Journal for Geometry and Graphics
Volume 12 (2008), No. 2, 123–132.

Orthogonal Line Congruences with Common Middle Surface

Pelagia Koltsaki, Despina Papadopoulou

Department of Mathematics, Faculty of Science
Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece
email: kopel@math.auth.gr, papdes@math.auth.gr

Abstract. Let S, S' be two orthogonal line congruences with common middle surface $P(u,v)$. We study S and S' firstly in the case that S is isotropic and then in the case that S is the normal line congruence of $P(u,v)$.

Key Words: Orthogonal line congruences, middle surface, middle envelope.

MSC 2000: 53A25

1. Introduction

In a three-dimensional Euclidean space E^3 two line congruences, whose straight lines correspond one-to-one, are called *orthogonal* iff the corresponding straight lines are orthogonal to one another. Orthogonal line congruences with common middle surface have been studied by N.K. Stephanides [3] and G. Stamou [2]. In this paper we also study the line congruences, which are orthogonal to a given line congruence S and have the same middle surface with S. First, we deal with the case that the middle envelope and the middle surface of S are different and then with the case that the above surfaces coincide.

Let S be an oriented line congruence in E^3, defined by the equation

$$\overline{x}(u,v,t) = \overline{OP} + t\overline{e}_3, \quad -\infty < t < +\infty, \tag{1.1}$$

where $\overline{OP} = P(u,v)$ is the coordinate vector for the surface of reference and $\overline{e}_3(u,v)$ is the unit vector in the direction of the straight lines of S. Suppose $\mathcal{D} = \{\overline{e}_i(u,v) \mid i = 1,2,3\}$ is an orthonormal, positively oriented moving frame of S and $\overline{OM} = M(u,v)$ is the middle envelope of S.

We assume that S satisfies the following conditions:

(a) The functions $P(u,v)$, $M(u,v)$ and $\overline{e}_i(u,v)$, $i = 1,2,3$, are defined on a simply connected domain G in the (u,v)-plane and are of class C^4.

(b) The spherical representation of S is one-to-one.

(c) The middle envelope $M(u,v)$ is a regular surface having no parabolic or umbilical points.

ISSN 1433-8157/\$ 2.50

(d) There is a one-to-one mapping between the points of the middle surface and the points of the middle envelope.

There exist linear differential forms σ_i, ω_{ij}, $i, j = 1, 2, 3$, such that

$$dP = \sum_{i=1}^{3} \sigma_i \overline{e}_i, \tag{1.2}$$

$$d\overline{e}_j = \sum_{i=1}^{3} \omega_{ji} \overline{e}_i, \quad \omega_{ij} + \omega_{ji} = 0, \quad i, j = 1, 2, 3. \tag{1.3}$$

According to condition (b) the differential forms ω_{31}, ω_{32} are linearly independent, i.e.,

$$\omega_{31} \wedge \omega_{32} \neq 0, \tag{1.4}$$

where '$\wedge$' denotes the exterior product of two differential forms.

For the exterior derivatives $d\omega_{31}$, $d\omega_{32}$ of the forms ω_{31}, ω_{32} respectively we may set

$$d\omega_{31} = q\, \omega_{31} \wedge \omega_{32}, \quad d\omega_{32} = \widetilde{q}\, \omega_{32} \wedge \omega_{31}, \tag{1.5}$$

where $q, \widetilde{q}$ are functions of u and v defined on G. Then it is well-known [3, p. 319], that

$$\omega_{12} = q\, \omega_{31} - \widetilde{q}\, \omega_{32}. \tag{1.6}$$

The surface $\overline{OP} = P(u, v)$ is the middle surface of S if and only if [3, p. 319]

$$\omega_{31} \wedge \sigma_2 + \sigma_1 \wedge \omega_{32} = 0. \tag{1.7}$$

From now on, we assume that $P(u, v)$ is the middle surface of S. There exist functions l, m, n of u and v defined on G such that

$$\sigma_1 = -m\omega_{31} - n\omega_{32}, \quad \sigma_2 = l\omega_{31} + m\omega_{32}. \tag{1.8}$$

The curvature k, the mean curvature h and the limit distance $2z$ of S are given by the formulae

$$k = l\, n - m^2, \quad 2h = l + n, \tag{1.9}$$

$$2z = \sqrt{(l - n)^2 + 4m^2} = 2\sqrt{h^2 - k}. \tag{1.10}$$

Considering $\overline{e}_3(u, v)$ as the unit normal vector of $M(u, v)$ and $\mathcal{D}$ as the moving frame on $M(u, v)$, there exist linear differential forms ρ, σ such that

$$dM = \rho\, \overline{e}_1 + \sigma\, \overline{e}_2. \tag{1.11}$$

We set

$$\overline{OP} = \overline{OM} + a\, \overline{e}_1 + b\, \overline{e}_2, \tag{1.12}$$

where the functions $a = a(u, v)$, $b = b(u, v)$, $(u, v) \in G$, satisfy the condition [3, p. 321]

$$\nabla_1 a + \nabla_2 b - \widetilde{q} a - q b = r_1 + r_2. \tag{1.13}$$

Here ∇_i, $i = 1, 2$, denote the Pfaffian derivatives with respect to the forms ω_{31}, ω_{32} and r_1, r_2 the principal radii of curvature of $M(u, v)$. Besides, it is known [3, p. 320] that the relation

$$\sigma_3 = -a\, \omega_{31} - b\, \omega_{32} \tag{1.14}$$

is valid.

2. Middle envelope different from middle surface

Suppose S is a line congruence in E^3 defined on G by (1.1), where $\overline{OP} = P(u,v)$ is its middle surface. Let $\mathcal{D} = \{\overline{e}_i(u,v) \mid i = 1,2,3\}$ be an orthonormal, positively oriented moving frame of S and $\overline{OM} = M(u,v)$ be its middle envelope. At every point $P(u,v)$ of the middle surface of S we consider a positively oriented orthonormal frame $\mathcal{D}' = \{\overline{e}'_i(u,v) \mid i = 1,2,3\}$ such as

$$\overline{e}'_1 = \overline{e}_3, \tag{2.1}$$

$$\overline{e}'_2 = \sin\varphi\,\overline{e}_1 - \cos\varphi\,\overline{e}_2, \tag{2.2}$$

$$\overline{e}'_3 = \cos\varphi\,\overline{e}_1 + \sin\varphi\,\overline{e}_2, \tag{2.3}$$

where $\varphi = \varphi(u,v)$ is the oriented angle between $\overline{e}_1(u,v)$ and $\overline{e}'_3(u,v)$.

In this paragraph, we assume that *the middle envelope $\overline{OM} = M(u,v)$ of S is different from its middle surface $\overline{OP} = P(u,v)$, that is, we study the case that S is not the normal line congruence of a minimal surface.*

It is already known [3, p. 322] that in a neighborhood of each point $(u_0,v_0) \in G$, there are infinitely many line congruences which are orthogonal to S and have the same middle surface $P(u,v)$. All these congruences are defined by the equation

$$b\,\nabla_1\varphi - a\,\nabla_2\varphi - m\cos 2\varphi + \frac{l-n}{2}\sin 2\varphi + \widetilde{q}a + qb = 0. \tag{2.4}$$

The solutions of (2.4) depend on an arbitrary function of one variable.

If we put

$$\Gamma = \widetilde{q} - \nabla_2\varphi, \quad \Delta = q + \nabla_1\varphi, \tag{2.5}$$

then the equation (2.4) may be written in the form

$$a\Gamma + b\Delta - m\cos 2\varphi + \frac{l-n}{2}\sin 2\varphi = 0. \tag{2.6}$$

We consider a line congruence $S'(\varphi)$, the straight lines of which are directed by the unit vector $\overline{e}'_3(u,v)$. The orthogonal line congruences S, $S'(\varphi)$ have the same middle surface iff (2.6) is valid. Referring to the moving frame $\mathcal{D}'$, according to the relations (1.2), (1.3), we may write

$$dP = \sum_{i=1}^{3} \sigma'_i\,\overline{e}'_i, \tag{2.7}$$

$$d\overline{e}'_j = \sum_{i=1}^{3} \omega'_{ji}\overline{e}'_i, \quad \omega'_{ij} + \omega'_{ji} = 0, \quad i,j = 1,2,3. \tag{2.8}$$

By the equations (1.2), (2.7) and applying the equations (2.1)–(2.3), we can conclude that

$$\sigma'_1 = \sigma_3, \tag{2.9}$$

$$\sigma'_2 = \sin\varphi\,\sigma_1 - \cos\varphi\,\sigma_2, \tag{2.10}$$

$$\sigma'_3 = \cos\varphi\,\sigma_1 + \sin\varphi\,\sigma_2. \tag{2.11}$$

Besides, using (1.6), (2.1)–(2.3), (2.5), from (2.8) we find out that

$$\omega'_{31} = -\cos\varphi\,\omega_{31} - \sin\varphi\,\omega_{32}, \tag{2.12}$$

$$\omega'_{32} = -d\varphi - \omega_{12} = -\Delta\omega_{31} + \Gamma\omega_{32}, \tag{2.13}$$

$$\omega'_{12} = \sin\varphi\,\omega_{31} - \cos\varphi\,\omega_{32}. \tag{2.14}$$

As for the exterior product of the linear differential forms $\omega'_{31}, \omega'_{32}$ takes the form

$$\omega'_{31} \wedge \omega'_{32} = D\,\omega_{31} \wedge \omega_{32}, \tag{2.15}$$

where

$$D = -\left(\cos\varphi\,\Gamma + \sin\varphi\,\Delta\right). \tag{2.16}$$

The differential forms $\omega'_{31}, \omega'_{32}$ are linearly independent iff $D \neq 0\ \forall (u,v) \in G$.
From now on, we assume

$$\cos\varphi\,\Gamma + \sin\varphi\,\Delta \neq 0 \quad \forall (u,v) \in G. \tag{2.17}$$

Then, there exist functions l', m', n', q', $\widetilde{q}'$ of u and v defined on G such that

$$\sigma'_1 = -m'\omega'_{31} - n'\omega'_{32}, \tag{2.18}$$

$$\sigma'_2 = l'\omega'_{31} + m'\omega'_{32}, \tag{2.19}$$

$$\omega'_{12} = q'\omega'_{31} - \widetilde{q}'\omega'_{32}. \tag{2.20}$$

From the relations (2.9), (2.18), by virtue of (1.14), (2.12), (2.13), we obtain

$$m'\cos\varphi + n'\Delta = -a, \tag{2.21}$$

$$m'\sin\varphi - n'\Gamma = -b. \tag{2.22}$$

Solving the preceding equations and using (2.6), (2.16), we get

$$m' = \frac{1}{D}\left(a\Gamma + b\Delta\right) = \frac{1}{D}\left(m\cos 2\varphi - \frac{l-n}{2}\sin 2\varphi\right), \tag{2.23}$$

$$n' = \frac{1}{D}\left(a\sin\varphi - b\cos\varphi\right). \tag{2.24}$$

Besides, from (2.10), (2.19), by the relations (1.8), (2.12), (2.13), we find out the system

$$l'\cos\varphi + m'\Delta = l\cos\varphi + m\sin\varphi, \tag{2.25}$$

$$l'\sin\varphi - m'\Gamma = m\cos\varphi + n\sin\varphi. \tag{2.26}$$

In view of (2.16), from the latter system, it follows

$$l' = -\frac{1}{D}\left[(m\Gamma + n\Delta)\sin\varphi + (m\Delta + l\Gamma)\cos\varphi\right]. \tag{2.27}$$

Let us now denote by k', h' and $2z'$ the curvature, the mean curvature and the limit distance of $S'(\varphi)$ respectively. Similarly to the formulae (1.9), (1.10), we have

$$k' = l'n' - m'^2, \tag{2.28}$$

$$2h' = l' + n', \tag{2.29}$$

$$4z'^2 = (l' - n')^2 + 4m'^2. \tag{2.30}$$

When m', n', l' from (2.23), (2.24), (2.27) are substituted in (2.28), (2.29), (2.30), we find

$$k' = \frac{1}{D}\left[(an - bm)\sin\varphi + (am - bl)\cos\varphi\right], \tag{2.31}$$

$$2h' = -\frac{1}{D}\left[(m\Gamma + n\Delta - a)\sin\varphi + (l\Gamma + m\Delta + b)\cos\varphi\right], \tag{2.32}$$

$$4z'^2 = \frac{1}{D^2}\left\{\left[(-m\Delta - l\Gamma + b)\cos\varphi - (m\Gamma + n\Delta + a)\sin\varphi\right]^2 + 4(a\Gamma + b\Delta)^2\right\}. \tag{2.33}$$

Moreover, similarly to (1.14), there are functions $a' = a'(u,v)$, $b' = b'(u,v)$ so that

$$\sigma_3' = -a'\omega_{31}' - b'\omega_{32}' \tag{2.34}$$

holds. From (2.34) we deduce

$$a' = -\frac{\sigma_3' \wedge \omega_{32}'}{\omega_{31}' \wedge \omega_{32}'}, \quad b' = -\frac{\omega_{31}' \wedge \sigma_3'}{\omega_{31}' \wedge \omega_{32}'}. \tag{2.35}$$

By substituting (2.11)–(2.13) into (2.35) and using (1.8), (2.16) we get

$$a' = \frac{1}{D}\left[(m\Gamma + n\Delta)\cos\varphi - (l\Gamma + m\Delta)\sin\varphi\right], \tag{2.36}$$

$$b' = \frac{1}{D}\left(m\sin 2\varphi - n\cos^2\varphi - l\sin^2\varphi\right). \tag{2.37}$$

If $\overline{OM'} = M'(u,v)$ is the middle envelope of the line congruence $S'(\varphi)$, then similarly to (1.12) we have

$$\overline{OM'} = \overline{OP} - a'\overline{e}_1' - b'\overline{e}_2', \tag{2.38}$$

in which the functions $a'(u,v)$, $b'(u,v)$ are defined by (2.36), (2.37) respectively. Thus, making use of the relations (2.1), (2.2) eq. (2.38) can also be written as

$$\overline{M'P} = b'\sin\varphi\,\overline{e}_1 - b'\cos\varphi\,\overline{e}_2 + a'\overline{e}_3. \tag{2.39}$$

Remark 2.1. We assume, without loss of generality, that the $S-$principal ruled surfaces of the line congruence S are the parameter surfaces $\omega_{31} = 0$, $\omega_{32} = 0$, which happens iff

$$m = 0 \quad \forall (u,v) \in G. \tag{2.40}$$

According to (2.23), (2.40) we have the equivalent relations

$$m' = 0 \iff a\Gamma + b\Delta = 0 \iff \text{ either } l = n \text{ or } \varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \quad \forall (u,v) \in G. \tag{2.41}$$

Hereafter, we continue to have only the values $\varphi = 0$, $\varphi = \frac{\pi}{2}$ in the second equivalence of (2.41), because the straight lines of the line congruences $S'(0)$ and $S'(\pi)$ (resp. $S'\left(\frac{\pi}{2}\right)$ and $S'\left(\frac{3\pi}{2}\right)$) have the same direction.

Besides, by (1.12), we lead up to

$$\frac{\overline{MP}}{|\overline{MP}|} = \frac{a}{\sqrt{a^2 + b^2}}\,\overline{e}_1 + \frac{b}{\sqrt{a^2 + b^2}}\,\overline{e}_2, \quad (a^2 + b^2 \neq 0 \quad \forall (u,v) \in G). \tag{2.42}$$

The line congruence, whose straight lines are directed by the unit vector $\overline{e}_3' = \dfrac{\overline{MP}}{|\overline{MP}|}$ is orthogonal to S and because of (2.3), (2.42) we can set

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}\,, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}\,. \tag{2.43}$$

By using (2.40), (2.41),(2.43), as well as taking into consideration the hypothesis (2.17), we come to the following conclusion:

If the line congruence S is isotropic ($m \equiv 0$, $l - n \equiv 0$) or we obtain $S'(\varphi)$ for the values $\varphi = 0$, $\varphi = \frac{\pi}{2}$, we always assume $\overline{e}_3' \neq \pm\dfrac{\overline{MP}}{|\overline{MP}|}$.

Line congruences whose straight lines are directed by $\overline{e}_3' = \pm\dfrac{\overline{MP}}{|\overline{MP}|}$ have been studied by N.K. Stephanidis [3], L. Vanhecke, L. Vermeire [4] and G. Stamou [2].

We suppose now, that *the line congruence S is isotropic.* Then we have

$$l = n, \quad m = 0 \quad \forall (u,v) \in G. \tag{2.44}$$

The relations (2.31), (2.37), because of (2.44), can be written as

$$k' = \frac{l}{D}\left(a\sin\varphi - b\cos\varphi\right), \tag{2.45}$$

$$b' = -\frac{l}{D}\,. \tag{2.46}$$

Besides, by making use of (2.45), (2.46), from (1.12), (2.39) we conclude

Proposition 2.1 *Let S, $S'(\varphi)$ be orthogonal line congruences with the same middle surface. If S is an isotropic line congruence, then*

$$k' = -\left\langle \overline{MP}, \overline{M'P} \right\rangle \tag{2.47}$$

is valid.

(A) The line congruence $S'(\varphi)$ is parabolic iff

$$k' = 0 \quad \forall (u,v) \in G. \tag{2.48}$$

According to the conclusion of Remark 2.1 we have $\overline{e}_3' \neq \dfrac{\overline{MP}}{|\overline{MP}|} \iff a\sin\varphi - b\cos\varphi \neq 0$ $\forall (u,v) \in G$. From this and the relations (2.45), (2.48), we can get the following:

Proposition 2.2 *If S is an isotropic line congruence, then there is no parabolic line congruence orthogonal to S sharing the middle surface with S.*

(B) The line congruence $S'(\varphi)$ is isotropic iff

$$l' = n', \quad m' = 0 \quad \forall (u,v) \in G. \tag{2.49}$$

Taking into account the relations (2.23), (2.24), (2.27), (2.44), we derive from (2.49)

$$l = \frac{1}{D}\left(a\sin\varphi - b\cos\varphi\right), \quad a\Gamma + b\Delta = 0. \tag{2.50}$$

Then, by virtue of (2.5), (2.16), the equations (2.50) lead to the

Proposition 2.3 *Let S be an isotropic line congruence. The isotropic line congruences $S'(\varphi)$, which are orthogonal to S and have the same middle surface with S, are defined by the equations*

$$l\,\nabla_1\varphi + ql + a = 0, \quad l\,\nabla_2\varphi - \widetilde{q}l + b = 0.$$

(C) The line congruence $S'(\varphi)$ is normal iff $h' = 0\ \forall(u,v) \in G$. Using the relations (2.5), (2.16), (2.32), (2.44) and taking into account the result of the second equivalence in (2.41), we deduce

Proposition 2.4 *If S is an isotropic line congruence, then the normal line congruences $S'(\varphi)$, which are orthogonal to S and have the same middle surface with S, are defined by the equations*

$$l\,\nabla_1\varphi + ql - a = 0, \quad l\,\nabla_2\varphi - \widetilde{q}l - b = 0.$$

3. Coincided middle envelope and middle surface

In this paragraph, we consider that *the middle envelope $M(u,v)$ of the line congruence S coincides with its middle surface $P(u,v)$*. In other words $P(u,v) = M(u,v)\ \forall(u,v) \in G$. This makes the line congruence S to be the normal line congruence of a minimal surface $P(u,v)$. Therefore we have

$$a = b = 0, \quad 2h = l + n = 0, \quad r_1 + r_2 = 0 \quad \forall(u,v) \in G. \tag{3.1}$$

It is well-known [3, p. 324] that in such a case, there are exactly two line congruences S', S'' orthogonal to S, the middle surface of which coincides with the minimal surface $P(u,v)$. Assuming, without loss of generality, an $S-$canonical frame, i.e., $m \equiv 0$, the line congruences S' and S'' are defined for the values $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ respectively.

Taking into account (1.6), (2.1) and substituting $\varphi = 0$ in the relations (2.2), (2.3), (2.5), (2.12)–(2.16) we deduce for the line congruence S'

$$\overline{e}'_1 = \overline{e}_3, \quad \overline{e}'_2 = -\overline{e}_2, \quad \overline{e}'_3 = \overline{e}_1, \tag{3.2}$$

$$\omega'_{31} = -\omega_{31}, \quad \omega'_{32} = -\omega_{12}, \quad \omega'_{12} = -\omega_{32}, \tag{3.3}$$

$$\omega'_{31} \wedge \omega'_{32} = -\widetilde{q}\,\omega_{31} \wedge \omega_{32}. \tag{3.4}$$

Similarly, for the value $\varphi = \frac{\pi}{2}$, we obtain that for the line congruence S'' the relations

$$\overline{e}''_1 = \overline{e}_3, \quad \overline{e}''_2 = \overline{e}_1, \quad \overline{e}''_3 = \overline{e}_2, \tag{3.5}$$

$$\omega''_{31} = -\omega_{32}, \quad \omega''_{32} = -\omega_{12}, \quad \omega''_{12} = \omega_{31}, \tag{3.6}$$

$$\omega''_{31} \wedge \omega''_{32} = -q\,\omega_{31} \wedge \omega_{32} \tag{3.7}$$

are valid. Considering $\varphi = 0$ (resp. $\varphi = \frac{\pi}{2}$), the condition (2.17), for which the differential forms ω'_{31}, ω'_{32} (resp. ω''_{31}, ω''_{32}) are linearly independent, becomes

$$\widetilde{q} \neq 0 \quad (\text{resp. } q \neq 0) \quad \forall(u,v) \in G.$$

By (3.2), (3.5), it is obvious that the line congruences S', S'' are directed by the vectors $\overline{e}_1(u,v)$ and $\overline{e}_2(u,v)$ respectively. They are parabolic congruences and their straight lines are tangent to the asymptotic lines of $P(u,v)$ [3, p. 324].

Proposition 3.1 *The following properties are valid*

(i) $h' = -h''$, *where* h' *and* h'' *are the mean curvatures of* S' *and* S'' *respectively.*

(ii) $2z = 4z' = 4z''$, *where* $2z'$ *and* $2z''$ *are the limit distances of* S' *and* S'' *respectively.*

(iii) $k = -4h'^2 = -4h''^2$.

Proof: From the equations (2.5), for the constant values 0 and $\frac{\pi}{2}$ of φ, we find

$$\Gamma = \widetilde{q}, \quad \Delta = q, \tag{3.8}$$

meanwhile, by (2.16) for $\varphi = 0$ (resp. $\varphi = \frac{\pi}{2}$) and by applying (3.8), we have

$$D = -\widetilde{q} \quad (\text{resp. } D = -q). \tag{3.9}$$

Besides, taking into account the equations (3.1), (3.8), (3.9), the relations (2.23), (2.24), (2.27) for $\varphi = 0$ become

$$m' = 0, \tag{3.10}$$

$$n' = 0, \tag{3.11}$$

$$l' = l = -n. \tag{3.12}$$

Similarly, for $\varphi = \frac{\pi}{2}$, we obtain

$$m'' = 0, \tag{3.13}$$

$$n'' = 0, \tag{3.14}$$

$$l'' = n = -l. \tag{3.15}$$

(i) It is easily proved from the equation (2.29), by virtue of (3.11), (3.12), (3.14), (3.15).

(ii) Applying the formula (1.10) to S', S'' it follows

$$2z' = 2\sqrt{h'^2 - k'}, \tag{3.16}$$

$$2z'' = 2\sqrt{h''^2 - k''}, \tag{3.17}$$

where k', k'' denote the curvatures of S', S'' respectively. However $S'$and S'' are parabolic line congruences. Hence we have

$$k' = k'' \equiv 0. \tag{3.18}$$

Using the relations (3.16), (3.17), (3.18) and the property (i) we obtain

$$2z' = 2z''. \tag{3.19}$$

Since $m = 0$ and $2h = l + n = 0 \ \forall (u, v) \in G$, the relation (1.10) becomes $2z = \sqrt{4l^2}$. Moreover, because of (2.29), (3.11), (3.12), (3.16), (3.18) we have

$$2z = 2\sqrt{l^2} = 2\sqrt{(l' + n')^2} = 4\sqrt{h'^2} = 4z'.$$

Therefore

$$2z = 4z', \tag{3.20}$$

which, using (3.19), can turn into

$$2z = 4z' = 4z''. \tag{3.21}$$

(iii) From (1.10) again, by means of (3.1), (3.16), (3.18), (3.20) we find that

$$k = -z^2 = -(2z')^2 = -4h'^2. \tag{3.22}$$

The latter, via conclusion (i), becomes

$$k = -4h'^2 = -4h''^2. \quad \square \tag{3.23}$$

Remark 3.1. The middle surface $P(u,v)$ of the line congruence S is the focal surface of the parabolic line congruences $S'$and S''. For the curvature K of $P\,(u,v)$ we have $K = -1/(2z')^2$, which, by making use of (3.16), (3.23), can be written

$$K = -\frac{1}{4h'^2}\,.$$

Hence, according to the property (iii) of the Proposition 3.1 $K = 1/k$. This result is already known, since S is a normal line congruence of a minimal middle surface.

Now, we assume an arbitrary point P on the middle surface $P(u,v)$ and the lines g, g', g'' of the line congruences S, S', S'' respectively that pass through P. Let Z_i, Z'_i, Z''_i, $i = 1, 2$, be the limit points of g, g', g'' respectively (see Fig. 1). Since S is the normal line congruence of $P(u,v)$, the middle plane of S and the tangent plane Π to the surface $P(u,v)$ at the point P coincide. The lines g', g'' lie on Π, they are perpendicular to each other and the points Z'_i, Z''_i, $i = 1, 2$, via conclusion (ii) of the Proposition 3.1, define a square. In addition, the line g is perpendicular to Π and the limit points Z_1, Z_2 of g are symmetrical to Π. As a consequence, taking into account (3.21), (3.22), we have the following

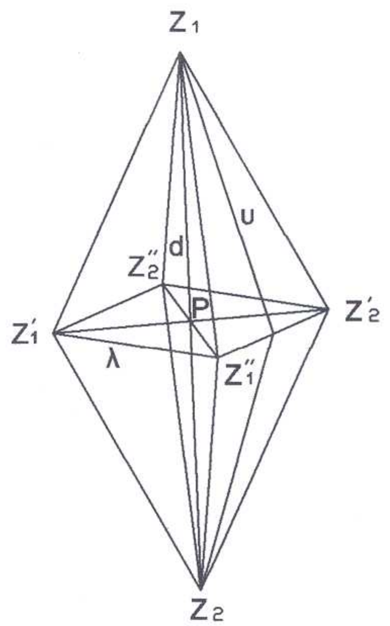

Figure 1: Two square pyramids, symmetrical to the middle plane of the line congruence S

Proposition 3.2 *The points Z_1, Z'_i, Z''_i and Z_2, Z'_i, Z''_i, $i = 1, 2$, define two canonical square pyramids P_1, P_2, which are symmetrical to the middle plane of the line congruence S. The center of their common base $Z'_1Z''_1Z'_2Z''_2$ is the middle point P. The length of its diagonals as well as the heights from the base to the apex of P_1, P_2 are equal to the semi limit distance of S.*

We denote with d the height from the base to the apex Z_1 (or Z_2), with υ the slant height of P_1, P_2 and with λ the length of a side of the base of P_1, P_2 (Fig. 1). According to Proposition 3.2 and the property (ii) of Proposition 3.1

$$d^2 = z^2 = -k, \tag{3.24}$$

$$\upsilon^2 = -\frac{9}{8}\,k, \tag{3.25}$$

$$\lambda^2 = -\frac{k}{2} \tag{3.26}$$

are valid. Thus:
The lateral surface area E and the volume V of each of the pyramids P_1, P_2 are given by the formulae

$$E = -\frac{3}{2}\,k, \tag{3.27}$$

$$V = \frac{1}{6}(-k)^{\frac{3}{2}}. \tag{3.28}$$

However, $K = 1/k$ holds. Thus, the relations (3.27), (3.28) may be written

$$E = -\frac{3}{2K}\,, \tag{3.29}$$

$$V = \frac{1}{6(-K)^{3/2}}\,. \tag{3.30}$$

We focus now on the middle envelopes $M'(u,v)$ and $M''(u,v)$ of the line congruences S' and S'' respectively. Setting $\varphi = 0$ in the relations (2.36), (2.37) and using (2.5), (2.6), (3.1) we obtain from (2.39)

$$\overline{M'P} = \frac{l}{\widetilde{q}}\,(\overline{e}_2 + q\overline{e}_3). \tag{3.31}$$

Similarly, setting $\varphi = \frac{\pi}{2}$, we find

$$\overline{M''P} = \frac{l}{q}\,(\overline{e}_1 + \widetilde{q}\,\overline{e}_3). \tag{3.32}$$

An immediate consequence of the relations (3.31), (3.32) is

Proposition 3.3 *The formulae*

$$k = -\left\langle \overline{M'P}, \overline{M''P} \right\rangle, \tag{3.33}$$

$$\left|\overline{M'M''}\right|^2 = \left|\overline{M'P}\right|^2 + \left|\overline{M''P}\right|^2 + 2k \tag{3.34}$$

are valid.

References

[1] J. Hoschek: *Liniengeometrie*. Bibliographisches Institut, Zürich 1971.

[2] G. Stamou: *Über die orthogonalen Strahlensysteme*. Arch. der Math. **27**, 221–224 (1976).

[3] N.K. Stephanidis: *Strahlensysteme mit gemeinsamer Mittenfläche*. Math. Nachr. **63**, 317–329 (1974).

[4] L. Vanhecke, L. Vermeire: *Focal surfaces of a rectilinear congruence of Guichard, Weingarten or Thybaut, as middle surface for a congruence*. Rad. Jugoslav. Akad. Znan. Umjet. **367**, 5–36 (1974).

Received July 16, 2008; final form December 15, 2008