied this equation in an unpublished notebook in 1958, as mentioned in [2, p. 243]. The Index of [2] lists the locus as *Yff conic*. Here we call it the *Yff conic of* U, denoted by $\mathcal{Y}(U)$. Clearly, $U \in \mathcal{Y}(U)$, and if X is any point on $\mathcal{Y}(U)$, then all six points

$$x : y : z, \quad y : z : x, \quad z : x : y, \quad x : z : y, \quad y : x : z, \quad z : y : x$$

lie on $\mathcal{Y}(U)$; moreover if x, y, z are distinct, then any five of the six points determine the conic, and $\mathcal{Y}(X) = \mathcal{Y}(U)$. We shall call the six points the *associates* of X.

YFF proved that the center of $\mathcal{Y}(U)$ is the point

$$G = g(a, b, c) : g(b, c, a) : g(c, a, b), \tag{2}$$

where

$$\begin{aligned} g(a, b, c) &= 2au_1 + (b + c - a)u_2, \\ u_1 &= vw + wu + uv, \\ u_2 &= u^2 + v^2 + w^2. \end{aligned}$$

If $u + v + w = 0$, then from $(u + v + w)^2 = 0$, we find

$$\frac{vw + wu + uv}{u^2 + v^2 + w^2} = -\frac{1}{2},$$

and $\mathcal{Y}(U)$ is in this case merely the line $x+y+z = 0$. Henceforth, we assume that $u+v+w \neq 0$.

If $vw + wu + uv = 0$, then (1) holds if and only if X lies on the ellipse $yz + zx + xy = 0$, represented by Fig. 1. This ellipse is the Yff conic $\mathcal{Y}(U)$ for each U on it, except for the vertices A, B, C. Such choices of U include X_{88}, X_{100}, X_{162}, and X_{190}.

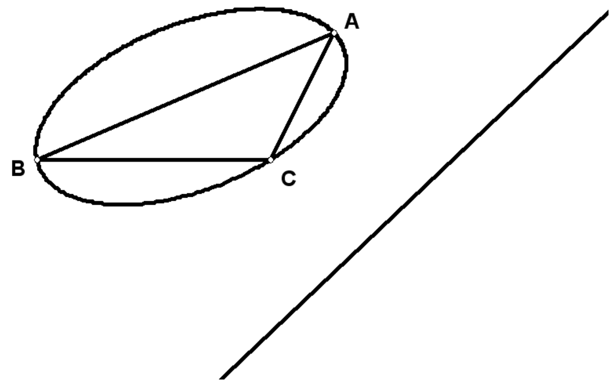

Figure 1: The line $x + y + z = 0$ and its isogonal conjugate, the ellipse $yz + zx + xy = 0$

2. Yff's description

At a lecture at the American University of Beirut in the fall of 2003, YFF showed that the conics which now bear his name constitute a pencil consisting of infinitely many ellipses, infinitely many hyperbolas, a single parabola, a degenerate point conic, and a line conic (two coincident lines). In a letter [4], YFF gives an elegant geometric description of these conics, reproduced with minor editing in this section.

I wrote the general equation in the form

$$x^2 + y^2 + z^2 - 2\lambda(yz + zx + xy) = 0,$$

in which

$$\lambda = \frac{u^2 + v^2 + w^2}{2(vw + wu + uv)}.$$

Therefore the system of conics is a one-parameter family. In fact it is a pencil of conics. For example, every conic in the pencil may be regarded as a linear combination of the imaginary conic $x^2 + y^2 + z^2 = 0$ and the circumconic $yz + zx + xy = 0$.

When $\lambda = 1$ the conic is an ellipse inscribed in the triangle. Its center is X_{37}, and it is the only tritangent conic in the system. The only circumconic is given by $\lambda = \infty$, and its center is X_9. There is one degenerate point conic, obtained when $\lambda = 1/2$, consisting only of the point X_1. The line $x + y + z = 0$ is the only line conic, given by $\lambda = -1$.

Considering the center of a conic as the pole of the line at infinity with respect to the conic, it is found that the center of the general conic in this pencil is

$$a(1 - \lambda) + b\lambda + c\lambda : a\lambda + b(1 - \lambda) + c\lambda : a\lambda + b\lambda + c1 - \lambda.$$

Since the coordinates are linear in λ, the locus of centers is a line, which passes through X_6 ($\lambda = 0$) and X_9 ($\lambda = 1$). Its equation is

$$(b - c)x + (c - a)y + (a - b)z = 0.$$

As the eccentricity of a conic approaches 1, its center approaches infinity. Therefore, in order to find a parabola in this system, we determine the intersection of the line of centers with the line at infinity. This is the point[1].

$$b^2 + c^2 - ca - ab : \ c^2 + a^2 - ab - bc : a^2 + b^2 - bc - ca,$$

which is X_{518}. The value of λ for the parabola is

$$\lambda_0 = \frac{-(a^2 + b^2 + c^2)}{2(bc + ca + ab) - (a^2 + b^2 + c^2)},$$

which is never greater than -1. Ellipses in the pencil are given by $\lambda < \lambda_0$ and by $\lambda_0 > 1/2$. When $\lambda_0 < \lambda < -1$ the conics are hyperbolas. For $-1 < \lambda < 1/2$ the conics are imaginary.

[1] YFF notes that at $(a, b, c) = (5, 10, 13)$, the function X_{1054} takes the same value as X_{518}. This observation provides a good example for understanding points, as defined in this paper, as functions; e.g., $X_{1054} \neq X_{518}$ although $X_{1054}(5, 10, 13) = X_{518}(5, 10, 13)$, just as sine $\neq$ cosine although $\sin \pi/4 = \cos \pi/4$.

Finally, all conics in the system are mutually tangent at two points. To find these points, we solve simultaneously the equations $x^2+y^2+z^2=0$ and $yz+zx+xy=0$. Elimination of x yields

$$y^2+yz+z^2=0,$$

with solutions $y/z \in \{\omega,\omega^2\}$, the complex cube roots of unity. Therefore the two points of intersection are

$$T_1 = 1:\omega:\omega^2 \text{ and } T_2 = 1:\omega^2:\omega.$$

Obviously these points are on every conic in the system. To complete the proof that all the conics are mutually tangent, it may be shown that the line

$$x+\omega y+\omega^2 z=0$$

is tangent to every conic (except for $\lambda=-1$) at T_1, and that the common tangent line at T_2 is

$$x+\omega^2 y+\omega z=0.$$

3. Two points on $\mathcal{Y}(X_{518})$

In addition to the point X_{518} on the parabola $\mathcal{Y}(X_{518})$, the point $X_1X_{518}\cap\mathcal{Y}(X_{518})$ will now be found. The line X_1X_{518} is given by

$$(b-c)x+(c-a)y+(a-b)z=0.$$

Substitute

$$z=\frac{(b-c)x+(c-a)y}{b-a}$$

into the equation for $\mathcal{Y}(X_{518})$ written as

$$\frac{yz+zx+xy}{x^2+y^2+z^2}+\frac{bc+ca+ab}{a^2+b^2+c^2}-\frac{1}{2}=0$$

and factor the expression on the left-hand side. The relevant factor simplifies to

$$-x\left(4b^2+c^2+a^2-4ca-ab-bc\right)+y\left(4a^2+b^2+c^2-4bc-ca-ab\right),$$

from which it follows that $X=X_{3246}$. We leave open the problem of finding other low-degree polynomial triangle centers on $\mathcal{Y}(X_{518})$. Also of interest would be coordinates for the vertex and focus.

4. Reflections

Suppose that $P=p:q:r$ is on an Yff conic $\mathcal{Y}(U)$ and that G as in (2) is the center of $\mathcal{Y}(U)$. The line GP meets $\mathcal{Y}(U)$ in two points, one of which is P. The other, which we denote by P_G, is the reflection of P in G. In order to find trilinears for P_G, it is helpful to abbreviate $g(a,b,c), g(b,c,a), g(c,a,b)$ as g_a, g_b, g_c, respectively. Temporarily restricting a,b,c

to be sidelengths of a euclidean triangle, and writing the area of ABC as σ, the actual trilinear distances of G are given by the ordered triple (hg_a, hg_b, hg_c), where

$$h = \frac{2\sigma}{ag_a + bg_b + cg_c}.$$

Actual trilinear distances for P are (kp, kq, kr), where $k = 2\sigma/(ap + bq + cr)$, so that actual trilinear distances for P_G are given by

$$P_G = (2hg_a - kp,\ 2hg_b - kq,\ 2hg_c - kr)$$

and a homogeneous first trilinear is

$$2g_a(ap + bq + cr) - p(ag_a + bg_b + cg_c),$$

which simplifies to

$$\begin{aligned}
& 2u_1\left((a^2 - b^2 - c^2)p + 2abq + 2acr\right) \\
& + u_2((b - a + c)(2bq + 2cr) - p(a + b - c)(a - b + c)) \\
= \ & 2p_1\left((a^2 - b^2 - c^2)p + 2abq + 2acr\right) \\
& + p_2((b - a + c)(2bq + 2cr) - p(a + b - c)(a - b + c)),
\end{aligned} \tag{3}$$

where

$$p_1 = qr + rp + pq, \quad p_2 = p^2 + q^2 + r^2.$$

Expanding (2) and letting

$$\begin{aligned}
t(a, b, c) \ = \ & p^3(a + b - c)(a - b + c) \\
& + 2p^2\left[(c - a)(a - b + c)q + (b - a)(a + b - c)r\right] \\
& - p\left[4a(q + r)(bq + cr) - (a^2 - (b - c)^2)(q^2 + r^2) + 2qr(a^2 - b^2 - c^2)\right] \\
& - (bq + cr)\left[4aqr + 2(b - a + c)(q^2 + r^2)\right],
\end{aligned}$$

we conclude that

$$P_G = t(a, b, c) : t(b, c, a) : t(c, a, b). \tag{4}$$

In the above derivation, the hypothesis that $u_1/u_2 = p_1/p_2$, used to establish (3), implies that the trilinears in (4) are invariant of u, v, w. Returning now to the more general case that a, b, c are indeterminates, we *define* P_G by the trilinears obtained for (4). It is easy to confirm that P_G, so defined, lies on $\mathcal{Y}(U)$.

5. The conic $\mathcal{Y}(X_6)$

Let $U = a : b : c$, the symmedian point, X_6. By (1), an equation for $\mathcal{Y}(X_6)$ is

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} = \frac{bc + ca + ab}{a^2 + b^2 + c^2}. \tag{5}$$

It is easy to check that the point

$$X_{45} = a - 2b - 2c : b - 2c - 2a : c - 2a - 2b$$

satisfies (5), that the center of $\mathcal{Y}(X_6)$ is the point $G = g_a : g_b : g_c$ given by

$$g_a = a^3 + a(b-c)^2 - (b+c)(3a^2 + b^2 + c^2),$$

and that X_{45} is the reflection (4) of X_6 in G. On writing X_{45} as $p : q : r$, one might expect that the six points

$$p : q : r, \quad q : r : p, \quad r : p : q, \quad p : r : q, \quad q : p : r, \quad r : q : p$$

are, in some order, the reflections in G of the six points

$$a : b : c, \quad b : c : a, \quad c : a : b, \quad a : c : b, \quad b : a : c, \quad c : b : a,$$

but the only such match is the one already recognized. Thus, we have, so far, 22 distinct points on $\mathcal{Y}(X_6)$.

Symbolic substitution, a method introduced in [3], maps $\mathcal{Y}(X_6)$ onto other Yff conics. As a first example, the substitution $(a, b, c) \to (bc, ca, ab)$ maps $\mathcal{Y}(X_6)$ onto $\mathcal{Y}(X_2)$, given, according to (5), by

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} = \frac{abc(a+b+c)}{b^2c^2 + c^2a^2 + a^2b^2}. \tag{6}$$

Specifically, each point $x : y : z = x(a, b, c) : x(b, c, a) : x(c, a, b)$ satisfying (5) maps to the point

$$x' : y' : z' = x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab)$$

on $\mathcal{Y}(X_2)$; e.g., X_{45} maps to the point

$$X_{3240} = bc - 2a(b+c) : ca - 2b(c+a) : ab - 2c(a+b). \tag{7}$$

It is easy to check that the center of $\mathcal{Y}(X_6)$ maps to a point that is not the center of $\mathcal{Y}(X_2)$, and that the point (7) is not the reflection of X_2 about the center of $\mathcal{Y}(X_2)$.

It will be convenient to write $\mathcal{S}$ for the substitution $(a, b, c) \to (bc, ca, ab)$. Recall that G denotes the center of $\mathcal{Y}(X_6)$, and let H denote the center of $\mathcal{S}(\mathcal{Y}(X_6))$; that is, of $\mathcal{Y}(X_2)$. Suppose that

$$P = p(a, b, c) : q(a, b, c) : r(a, b, c)$$

is a point on $\mathcal{Y}(X_6)$. We shall show that ordinarily H is not on the line $\mathcal{S}(P)\mathcal{S}(G)$. This property of symbolic substitution, we shall see, enables the production of infinite sequences of triangle centers on $\mathcal{Y}(X_6)$.

We have

$$\mathcal{S}(P) = p(bc, ca, ab) : q(bc, ca, ab) : r(bc, ca, ab), \tag{8}$$

and $\mathcal{S}$ maps the center of $\mathcal{Y}(X_6)$ to the point

$$\mathcal{S}(G) = g(bc, ca, ab) : g(ca, ab, bc) : g(ab, bc, ca).$$

Moreover, $H = h(a, b, c) : h(b, c, a) : h(c, a, b)$, where

$$h(a, b, c) = a^3(b-c)^2 - a^2\,(b+c)^3 + b^2c^2(a - b - c).$$

Let

$$D = \begin{vmatrix} x & y & y \\ g(bc, ca, ab) & g(ca, ab, bc) & g(ab, bc, ca) \\ h(a, b, c) & h(b, c, a) & h(c, a, b) \end{vmatrix},$$

so that the line $\mathcal{S}(G)H$ has equation $D = 0$. Straightforward computation shows that

$$\begin{aligned} D &= d[(b-c)f(a,b,c)p' + (c-a)f(b,c,a)q' + (a-b)f(c,a,b)r'], \text{ where} \\ d &= 2\left(a^2b^2 + a^2c^2 + b^2c^2 - abc(a+b+c)\right), \\ f(a,b,c) &= a^4\left(b^2+c^2\right) + 2a^3bc\,(b+c) + a^2bc\left(3b^2+3c^2+5bc\right) + b^3c^3. \end{aligned}$$

Thus, writing $\mathcal{S}(P)$ as $P' = p' : q' : r'$, we conclude that $\mathcal{S}(P)$ does not lie on the line $\mathcal{S}(G)H$ unless

$$(b-c)f(a,b,c)p' + (c-a)f(b,c,a)q' + (a-b)f(c,a,b)r' = 0.$$

It is easy to check, for example, that for $P = X_6$ and $P = X_{45}$ (i.e., for $P' = X_2$ and $P' = X_{3240}$), the point $\mathcal{S}(P)$ does not lie on the line $\mathcal{S}(G)H$. Consequently, the reflection of X_2 in H is a third point, which we denote by R, on $\mathcal{Y}(X_2)$, in addition to X_2 and X_{3240}. Using (4) with $(p,q,r) = (bc, ca, ab)$ gives

$$\begin{aligned} R &= \rho(a,b,c) : \rho(b,c,a) : \rho(c,a,b), \text{ where} \\ \rho(a,b,c) &= bc\left(b^2c^2 - 2abc^2 - 2ab^2c - 5a^4\right)(b-c)^2 \\ &\quad + 2a^3bc\,(b+2c)\,(2b+c)\,(b+c) \\ &\quad - a^2bc\left(bc+b^2+c^2\right)\left(5bc-b^2-c^2\right). \end{aligned}$$

As $\mathcal{S}$ is a self-inverse mapping, the point

$$\mathcal{S}(R) = \rho(bc,ca,ab) : \rho(ca,ab,bc) : \rho(ab,bc,ca)$$

is a third point on $\mathcal{Y}(X_6)$. In like manner, the reflection of X_{3240} in H is a fourth point on $\mathcal{Y}(X_2)$, and $\mathcal{S}$ maps this reflection to a fourth point on $\mathcal{Y}(X_6)$. Call the two new points P_3 and P_4.

We conjecture that the reflections of P_3 and P_4 in G are new points P_5 and P_6 on $\mathcal{Y}(X_6)$, that $\mathcal{S}(P_5)$ and $\mathcal{S}(P_6)$ are new on $\mathcal{Y}(X_2)$, that their reflections in H are new on $\mathcal{Y}(X_2)$, that $\mathcal{S}$ maps these reflections onto new points P_7 and P_8 on $\mathcal{Y}(X_6)$, and that this manner of production can be continued indefinitely, giving infinitely many triangle centers on $\mathcal{Y}(X_6)$, as well as $\mathcal{Y}(X_2)$.

Similar results can be found from other symbolic substitutions, of which we mention two, briefly. The substitution

$$(a,b,c) \to (b-c, c-a, a-b)$$

maps $\mathcal{Y}(X_6)$ onto $\mathcal{Y}(X_{100})$ and maps X_{45} to the point $X_{37} = b+c : c+a : a+b$. The substitution

$$(a,b,c) \to (b+c, c+a, a+b)$$

maps $\mathcal{Y}(X_6)$ onto $\mathcal{Y}(X_{37})$, and it maps X_{45} to the point

$$4a+b+c : 4b+c+a : 4c+a+b.$$

Note that this second substitution has an inverse:

$$(a,b,c) \to (b+c-a, c+a-b, a+b-c).$$

Consequently, we conjecture that sequences of triangle centers on $\mathcal{Y}(X_6)$, in addition to those mentioned above, can be found using reflections and substitutions, as above, between the conics $\mathcal{Y}(X_6)$ and $\mathcal{Y}(X_{37})$.

6. The conic $\mathcal{Y}(X_{244})$

The point

$$X_{244} = u : v : w = (b-c)^2 : (c-a)^2 : (a-b)^2$$

has a notable Yff conic. Indeed, on substituting for u, v, w, we find

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} - \frac{vw + wu + uv}{u^2 + v^2 + w^2} = -\frac{x^2 + y^2 + z^2 - 2xz - 2yz - 2xy}{2(x^2 + y^2 + z^2)},$$

which shows, by (1), that an equation for $\mathcal{Y}(X_{244})$ is simply

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0.$$

This is the inscribed ellipse $W(X_1)$ in [2, p. 238]. Its center is $X_{37} = b + c : c + a : a + b$, as already noted in Section 2.

For arbitrary $p : q : r \neq 1 : 1 : 1$, the symbolic substitution $(a, b, c) \to (p, q, r)$ maps $\mathcal{Y}(X_{244})$ to itself and in particular maps X_{244} to the point

$$(q-r)^2 : (r-p)^2 : (p-q)^2.$$

Thus, $\mathcal{Y}(X_{244})$ consists of the trilinear squares of points of the form

$$q - r : r - p : p - q,$$

that is, points on the line $x + y + z = 0$, the antiorthic axis. Among such squares are X_i for $i \in \{244, 678, 2310, 2632, 2638, 2643, 3222, 3248\}$.

7. The conic $\mathcal{Y}(X_{1054})$

Another special Yff conic is $\mathcal{Y}(X_{1054})$, where

$$X_{1054} = (a-b)(a-c) - (b-c)^2 : (b-c)(b-a) - (c-a)^2 : (c-a)(c-b) - (a-b)^2.$$

Taking these three trilinears as u, v, w, we find

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} - \frac{vw + wu + uv}{u^2 + v^2 + w^2} = \frac{x^2 + y^2 + z^2 + 3xy + 3xz + 3yz}{3(x^2 + y^2 + z^2)},$$

so that a simple equation for $\mathcal{Y}(X_{1054})$ is

$$x^2 + y^2 + z^2 + 3xy + 3xz + 3yz = 0,$$

or equivalently,

$$(x + y)(x + z) + (y + z)(y + x) + (z + x)(z + y) = 0. \tag{9}$$

The conic $\mathcal{Y}(X_{1054})$ is represented by Fig. 2.

For arbitrary $p : q : r \neq X_1$, the symbolic substitution $(a, b, c) \to (p, q, r)$ maps $\mathcal{Y}(X_{1054})$ to itself and in particular maps X_{1054} to the point

$$(p-q)(p-r) - (q-r)^2 : (q-r)(q-p) - (r-p)^2 : (r-p)(r-q) - (p-q)^2.$$

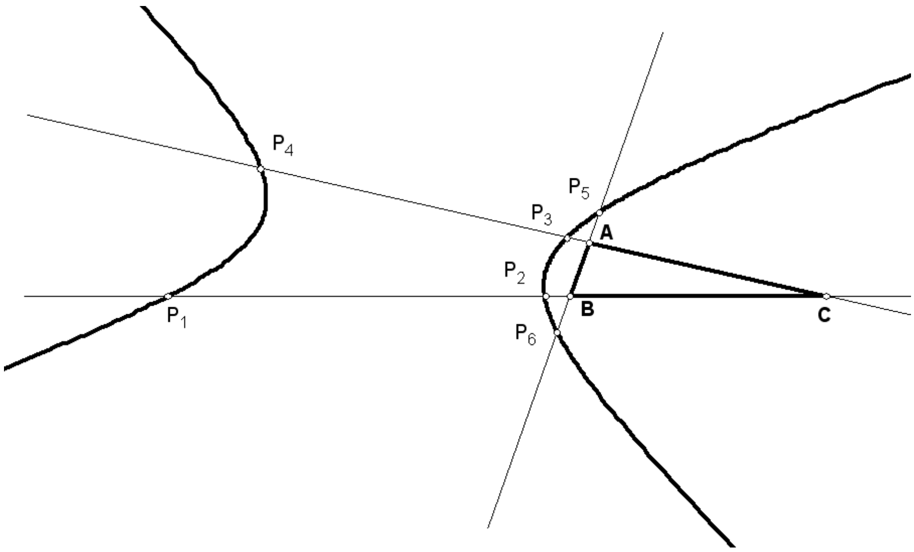

Figure 2: The Yff conic $\mathcal{Y}(X_{1054})$ and the six associates of the point $P_1 = 0 : -3+\sqrt{5} : 2$

Thus, each point on $\mathcal{Y}(X_{1054})$ is the P^*-Hirst inverse of X_1, where P^* is a point on the antiorthic axis (which is the trilinear polar of X_1), and conversely.

The center of $\mathcal{Y}(X_{1054})$ is the point

$$5a-3(b+c) : 5b-3(c+a) : 5c-3(a+b),$$

so that $\mathcal{Y}(X_{1054})$ is an ellipse, parabola, or hyperbola ([2, p. 234], according as

$$\Phi > 0, \quad \Phi = 0, \quad \text{or} \quad \Phi < 0,$$

where

$$\Phi = -5a^2 - 5b^2 - 5c^2 + 6bc + 6ca + 6ab.$$

For example, the conic is an ellipse for $(a,b,c) = (5,11,13)$, a parabola for $(a,b,c) = (5,10,13)$, and a hyperbola for $(a,b,c) = (5,12,13)$.

Putting $x = 0$ in (9) leads to two points of intersection of $\mathcal{Y}(X_{1054})$ with the line BC :

$$0 : -3+\sqrt{5} : 2 \quad \text{and} \quad 0 : -3-\sqrt{5} : 2.$$

These and their associates comprise a set of 6 (not 12) distinct noncentral points on $\mathcal{Y}(X_{1054})$.

8. General homogeneous coordinates

In connection with the equation (1) that defines Yff conics, suppose that the homogeneous coordinates given for U and X are barycentric rather than trilinear. Then, for example, the barycentric points $a : b : c$ and

$$a-2b-2c : b-2c-2a : c-2a-2b$$

are, in trilinears, $1 : 1 : 1$ and

$$(a-2b-2c)/a : (b-2c-2a)/b : (c-2a-2b)/c,$$

and (1), expressed in terms of trilinear coordinates, is

$$\frac{bcyz + cazx + abxy}{a^2x^2 + b^2y^2 + c^2z^2} = \frac{bcvw + cawu + abuv}{a^2u^2 + b^2v^2 + c^2w^2}. \tag{10}$$

The locus of a point X satisfying (10) is a conic closely related to the Yff conic $\mathcal{Y}(X_6 \cdot U)$, where $\cdot$ denotes trilinear product. Actually, $\mathcal{Y}(X_6 \cdot U)$ is given by

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} = \frac{bcvw + cawu + abuv}{a^2u^2 + b^2v^2 + c^2w^2}, \tag{11}$$

so that for each X on $\mathcal{Y}(X_6 \cdot U)$, the point X/X_6, where / denotes trilinear quotient, lies on the conic (10).

One may generalize from barycentric to general homogeneous coordinates with arbitrary base-point P having trilinear coordinates $p : q : r$. (For barycentrics, $P = X_2$; for trilinears, $P = X_1$.) As exemplified by (10) and (11), the resulting conic represented by (1) in general homogeneous coordinates takes the following form in trilinears:

$$\frac{qryz + rpzx + pqxy}{p^2x^2 + q^2y^2 + r^2z^2} = \frac{qrvw + rpwu + pquv}{p^2u^2 + q^2v^2 + r^2w^2}. \tag{12}$$

We denote the conic (12) by $\mathcal{Y}(P, U)$ and observe that it is simply the locus of X/P as X traverses the Yff conic $\mathcal{Y}(P \cdot U)$.

9. Related conics

It is obvious that the general symmetric polynomial of degree 2 in x, y, z is a linear combination

$$f(x, y, z) = r(x^2 + y^2 + z^2) + s(yz + zx + xy),$$

where r, s are real numbers or, in the present context, functions of a, b, c. For arbitrary $U = u : v : w$, the equation

$$f(x, y, z) = f(u, v, w) \tag{13}$$

is nonhomogeneous in a, b, c. In order to obtain a corresponding homogeneous equation, let

$$F(x, y, z) = \frac{f(x, y, z)}{(ax + by + cz)^2}$$

Then since

$$ax + by + cz = au + bv + cw = 2\sigma,$$

the equation

$$F(x, y, z) = F(u, v, w) \tag{14}$$

has the desired properties. We call the locus of X satisfying (14) the (r, s)-*Yff-like conic of* U, and denote it by $\mathcal{Y}(U; r, s)$. The form (14) lends itself to analytical methods as in [2, p. 234]. Let

$$\begin{aligned} N &= r(u^2 + v^2 + w^2) + s(vw + wu + uv), \\ D &= (au + bv + cw)^2. \end{aligned}$$

Then (14) can be written as

$$\begin{aligned}&(rD-Na^2)x^2+(rD-Nb^2)y^2+(rD-Nc^2)z^2\\&+(sD-2Nbc)yz+(sD-2Nca)zx+(sD-2Nab)xy=0\,.\end{aligned}$$

Define

$$\begin{array}{rclrclrcl}
u_1 &=& rD-Na^2 & v_1 &=& rD-Nb^2 & w_1 &=& rD-Nc^2\\
f &=& (sD-2Nbc)/2 & g &=& (sD-2Nca)/2 & h &=& (sD-2Nab)/2\\
U_1 &=& v_1w_1-f^2 & V_1 &=& w_1u_1-g^2 & W_1 &=& u_1v_1-h^2\\
F &=& gh-u_1f & G &=& hf-v_1g & H &=& fg-w_1h
\end{array}$$

The center of the conic is the point

$$aU_1+bH+cG : aH+bV_1+cF : aG+bF+cW_1,$$

where

$$aU_1+bH+cG=\left(-\frac{1}{4}\right)(au+bv+cw)^4\,(s-2r)\,(2ar+as-bs-cs)\,,$$

so that the center of the conic, in case $s\neq 2r$, is

$$s(b+c-a)-2ar : s(c+a-b)-2br : s(a+b-c)-2cr,$$

and this is invariant of U. Examples depending on r and s are tabulated here:

r	0	1	1	1	-2	$2bc+2ca+2ab-a^2-b^2-c^2$
s	1	0	-2	1	3	$2(a^2+b^2+c^2)$
center of $\mathcal{Y}(U;r,s)$	X_9	X_6	X_{37}	X_1	X_{3247}	X_{518}

Regarding the classification of $\mathcal{Y}(U;r,s)$, define

$$\Phi=U_1a^2+V_1b^2+W_1c^2+2Fbc+2Gca+2Hab$$

and find that this polynomial factors as

$$\begin{aligned}&(F_1)(2r\left(a^2+b^2+c^2\right)+sa^2+b^2+c^2-2ac-2bc-2ab)),\ \text{where}\\&F_1=(au+bv+cw)^4(2r-s)/4,\end{aligned}$$

so that classification depends on the factor

$$\widehat{\Phi}=2r\left(a^2+b^2+c^2\right)+sa^2+b^2+c^2-2ac-2bc-2ab$$

as follows: the conic is an ellipse, a parabola, or a hyperbola according as $\widehat{\Phi}>0$, $\widehat{\Phi}=0$, or $\widehat{\Phi}<0$.

For example, $\mathcal{Y}(U;1,0)$, given by

$$\frac{x^2+y^2+z^2}{(ax+by+cz)^2}=\frac{u^2+v^2+w^2}{(au+bv+cw)^2},$$

is an ellipse. As a second example, if

$$2r\left(a^2+b^2+c^2\right)+s(a^2+b^2+c^2-2bc-2ca-2ab)=0,$$

and $2r\neq s$, then $\widehat{\Phi}=0$, so that the conic is the parabola represented in the above table, with center X_{518}. Explicitly, the conic is a parabola if

$$\frac{r}{s}=\frac{bc+ca+ab}{a^2+b^2+c^2}-\frac{1}{2}$$

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