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Two Synthesis Methods for Non-Circular Cylindrical Gears

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Abstract. Two synthesis methods for the design of non-circular cylindrical gears are presented here. One of the methods generalizes the use of n-lobed ellipses as pitch curves. It is shown that n-lobed hyperbolas and parabolas can be used as well as internal rolling n-lobed ellipses. The resulting transmission ratio is determined by the two lobe numbers of the gears and one real design parameter. The other synthesis method solves the following problem: Prescribed input and output functions of revolution of two planes rotating about two fixed points are given. Determine centrodes (pitch curves) which realize this motion between the two planes by rolling on each other. The general solution takes account of additional mechanical constraints.

Key Words: mechanism, kinematics, non-circular gears, lobed conics, pitch curves, transmission.

MSC 2000: 53A17

1. Introduction

First known publications [2, 8, 13] on *non-circular* cylindrical gears are by H. Holditch (1842), H.T. Brown (1871), and F. Reuleaux (1875). The most common form of non-circular gears is *elliptical* because of the geometric shape of their centrodes (pitch curves). Such gears are congruent and rotate about focal points.

In [9], the design of so-called *modified elliptical gears* was presented by F.L. Litvin. Wunderlich [15] also investigated such gears in terms of centrodes of a special plane motion. He calls such centrodes n-lobed ellipses. D.B. Dooner [4] treats basic relations for generalized function transmission with toothed bodies.

In [7] the synthesis of external non-circular gears is shown where an *m-lobed ellipse* is rolling on an n-lobed ellipse given the lobe numbers n and m as well as the maximum and minimum radii of the m-lobed ellipse. The center distance is then determined. The resulting transmission ratio and the rack centrode are also calculated. The paper studies relations

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between lobe numbers and other geometric parameters by using extremal radii. Here, we will show that a restriction on positive radii is not necessary. In this way n-lobed hyperbolas and parabolas as well as internal running centrodes (internal non-circular gears) are obtained.

In [14], a Fourier series method is used to approximate either one of the pitch curves or the intended transmission ratio.

Continuing interest in the theory of non-circular gearing is shown by recent papers like [3, 5, 6, 7, 10, 11, 14]. One of the challenges is planar motion generation by link mechanisms coupled with circular elements. The problem statement is: Given the input and output functions of revolution of two planes rotating about two fixed points. Determine centrodes which realize this motion between the two planes by rolling on each other. The paper presents a general solution which considers additional mechanical constraints.

2. Basic relations for centrodes of non-circular cylindrical gears

Basic relations for centrodes of non-circular gears follow from the theorem by Aronhold-Kennedy [1] adapted to the pole-configuration of the relative motions of two planes that are rotating about two fixed points in a third plane. For this purpose, in Fig. 1 we consider coordinate systems which represent the planes involved: The coordinate system $(O_1;\, x_1, y_1)$ is attached to a first plane Σ_1 rotating about the point $O_1 = O_0$ with angular velocity $\dot{\varphi}_1(t)$. The coordinate system $(O_2;\, x_2, y_2)$ is attached to a second plane Σ_2 rotating about the point $O_2 = O_4$ with angular velocity $\dot{\varphi}_2(t)$.

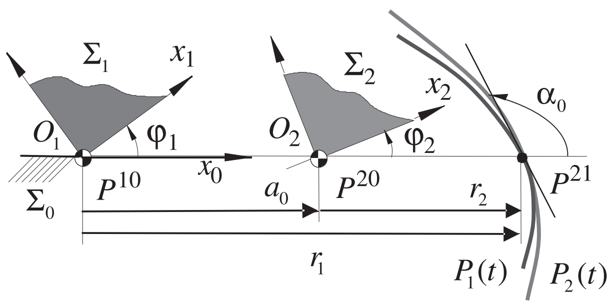


Figure 1: Two planes rotating about two fixed points of a third plane

The coordinate systems $(O_0;\, x_0, y_0)$ and $(O_4;\, x_4, y_4)$ are fixed to the ground plane and it is assumed that their origins have the *center distance*

$$a_0 = \overline{O_1O_2} \neq 0. \tag{1}$$

Let Σ_k denote a coordinate system $(O_k;\, x_k, y_k)$ as well as a plane which is represented by the system. The subscript k may be 0, 1, 2, 3, and 4. The subscr ipt k is used to indicate that any point X of the plane is described by the coordinate system Σ_k . For instance, X_1 indicates a point which is refered to Σ_1: Furthermore, representing the coordinates x_k and y_k of a point X_k as the real and imaginary parts of a complex number, respectively, we obtain

$$X_k = x_k + \mathrm{i}y_k = \rho_k \mathrm{e}^{\mathrm{i}\theta_k}$$

where $\mathrm{i}^2 = -1$. Here, the second equation is called the exponential representation of X_k where ρ_k is the absolute value (modulus) and Θ_k the argument of X_k . Let φ_k be the rotation angle of the frame Σ_k $(k = 1, 2)$.

With this notation [15, 1] we obtain the coordinate transformations

$$X_0 = X_1 e^{i\varphi_1}$$
$$X_4 = X_2 e^{i\varphi_2}$$
$$X_0 = a_0 + X_4.$$

Therefore, the relative displacement of Σ_2 with respect to Σ_1 is represented by

$$X_1 = a_0 e^{-i\varphi_1} + X_2 e^{i(\varphi_2-\varphi_1)}. \tag{2}$$

Now we assume that the three planes perform a composite one-parameter motion. Let $t \in I \subset \mathbb{R}$ be the parameter of this motion with *revolution angles* $\varphi_1(t)$ and $\varphi_2(t)$.

The relative motion of frame Σ_i with respect to frame Σ_0 is a rotation with respect to the pole P^{i0}. The relative motion of frame Σ_2 with respect to Σ_1 has the pole P^{21}. According to the Aronhold- Kennedy theorem the pole distances are related to the angular velocities as follows:

$$\dot\varphi_2(P_0^{20} - P_0^{21}) = \dot\varphi_1(P_0^{10} - P_0^{21}).$$

With our choice of the motion and the coordinate frames, we have

$$P_0^{10} = 0, \quad P_0^{20} = a_0,$$

and

$$\dot\varphi_2(a_0 - P_0^{21}) = -\dot\varphi_1 P_0^{21}. \tag{3}$$

When $\dot\varphi_1 = 0$ and $\dot\varphi_2 \neq 0$ the third pole P^{21} coincides with P^{20}.

In the case $\dot\varphi_1 \neq 0$ we define the instantaneous *transmission ratio* as

$$\omega(t) := \frac{\dot\varphi_2(t)}{\dot\varphi_1(t)} \tag{4}$$

and with (3) the third pole can be determined:

$$P_0^{21} = a_0 \frac{\omega}{\omega - 1}.$$

Then the centrodes $P_1(t)$ and $P_2(t)$ of the motion of frame Σ_2 with respect to frame Σ_1 are determined by the transformations of pole P^{21} into Σ_1 and Σ_2, respectively, at each moment t. In this way, the two centrodes (pitch curves) are determined and we find *polar representations of the centrodes*:

$$P_1(t) = a_0 \frac{\omega(t)}{\omega(t)-1} e^{-i\varphi_1 t)} = r_1(t)\, e^{-i\varphi_1(t)} \quad \text{where} \quad r_1(t) = a_0 \frac{\omega(t)}{\omega(t)-1} \tag{5}$$

and

$$P_2(t) = r_2(t)\, e^{-i\varphi_2(t)} \quad \text{where} \quad r_2(t) = a_0 \frac{1}{\omega(t)-1}. \tag{6}$$

The functions $r_1(t)$ and $r_2(t)$ describe the polar radii and $\varphi_1(t)$ and $\varphi_2(t)$ are the polar angles of the centrodes described with respect to Σ_1 and Σ_2, respectively.

With (3), (4), (5), (6) we can easily conclude

Proposition 1 *a) The pole P^{21} lies outside the line segment $\overline{O_1O_2}$ iff $\omega(t) > 0$.*
b) If $\omega(t) = 0$ then $P^{21} = O_1$. That means $r_1(t) = 0$ and $r_2(t) = -a_0$.

The parameter transformation $\varphi := \varphi_1(t),\ 0 \leq \varphi \leq 2\pi$, introduces the polar angle of P_1 and transforms eqs. (4),(5), (6) into

$$\omega(\varphi) := \frac{\mathrm{d}\varphi_2}{\mathrm{d}\varphi} \tag{7}$$

$$P_1(\varphi) = a_0\,\frac{\omega(\varphi)}{\omega(\varphi)-1}\,\mathrm{e}^{-\mathrm{i}\varphi} = r_1(\varphi)\mathrm{e}^{-\mathrm{i}\varphi} \tag{8}$$

where

$$r_1(\varphi) = a_0\,\frac{\omega(\varphi)}{\omega(\varphi)-1} \tag{9}$$

$$P_2(\varphi) = r_2(\varphi)\mathrm{e}^{-\mathrm{i}\varphi_2(\varphi)} \tag{10}$$

where

$$r_2(\varphi) = a_0\,\frac{1}{\omega(\varphi)-1}\,. \tag{11}$$

By dividing the polar radii, it follows that

$$\omega(\varphi) = \frac{r_1(\varphi)}{r_2(\varphi)}\,. \tag{12}$$

Furthermore, the center distance satisfies

$$a_0 = r_1(\varphi) - r_2(\varphi). \tag{13}$$

A plane motion described by the equations (1) to (13), together with the given assumptions, is called a *gear motion* (GM). It is called n_1/n_2-periodic if n_1 full revolutions of the frame Σ_1 about O_1 and n_2 full revolutions of the frame Σ_2 about O_2 result in the initial position $\varphi_1 = \varphi_2 = 0 \bmod 2\pi$.

Proposition 2 *A GM is n_1/n_2-periodic iff*

$$\int\limits_0^{2\pi n_1} \omega(x)\,\mathrm{d}x = 2\pi n_2\,.$$

Proof: If a GM is n_1/n_2-periodic, then $2\pi n_1$ is the revolution angle for n_1 full revolutions of the frame Σ_1 about O_1. The revolution angle for n_2 full revolutions of the frame Σ_2 about O_2 is $\varphi_2(2\pi n_1) = 2\pi n_2$. Applying (7) to the left hand side of this equation yields the n_1/n_2-periodic condition. If the n_1/n_2-periodic condition holds for a GM then by (7) it follows

$$\varphi_2(2\pi n_1) = \int\limits_0^{2\pi n_1} \omega(x)\,\mathrm{d}x = 2\pi n_2\,.$$

Therefore, n_1 full revolutions of the frame Σ_1 result in n_2 full revolutions of the frame Σ_2. □

Let us consider the case in which the polar radius $r_j(\varphi)$ is $\frac{2\pi}{n_j}$-periodic for some index j. That means

$$r_j(\varphi) = r_j\left(\varphi + \frac{2\pi}{n_j}\right), \quad j = 1, 2.$$

If $r_j(\varphi)$ is $\frac{2\pi}{n_j}$-periodic then it is also $k\,\frac{2\pi}{n_j}$-periodic with any integer k, as well as 2π-periodic.

By Eq. (13) it follows

Proposition 3 *For each integer $k \in \mathbb{Z}$, $r_1(\varphi)$ is $\frac{2\pi}{k}$-periodic iff $r_2(\varphi)$ is $\frac{2\pi}{k}$-periodic.*

By Eq. (8) we can easily show

Proposition 4 *The centrode P_1 is a closed curve when $r_1(\varphi)$ is 2π-periodic and finite for all φ.*

Furthermore, we find

Proposition 5 *For $j = 1, 2$, the polar radius $r_j(\varphi)$ is $\frac{2\pi}{n_j}$-periodic iff the transmission ratio $\omega(\varphi)$ is $\frac{2\pi}{n_j}$-periodic.*

Proof: Eqs. (9) and (11) yield $\omega(\varphi) = \dfrac{r_1(\varphi)}{r_1(\varphi) - a_0}$ and $\omega(\varphi) = \dfrac{r_2(\varphi) + a_0}{r_2(\varphi)}$. □

In order to become more familiar with the relations we study the following

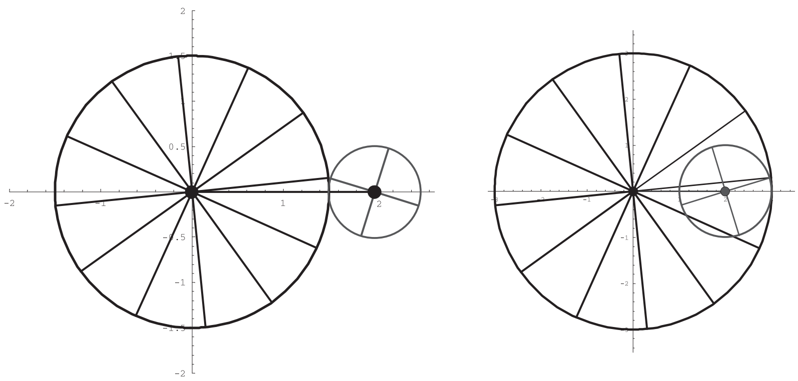
Question: Determine the centrodes for given input and output functions of revolution and the centre distance.

Solution: The first derivatives of the given functions $\varphi_j(t)$ determine the transmission ratio with (4). Choosing $\varphi := \varphi_1(t)$ and introducing the transmission-ratio into equations (9) and (11), the polar radii $r_j(\varphi)$ are determined. However, for the polar representation of the centrode (10) we need the revolution-angle function explicitly. It can be obtained by integration of the transmission ratio (7):

$$\varphi_2(\varphi) = \int_0^{\varphi} \omega(x)\,\mathrm{d}x + \varphi_{20} \tag{14}$$

with an integration constant φ_{20}.

Therefore, the solution of the problem depends on the integrability in Eq. (14). This is not necessary fulfilled for given arbitrary input and output functions $\varphi_j(t)$. If Eq. (14) is integrable then $\varphi_2(\varphi)$ is obtained. Inserting the revolution angles into Eqs. (8) and (10), the centrodes are uniquely determined.



ω = −3 < 0 ω = 3 > 0

Figure 2: Constant transmission ratios

Examples: Two numerical examples are given in Fig. 2. In each case the transmission ratio is chosen to be constant. In this way, the resulting centrodes are circles, rolling externally ($\omega < 0$) and internally ($\omega > 0$), respectively, corresponding to Proposition 1.

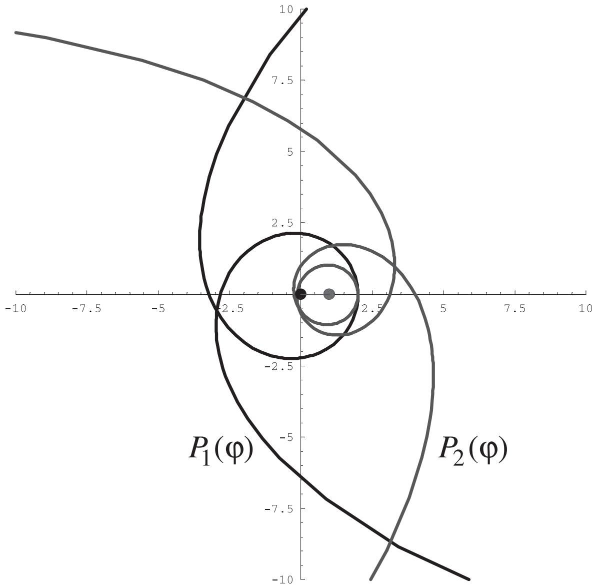


Figure 3: Example centrodes $P_j(\varphi)$ for the position $\varphi = 0.1$

In the example illustrated in Fig. 3, the transmission ratio is the 2π-periodic function $\omega(\varphi) = \dfrac{3+\cos\varphi}{2}$. Then, the resulting polar radii are $r_1(\varphi) = a_0\,\dfrac{3+\cos\varphi}{1+\cos\varphi}$ and $r_2(\varphi) = r_1(\varphi) - a_0$ which are also 2π-periodic. The second revolution angle is $\varphi_2(\varphi) = \frac{3}{2}\varphi + \frac{1}{2}\sin\varphi$. The GM is 2/3-periodic but the centrodes are not closed because $r_1(\pi)$ is not finite.

Proposition 6 *The common tangent to the centrodes at a pole $P^{21}(\varphi)$ has the slope angle*

$$\alpha_1(\varphi) = \alpha_0(\varphi) - \varphi \tag{15}$$

with respect to the x_1-axis, and the slope angle

$$\alpha_0(\varphi) = \arctan\frac{(\omega(\varphi)-1)\omega(\varphi)}{\dot\omega(\varphi)}$$

with respect to the x_0-axis, respectively. If the pole $P^{21}(\varphi_\infty)$ is a point at infinity and $\dot\omega(\varphi_\infty) \neq 0$ then $\alpha_0(\varphi_\infty) = 0$.

Proof. The slope angle of the tangent line at a pole $P^{21}(\varphi)$ with respect to the x_1-axis is $\alpha_1(\varphi) = \arg\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}P_1(\varphi)\right) = (\dot r_1(\varphi) - \mathrm{i}r_1(\varphi))\exp(-\mathrm{i}\varphi)$.
Therefore, $\alpha_1(\varphi) = \alpha_0(\varphi) - \varphi$

$$\alpha_0(\varphi) = \arg(\dot r_1(\varphi) - \mathrm{i}r_1(\varphi)) = \arctan\frac{(\omega(\varphi)-1)\omega(\varphi)}{\dot\omega(\varphi)}. \qquad \square$$

A geometrical interpretation of Equation (15) is given in Fig. 1. The slope angle in question is shown as α_0.

By (9) and (11) we see that each point at infinity of a centrode is also a point on the other centrode. Such a point at infinity is given by a parameter $\varphi_\infty \in [0, 2\pi]$ with

$$\omega(\varphi_\infty) = \dot\varphi_2(\varphi_\infty) = 1. \tag{16}$$

In this case with $\dot\omega(\varphi_\infty) \neq 0$ Equation (15) yields $\alpha_0(\varphi_\infty) = 0$.

3. Lobed Conics

Let us consider the following problem: Determine the centrode $P_2(\varphi)$ and also the transmission ratio when the centrode $P_1(\varphi)$ is given by a generalized non-degenerate conic section, a so-called n_1-*lobed conic*.

Solution: Let the n_1-lobed conic centrode $P_1(\varphi)$ be given by the polar equation

$$r_1(\varphi_1) = \frac{h_1}{1 - \varepsilon_1 \cos n_1\varphi_1}\,. \tag{17}$$

In the case $n_1 = 1$ it is well known that this polar equation describes a non-degenerate conic where φ_1 is the polar angle, h_1 is the half parameter, and ε_1 is the numerical eccentricity.

In particular, the non-degenerate conics are the ellipse ($|\varepsilon_1| < 1$), parabola ($|\varepsilon_1| = 1$), and hyperbola ($|\varepsilon_1| > 1$).

In the general case n_1 is the number of lobes. We see

$$r_1\left(\varphi_1 + \frac{2\pi}{n_1}\right) = \frac{h_1}{1 - \varepsilon_1 \cos n_1(\varphi_1 + \frac{2\pi}{n_1})} = r_1(\varphi_1).$$

Hence, the polar radius $r_1(\varphi_1)$ is $\frac{2\pi}{n_1}$-periodic with the lobe number n_1. Now we proceed to the solution of the given problem. By Eq. (13) we obtain $r_2(t) = r_1(t) - a_0$. Then $\omega(t) = \frac{r_1(t)}{r_2(t)}$ is determined. Then we try to integrate Eq. (14). In the case of integrability the desired centrode $P_2(\varphi)$ is given by Eq. (10).

We can ask for a solution centrode $P_2(\varphi)$ which has a periodic number n_2. Then Propositions 2 and 5 tell us that the following equation has to be fulfilled:

$$\frac{2\pi}{n_2} = \frac{1}{n_1}\int_0^{2\pi} \omega(\varphi)\,\mathrm{d}\varphi = \frac{1}{n_1}\int_0^{2\pi} \frac{r_1(\varphi)}{r_1(\varphi) - a_0}\,\mathrm{d}\varphi. \tag{18}$$

This condition can be used to determine the eccentricity and half parameter for $r_1(\varphi_1)$ according to (17) when the center distance is fixed and the GM is specified to be n_1/n_2-periodic. Eq. (18) is integrable in all cases. In the elliptic and hyperbolic case we obtain

$$\begin{aligned}
r_1(\varphi_1) &= a_0\,\frac{n_1(1-m^2)}{n_1(1-m^2) + n_2(1+m^2) + 2mn_2\cos n_1\varphi_1}\\
r_2(\varphi_2) &= a_0\,\frac{-n_2(1-m^2)}{n_1(1+m^2) + n_2(1-m^2) - 2mn_1\cos n_2\varphi_2}\\
\varphi_2(\varphi_1) &= \frac{2}{n_2}\arctan\left(\frac{m-1}{m+1}\tan n_1\frac{\varphi_1}{2}\right).
\end{aligned}$$

The parameter $m \in \mathbb{R}\backslash\{-1, 0, 1\}$ determines the class of the lobed centrode as given in Table 1. Note that an arbitrary center distance can be used.
Different cases are illustrated in Figs. 4–7.

In the case of lobed hyperbolas the number of lobes corresponds to the number of pairs of branches that is also the number of pairs of asymptotes. From the condition for infinite points,

$$\omega(\varphi_1) = \frac{n_1}{n_2}\,\frac{(m-1)(1+m)}{1+m^2+2m\cos n_1\varphi_1} = 1\,,$$

Table 1: Parameter and class of conic

Parameter	n_1*-lobed centrode* $P_1(\varphi)$	n_2*-lobed centrode* $P_2(\varphi)$
$0 < \lvert m\rvert < 1$	ellipse	ellipse and $\omega(\varphi) < 0$
$1 < \lvert m\rvert < m^* = \left\lvert\dfrac{n_1+n_2}{n_1-n_2}\right\rvert$	hyperbola	hyperbola
$m = m^*$ and $n_1 \neq n_2$	parabola	parabola
$m^* < \lvert m\rvert$	ellipse	ellipse and $\omega(\varphi) > 0$

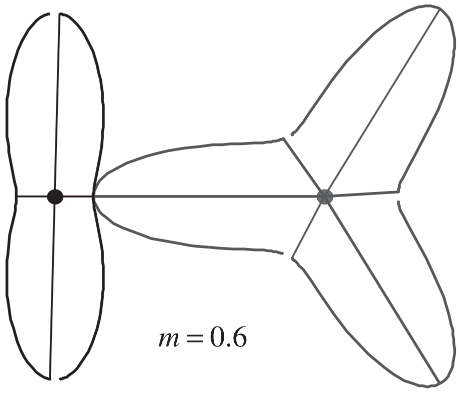


Figure 4: 2-lobed with 3-lobed ellipse

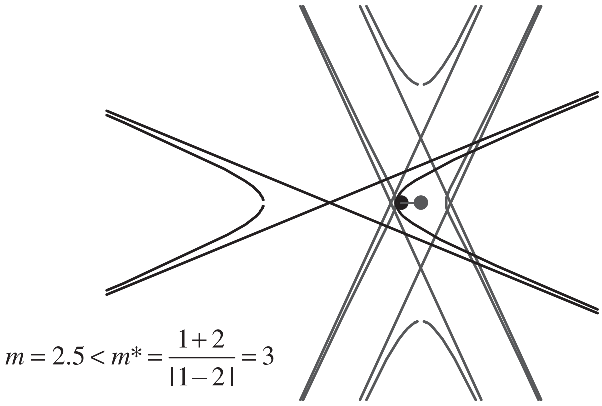


Figure 5: 1-lobed with 2-lobed hyperbola

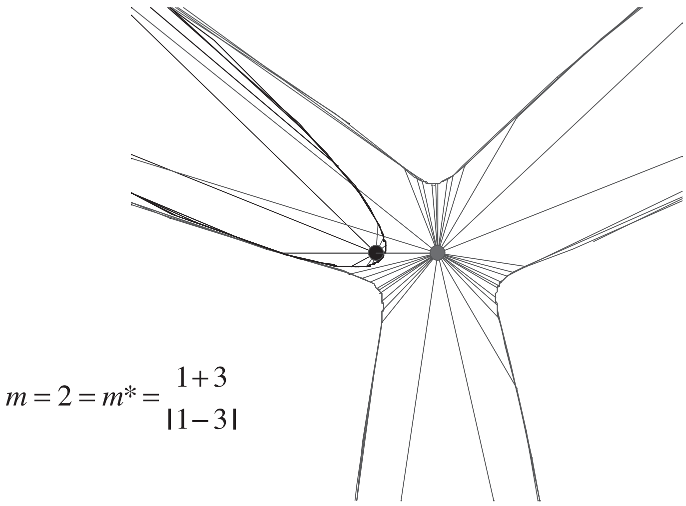


Figure 6: 1-lobed with 3-lobed parabola

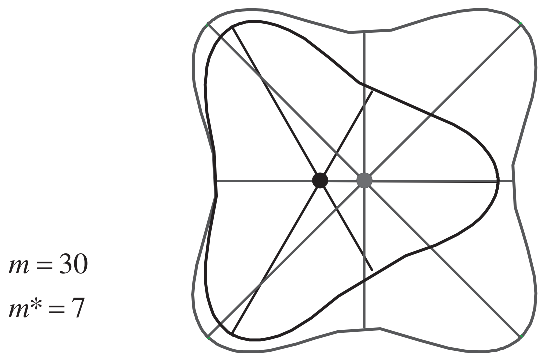


Figure 7: 3-lobed with 4-lobed ellipse

we find

$$\cos n_1\varphi_1 = \frac{n_1(1-m^2)+n_2(1+m^2)}{2mn_2}\,.$$

As in the considered case the absolute value of the denominator is not smaller than the absolute value of the numerator, the condition has two principal solutions $\varphi_1^{(1,2)}$. So, we have

Proposition 7 *Both centrodes have infinite polar radii for the parameters*

$$\varphi_1^{(1,2,\ldots,2n_1)} = \varphi_1^{(1,2)} + \sum_{k=0}^{n_1-1} 2\pi k/n_1.$$

Figure 5 illustrates this proposition. We can summarize:

Theorem 1 *For prescribed lobe numbers n_1, n_2 and each real number $m \in \mathbb{R}\backslash\{-1,0,1\}$ with restrictions according to Table 1, lobed conics of the same class are determined in order to represent the pitch curves of an n_1/n_2-periodic motion composed of two rotations about focal points of the lobed conics.*

4. Synthesis of periodic GM with given revolution-angle functions

It may be considered a disadvantage in technological application of lobed conics that the transmission ratio is determined by the three parameters n_1, n_2, and m of the design. Motivated by mechanical applications as shown in the introduction, we are now facing a problem where revolution-angle functions $\varphi_1(t)$ and $\varphi_2(t)$ of a motion composed of two rotations, are given. We want to find centrodes that generate these revolution-angle functions. The following ansatz (approach) to the revolution-angle functions is proposed

$$\Phi_j(t) := n_j t + S_j(t) \tag{19}$$

where

$$S_j(t) = \frac{a_{j0}}{2} + \sum_{k=1}^{Nj} \left(a_{jk}\cos m_{jk}t + b_{jk}\sin m_{jk}t\right), \quad j = 1,2 \tag{20}$$

with so-called *rotation numbers* n_j, *amplitudes* a_{jk}, b_{jk}, *periods* T_j, *frequencies* $\frac{2\pi}{T_j}$, and *frequency multiples* $m_{jk} := \frac{2\pi}{T_j}k$.

The function $S_j(t)$ is known as Fourier partial sum of the first N_j terms. The functions $S_j(t)$ and their derivatives are T_j-periodic. So, the first derivative is

$$\dot\Phi_j(t) = n_j + \sum_{k=1}^{N_j}\left(-a_{jk}m_{jk}\sin(m_{jk}t) + b_{jk}m_{jk}\cos(m_{jk}t)\right) \quad \text{for} \quad j = 1,2. \tag{21}$$

Therefore, the transmission ratio is

$$\omega(t) = \frac{\dot\Phi_2(t)}{\dot\Phi_1(t)}\,. \tag{22}$$

Looking at Proposition 5, for $\dot\Phi_j(t)$ we choose the period

$$T_j = \frac{2\pi}{z_j} \tag{23}$$

with an integer z_j, in order to obtain a periodic transmission ratio. Then, the frequency multiples are

$$m_{jk} = z_j k. \tag{24}$$

Now it is easy to show:

Proposition 8 *With $T_j = \frac{2\pi}{z_j}$, $j = 1,2$, the transmission ratio $\omega(t) = \frac{\dot\Phi_2(t)}{\dot\Phi_1(t)}$ has the period $T = \frac{2z}{z_1 z_2}\pi$ where $z = LCM(z_1, z_2)$ is the least common multiple of the integers z_1 and z_2.*

A periodic transmission ratio is necessary but not sufficient for a periodic GM. So, we now require the period of the revolution-angle function, assuming a common period T for the Fourier partial sums. For any integer g_j we have

$$\Phi_j(t + g_jT) = \Phi_j(t) + g_jn_jT = \Phi_j(t) + g_jn_j\frac{2\pi z}{z_1z_2}\,.$$

For $t = 0$ it follows

$$\Phi_j(g_jT) = \Phi_j(0) + g_jn_j\frac{2\pi z}{z_1z_2}$$

and there is always a g'_j with

$$\Phi_j(g'_jT) = \Phi_j(0) + 2\pi n_j \quad (j = 1, 2).$$

For convenience let us call the approach (19) the *Fourier ansatz* (FA). Then it is proven:

Proposition 9 *Revolution-angle functions of the FA result in an n_1/n_2-periodic GM if the Fourier partial sums have the common period of the transmission ratio according to Proposition 8.*

Remark: With arbitrary revolution angle functions we get an n_1/n_2-periodic GM only in the case in which the integral in Proposition 2 exists. With the proposed FA, a n_1/n_2-periodic GM is easily obtained by choosing coefficients n_j; a_{jk}, b_{jk} and a common period of the transmission ratio.

The Figures 8–10 display some interesting examples.

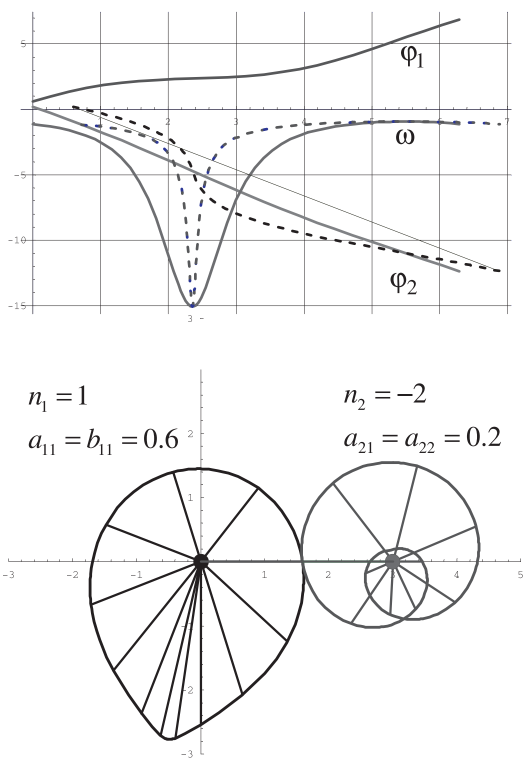


Figure 8: Functions and centrodes of a Fourier ansatz with $z_1 = z_2 = z$

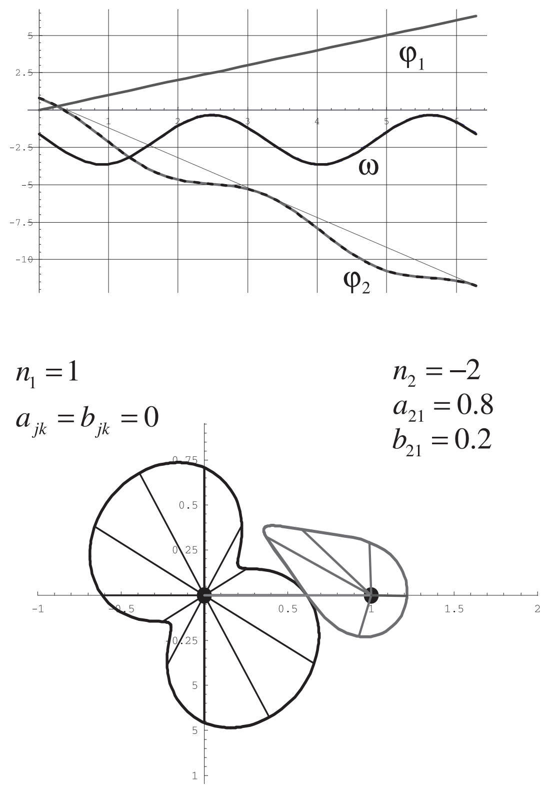


Figure 9: Functions and centrodes of a Fourier ansatz with $z_1 = 1$, $z_2 = z = 2$

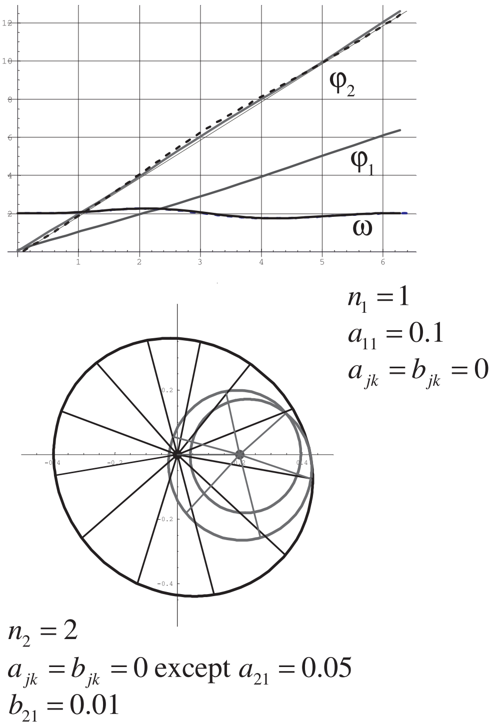


Figure 10: Functions and centrodes of a Fourier ansatz with $z_1 = 1$, $z_2 = z = 2$

Theorem 2 *For each given pair of piecewise continously differentiable revolution-angle functions $\varphi_1(t)$ and $\varphi_2(t)$ on an nterval $0 < t < T$, where T is a common period for the Fourier partial sums, the centrodes of a n_1/n_2-periodic GM are uniquely determined in such a way that their rolling generates the given revolution-angle functions.*

The proof of Theorem 2 is a conclusion drawn from the Dirichlet theorem: For given $\varphi_j(t)$, $0 < t < T$, we satisfy the condition

$$\Phi_j(t) = n_j t + S_j(t) \doteq \varphi_j(t)$$

in terms of Fourier-series convergence determining the coefficients a_{jk}, b_{jk} by the Euler formula applied to the function

$$S_j(t) = \varphi_j(t) - n_j t.$$

Conclusions

With Theorem 1, the paper revealed all types of lobed conics which represent the pitch curves of an n_1/n_2-periodic GM. Lobed ellipses, hyperbolas, and parabolas can be used as well as internal rolling lobed ellipses. The transmission ratio is determined by the two lobe numbers of the gears and one real design parameter.

With Theorem 2, a second synthesis method is presented: For each everywhere-finite and piecewise monotone driving and driven functions of revolution, an approximating function is determined which includes a Fourier series. Then, the centrodes can be calculated so as to realize an n_1/n_2-periodic GM. Thereby, the coefficients of the driving and driven functions can be explained geometrically. In an application, the coefficients of the Fourier-series can be easily calculated by the help of equally spaced control points for the desired driving and driven functions.

It is easily possible to obtain closed centrodes. However, segments of centrodes are also interesting. Engineering requirements like

- Rotation index of a centrode is to be 1
- Curvature of a centrode is to be bounded
- Centrode is to be convex suggest further investigation.

What are appropriate relations for the coefficients of the Fourier-series?

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